Explicit Definitions for the Electromagnetic Energies in Electromagnetic Radiation

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Abstract: The total electromagnetic energy of a pulse radiator in free space is separated into a Coulomb-velocity energy, a radiative energy, and a macroscopic Schott energy. The Coulomb-velocity energy becomes zero as soon as its sources have disappeared. The radiative energy leaves the radiator and propagates to the surrounding space. The macroscopic Schott energy continues to exist for a short time after the sources have disappeared. The Poynting vector describes the total power flux density related to the total electromagnetic energy and should include a real radiative power flux by the propagation of the radiative energy and a pseudo power flux caused by the fluctuation of the reactive energy. All energies are defined with explicit expressions in which the vector potential plays an essential role. The time domain formulation and the frequency domain formulation of the theory are in consistent with each other. The theory is verified with Hertzian dipole. Numerical examples demonstrate that the theory may provide insightful interpretation for electromagnetic radiation problems.

Index Terms: Reactive energy, Schott energy, radiative energy, electromagnetic coupling, Poynting vector.

1. Introduction

The electromagnetic radiation problems have been intensively investigated for more than a hundred years. There are still no widely accepted explicit expressions for the macroscopic electromagnetic reactive energy and the radiative energy of a radiator [1]-[15]. In classical charged particle theory, the fields associated with charged particles can be divided into Coulomb fields, velocity fields and radiative fields [16], [17]. The Coulomb fields and velocity fields carry energies in the space, and are jointly referred to as Coulomb-velocity energy in this paper. The radiative fields are generated by acceleration of charged particles, emitting radiative energy to the surrounding space. In free space, the Coulomb-velocity energies are considered to be attached to their charged particles. They become zero when the charged particles have disappeared. On the contrary, after being radiated by the charged particles, the radiative energies depart from the sources and keep propagating to the remote infinity. They continue to exist after their generation sources have disappeared and can couple with other sources they encounter in their journey. Schott energy was first introduced in 1912 by Schott [18]-[20]. It is reversible and is considered to be responsible for energy exchange between the Coulomb-velocity energies and the radiative energies. Although it is natural to consider that the reactive energy in macroscopic electromagnetics is similar to the Coulomb-velocity energy and the Schott energy, no successful attempt has been found in published papers to handle the reactive energy in this manner.

For a harmonic field over the time interval $(-\infty < t < \infty)$, the radiative energy expands to the whole space and the total radiative energy is infinitely large [14], so is the total electromagnetic energy. As there is no explicit for the reactive energy, it is conventionally evaluated by subtracting an additional term associated with the radiative power from the total electromagnetic energy density. However, the additional term is obtained with some approximation and it is not easy to give a general expression for it because the propagation patterns are quite different for different radiators [1], [5].

A better strategy is to examine a pulse radiator because its radiative energy and its total electromagnetic energy are all finite. Consider in free space a radiator with charge density $\rho(r_1, t)$ and current density $\mathbf{J}(\mathbf{r}_1, t)$, where $\mathbf{r}_1 \in V_s$, and $t \in [0, T]$. The total electric energy and the total magnetic energy can be written as

$$W_{tot}^{e}(t) = \int_{V_\infty} \left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) d\mathbf{r}_1 = \int_{V_s} \left(\frac{1}{2}\rho\phi\right) d\mathbf{r}_1 + \int_{V_\infty} \left(-\frac{1}{2}\mathbf{D} \cdot \frac{\partial \mathbf{A}}{\partial t}\right) d\mathbf{r}_1 \qquad (1)$$

$$W_{tot}^{m}(t) = \int_{V_\infty} \left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) d\mathbf{r}_1 = \int_{V_s} \left(\frac{1}{2}\mathbf{J} \cdot \mathbf{A}\right) d\mathbf{r}_1 + \int_{V_\infty} \left(\frac{1}{2}\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{A}\right) d\mathbf{r}_1 \qquad (2)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electromagnetic fields, $\mathbf{D}$ and $\mathbf{B}$ are the flux densities. The scalar potential $\emptyset$ and the vector potential $\mathbf{A}$ are subject to the Lorentz Gauge and their reference zero points are put at the infinity. Under these conditions, [21] has shown that the potentials are Gauge-invariant. Note that in deriving (1) and (2), the volume integrations with respect to the divergence terms have been transformed to surface integrals, which are zeros because we can always put the integration surface in the region that the fields of the pulse radiator cannot reach.

Denote

$$W_\rho(t) = \int_{V_s} \frac{1}{2}\rho(\mathbf{r}_1, t)\phi(\mathbf{r}_1, t)\, d\mathbf{r}_1 \qquad (3)$$

$$W_J(t) = \int_{V_s} \frac{1}{2}\mathbf{J}(\mathbf{r}_1, t) \cdot \mathbf{A}(\mathbf{r}_1, t)\, d\mathbf{r}_1 \qquad (4)$$

Hereafter, $W_{\rho J} = W_\rho + W_J$ is referred to as the Coulomb-velocity energy. As can be checked clearly from (3) and (4), the Coulomb-velocity energy appears and disappears simultaneously with its sources. It seems natural to choose the Coulomb-velocity energy alone as the reactive energy. However, we will see that this choice is not proper. Since the total electric (magnetic) energy consists of the electric (magnetic) radiative energy and the electric (magnetic) reactive energy. If we take the Coulomb-velocity energy as the reactive energy, then we have to define the second integral at the RHS of (1), i.e., $\int_{V_\infty}(-0.5\mathbf{D} \cdot \partial\mathbf{A}/\partial t)d\mathbf{r}_1$, as the electric radiative energy and that in (2), i.e., $\int_{V_\infty}(0.5\, \partial\mathbf{D}/\partial t \cdot \mathbf{A})d\mathbf{r}_1$, as the magnetic radiative energy. They are not equal to each other. This is not acceptable because in free space the electric radiative energy of a radiator usually equals to its magnetic energy, which can be justified from the far-fields of a radiator with finite size. We have further checked in the supplementary material that it is not proper to define W_J in (4) as the magnetic reactive energy, while it may be proper to define W_ρ in (3) as the electric reactive energy. Therefore, in order to overcome the inconsistency, we make the magnetic radiative energy equal to the electric radiative energy in free space,

$$W_{rad}^{m}(t) = W_{rad}^{e}(t) = \frac{1}{2}W_{rad}(t) = \int_{V_\infty} \left(-\frac{1}{2}\mathbf{D} \cdot \frac{\partial \mathbf{A}}{\partial t}\right) d\mathbf{r}_1 \qquad (5)$$

Consequently, the total electric energy in (1) can be expressed as

$$W_{tot}^{e}(t) = W_\rho(t) + W_{rad}^{e}(t) \quad (6)$$

and the total magnetic energy in (2) includes an additional term except the velocity energy and the magnetic radiative energy,

$$W_{tot}^{m}(t) = W_J(t) + W_{rad}^{m}(t) + \int_{V_\infty} \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{D} \cdot \mathbf{A})d\mathbf{r}_1 \qquad (7)$$

It has been demonstrated in [22] that the last term in the RHS of (7) is corresponding to the Schott energy in the charged particle theory [18]-[20] by applying the Lienard-Wiechert potentials [16] to a moving charge. We denote it as the macroscopic Schott energy,

$$W_S(t) = \int_{V_\infty} \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{D}\cdot\mathbf{A})d\mathbf{r}_1 \quad (8)$$

The total electromagnetic energy is the sum of (1) and (2). Substituting (5)-(8) into them gives [23], [24],

$$W_{tot}(t) = W_{\rho J}(t) + W_{rad}(t) + W_S(t) \quad (9)$$

By subtracting the radiative energy from the total electromagnetic energy, the total reactive energy is consequently expressed by

$$W_{react}(t) = W_{tot} - W_{rad} = W_{\rho J}(t) + W_S(t) \quad (10)$$

(9) is the proposed energy separation equation for a pulse radiator. We want to emphasize three points at first:

(i) The energy separation equation (9) for pulse radiator is directly derived from Maxwell equations with no approximation.

(ii) The electric radiative energy equals the magnetic radiative energy, which is in consistent with the practical situation of a radiator in free space.

(ii) The macroscopic Schott energy is related to the Schott energy in the charged particle theory, both are full time derivatives [22].

We are to further verify the separation in the following sections:

(i) In Section 2, explicit expressions for the energies are provided, which show that the Coulomb-velocity energy is attached to its sources, and soon after the sources have disappeared, the macroscopic Schott becomes zero while the radiative energy keeps propagating with a constant amplitude.

(ii) In Section 3, by applying the energy separation formulation to harmonic waves, it is verified that the time domain formulation of the theory is in consistent with its frequency domain formulation, which has been discussed in [14], [15] and has been comprehensively compared with other frequency domain formulations with several examples.

(iii) In Section 4, Hertzian dipole is used as a standard validation example because the exact solutions for its fields and potentials are available both in time domain and in frequency domain, together with a well-established equivalent circuit model. It is checked that the results in both formulations are exactly in agreement with that of the circuit model. All expressions of the electromagnetic energies and powers corresponding to the Hertzian dipole are derived in the supplementary material.

On the other hand, Poynting vector is widely considered as the electromagnetic power flux density [25]. Poynting Theorem describes the relationship between the Poynting vector, the varying rate of the total electromagnetic energy densities, and the work rate done by the exciting source. It provides an intuitive description of the propagation of the electromagnetic energy. However, the interpretation of the Poynting vector has always been controversial [26]-[30], and some researchers have pointed out that Poynting Theorem may have not been used in the correct way in some situations [32], [33]. This difficulty is largely due to the fact that it is not easy to separate the real radiative power flux from the Poynting vector. As the Poynting vector is related to the total electromagnetic energy, it should include a real radiative power flux from the contribution of the radiative energy and a pseudo power flux due to the fluctuation of the reactive energy. Therefore, a new energy-power balance equation at a certain instant time is proposed in Section 2. It is based on the Poynting relationship, only with some substitutions and reorganizations that are derived from Maxwell equations.

Two kinds of time domain formulations for this issue can be found in published literatures. One was proposed by Shlivinski and Heyman [2], [3], the other was proposed by Vandenbosch [6], [7]. The first one is an approximate method, the second one is sometimes not in consistent with its counterpart in frequency domain. The formulation proposed in this paper will remedy these problems. In Section 5, a loop pulse radiator and a Yagi antenna are analyzed with the proposed theory. They are not for comparison with the other time domain formulations but for the purpose to show what we can do with the proposed expressions. Numerical examples for comparison among various formulations in frequency domain can be found in [14], [15], [24].

The theory is briefly discussed in Section 6, where it is emphasized that the theory is neither a static limit formulation nor a kind of updated version of the Carpenter formulation [28].

2. Explicit Expressions for Energies of a Pulse Radiator in Free Space

For a pulse radiator in free space, its total electromagnetic energy is separated into three parts, as shown in (9). Substituting (3)~(5) and (8) into (9), we can write the energy separation expression explicitly as,

$$W_{tot}(t) = \int_{V_S} (\tfrac{1}{2}\rho\phi + \tfrac{1}{2}\mathbf{J}\cdot\mathbf{A})\, d\mathbf{r}_1 + \int_{V_\infty} \left(-\mathbf{D}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) d\mathbf{r}_1 + \int_{V_\infty} \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{D}\cdot\mathbf{A}) d\mathbf{r}_1 \qquad (11)$$

where the retarded scalar potential $\phi(\mathbf{r},t)$ and the retarded vector potential $\mathbf{A}(\mathbf{r},t)$ are evaluated at the observation point $\mathbf{r}$ and the time t in their usual way,

$$\phi(\mathbf{r},t) = \int_{V_S} \frac{\rho(\mathbf{r}_1,t')}{4\pi\varepsilon_0 R_1} d\mathbf{r}_1 \qquad (12)$$

$$\mathbf{A}(\mathbf{r},t) = \mu_0 \int_{V_S} \frac{\mathbf{J}(\mathbf{r}_1,t')}{4\pi R_1} d\mathbf{r}_1 \qquad (13)$$

In the above equations, $t' = t - R_1/c$ is the retarded time, and c is the light velocity in vacuum and $R_1 = |\mathbf{r}-\mathbf{r}_1|$ is the distance between the two positions. μ_0 and ε_0 are respectively the permeability and permittivity in free space. It can be seen from (11) that the total energy consists of integration over the source region or the whole three-dimensional space. The integrands are all products of two quantities, the left ones are source distributions or their fields, while the right ones are potentials.

The electric field can be expressed in terms with potentials as $\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t) - \partial\mathbf{A}(\mathbf{r},t)/\partial t$. With some tedious derivations detailed in the supplementary material, all energies can be expressed with integrations over the source region, which is much more efficient than to evaluate them with integrations over the whole three-dimensional space. Specifically, the macroscopic Schott energy can be expressed with a three-fold integration,

$$W_S(t) = \int_{V_\infty} \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{D}\cdot\mathbf{A}) d\mathbf{r}_1 = -\frac{1}{8\pi\varepsilon_0}\int_{V_S}\int_{V_S}\frac{1}{r_{21}}\int_{t-r_{21}/c}^{t} \times$$

$$\left[\rho(\mathbf{r}_1,\tau)\dot{\rho}\left(\mathbf{r}_2, 2t-\tau-\frac{r_{21}}{c}\right) + c^{-2}\dot{\mathbf{J}}\left(\mathbf{r}_1, 2t-\tau-\frac{r_{21}}{c}\right)\cdot\mathbf{J}(\mathbf{r}_2,\tau)\right] d\tau d\mathbf{r}_2 d\mathbf{r}_1 \qquad (14)$$

where $r_{21} = |\boldsymbol{r}_2 - \boldsymbol{r}_1|$, and the upper script "·" represents time derivative. Note that $\rho(\boldsymbol{r}_1,t_1)$, $\boldsymbol{J}(\boldsymbol{r}_1,t_1)$ and $\rho(\mathbf{r}_2,t_2)$, $\mathbf{J}(\mathbf{r}_2,\boldsymbol{t}_2)$ stand for the sources at $(\mathbf{r}_1,t_1)$ and $(\mathbf{r}_2,t_2)$, respectively. They are the same sources on the same radiator. As checked in the supplementary material, for a pulse source in $[0,T]$, the innermost integral becomes zero when $t \geq T + 0.5t_{max}$, where $t_{max} = r_{21,max}/c$ is the largest travelling time between two source points. This means that, soon after the sources have disappeared, the volume integral of $0.5\,\partial(\mathbf{D}\cdot\mathbf{A})/\partial t$ over the whole space becomes zero even although $\partial(\mathbf{D}\cdot\mathbf{A})/\partial t$ is not zero everywhere in the space.

In order to illustrate the property of the radiative energy in a simpler way, we introduce a principal radiative energy as follows,

$$W_{rad}^{pri}(t) = \int_{V_\infty} \frac{1}{2}\left(\frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{A} - \mathbf{D}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) d\mathbf{r}_1 \qquad (15)$$

Making use of (5) and (15) , we can write the radiative energy as

$$W_{rad}(t) = W_{rad}^{pri}(t) - W_S(t) \quad (16)$$

It is straightforward to check that $W_{tot}(t) = W_{rad}(t) + W_{react}(t) = W_{rad}^{pri}(t) + W_{\rho J}(t)$.

As shown in the supplementary material, the principal radiative energy can be evaluated with the integration over the source region,

$$W_{rad}^{pri}(t) = \frac{1}{8\pi\varepsilon_0}\int_{V_s}\int_{V_s}\frac{1}{r_{21}}\int_{r_{21}/c}^{t} \times$$
$$\left\{\begin{array}{l}\left[\dot{\rho}(\mathbf{r}_1,\tau)\rho\left(\mathbf{r}_2,\tau-\frac{r_{21}}{c}\right)-\dot{\rho}\left(\mathbf{r}_1,\tau-\frac{r_{21}}{c}\right)\rho(\mathbf{r}_2,\tau)\right] \\ +c^{-2}\left[\mathbf{J}(\mathbf{r}_1,\tau)\dot{\mathbf{J}}\left(\mathbf{r}_2,\tau-\frac{r_{21}}{c}\right)-\mathbf{J}\left(\mathbf{r}_1,\tau-\frac{r_{21}}{c}\right)\dot{\mathbf{J}}(\mathbf{r}_2,\tau)\right]\end{array}\right\} d\tau d\mathbf{r}_2 d\mathbf{r}_1 \quad (17)$$

For a pulse source over $[0,T]$, the principal radiative energy becomes constant after the sources have disappeared, i.e., $W_{rad}^{pri}(t) = W_{rad}^{pri}(T)$ for $t \ge T$. However, it can be seen from (16) that the radiative energy $W_{rad}(t)$ continues to vary within the small time period $[T, T + 0.5t_{max}]$ due to the effect of $W_S(t)$. The temporal evolution behaviors of the energies are sketched in Fig.1(a). For static electromagnetic fields, the radiative energy is zero, and the reactive electric (magnetic) energy is exactly the stored electric (magnetic) energy associated with the static sources.

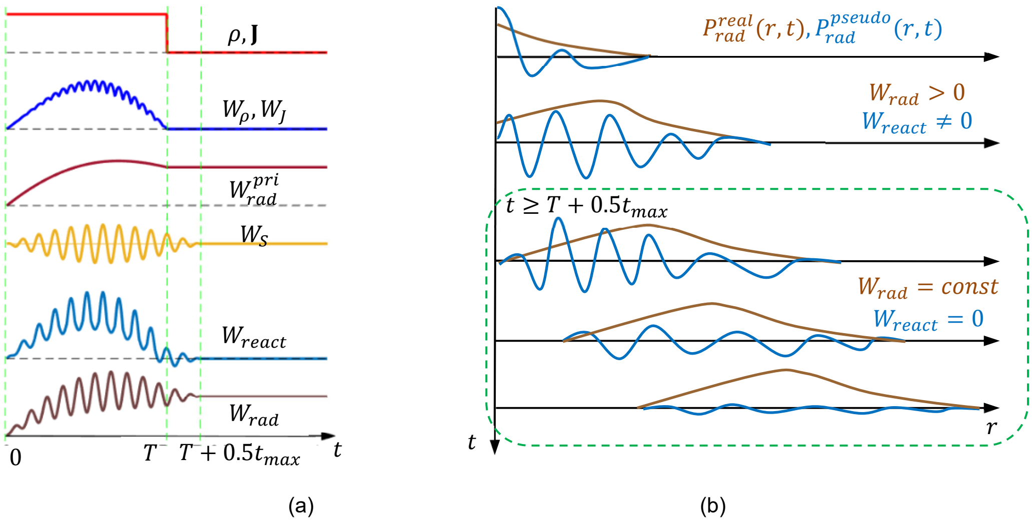


Fig. 1. Sketch diagram of the energies. (a) Non zero period of the energies for a pulse radiator with sources existing in $[0, T]$. (b) Real radiative power flux and the pseudo radiative power flux at concentric observation surfaces.

The Poynting Theorem correctly describes the relationship between the work rate done by the source, the total electromagnetic energy in region $V_a \supseteq V_s$, and the total electromagnetic power flux crossing the boundary S_a of the region,

$$-\int_{V_s}\mathbf{J}\cdot\mathbf{E}d\mathbf{r}_1 = \frac{\partial}{\partial t}\int_{V_a}\left(\frac{1}{2}\mathbf{D}\cdot\mathbf{E}+\frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right)d\mathbf{r}_1 + \oint_{S_a}\mathbf{S}\cdot\hat{\mathbf{n}}dS \quad (18)$$

where the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ is the electromagnetic power flux density related to the total electromagnetic energy. $\hat{\mathbf{n}}$ is the outward unit vector of S_a. Using the vector identities in deriving (1) and (2), we can write that

$$-\int_{V_s}\mathbf{J}\cdot\mathbf{E}\,d\mathbf{r}_1 - \frac{\partial}{\partial t}W_{\rho J}(t) = \frac{\partial}{\partial t}\int_{V_a}\frac{1}{2}\left(\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{A}-\mathbf{D}\cdot\frac{\partial\mathbf{A}}{\partial t}\right)d\mathbf{r}_1$$
$$+\oint_{S_a}\left[\mathbf{E}\times\mathbf{H}-\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\times\mathbf{A}+\mathbf{D}\phi)\right]\cdot\hat{\mathbf{n}}dS \quad (19)$$

As S_a is a fixed observation surface, the surface integral cannot be discarded like in deriving (1) and (2). Integrating both side of (19) with respect to t over $[0,t]$ gives

$$W_{exc}(t) - W_{\rho J}(t) = W_{rad}^{pri}(t) + \int_0^t P_{rad}^{pri}(\tau)d\tau \quad (20)$$

where $W_{exc}(t)$ is the total work done by the source. $W_{rad}^{pri}(t)$ is the principal radiative energy in V_a. It is defined using (15) but the integration domain is replaced by V_a. $\mathbf{S}_{rad}(\mathbf{r},t)$ is the integrand of the surface integral in (19),

$$\mathbf{S}_{rad}(\mathbf{r},t) = \mathbf{E}\times\mathbf{H} - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H}\times\mathbf{A} + \frac{1}{2}\mathbf{D}\phi\right) \quad (21)$$

and $P_{rad}^{pri}(t)$ is its surface integral, $P_{rad}^{pri}(t) = \oint_{S_a} \boldsymbol{S}_{rad}\cdot\hat{\mathbf{n}}dS$. Accordingly, $P_{rad}^{pri}(t)$ can be interpreted as the principal radiative power flux passing through the observation surface S_a. It is associated with the principal radiative energy $W_{rad}^{pri}(t)$ within the surface. Since it is not easy to find a general explicit expression for the radiative power flux passing through the observation surface, the principal radiative power flux $P_{rad}^{pri}(t)$ can provide a good measurement for it. As shown in harmonic cases and verified in Hertzian dipole, the principal radiative power flux $P_{rad}^{pri}(t)$ gives a kind of time averaged value of the total radiative power flux that passes through the observation surface.

For $t \geq T + 0.5t_{max}$, we have $W_{\rho J}(t) = W_S(t) = 0$. The total radiative energy can be expressed with the temporal integration of $P_{rad}^{pri}(t)$ on an arbitrary observation surface enclosing the radiator,

$$W_{rad}(t) = W_{rad}^{pri}(t) = \int_{t_{min}}^{t_{max}} P_{rad}^{pri}(t)dt = W_{exc}(T) \quad (22)$$

For pulse sources, $P_{rad}^{pri}(t)$ has nonzero values only over period $[t_{a,min,}t_{a,max}]$, in which $t_{a,min}$ and $t_{a,max}$ are respectively the earliest and the latest time that the fields pass through the observation surface S_a. As a special case, we may choose $V_a = V_s$, and put the observation surface S_a close to the surface of the sources. Assuming that all radiative fields coming out of S_a no longer interact with the sources in V_s and ignoring the radiative energy stored in V_s, we may obtain the power coming out of the surface of the sources as

$$P_{rad0}^{pri}(t) = -\int_{V_s}\left[\mathbf{J}\cdot\mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\phi + \frac{1}{2}\mathbf{J}\cdot\mathbf{A}\right)\right]d\mathbf{r}_1 \quad (23)$$

Apparently, $P_{rad}^{pri}(t)$ at different observation surfaces are not expected to be equal, but their integrations over the time interval $[t_{a,min,}t_{a,max}]$ are equal, and are approximately equal to that of $P_{rad0}^{pri}(t)$ since all the radiative energy of the pulse source in vacuum will eventually pass through all observation surfaces and propagate to infinity.

As is discussed in [20], the radiative energy is always nonnegative. It represents an irreversible loss of energy from the source and induces a real radiative power flux in space. The Schott energy can vary reversibly. For $t \geq T + 0.5t_{max}$, although $W_S(t) = 0$, its integrand is not necessary to be zero everywhere. It induces an energy oscillation and causes a pseudo power flux. Therefore, the Poynting vector should include both contributions. The integration of the normal component of the Poynting vector on concentric spherical surfaces S_r with radius r can be separated into a real radiative power flux and a pseudo radiative power flux,

$$P_{pv}(r,t) = \oint_{S_r}\mathbf{S}(\mathbf{r}_1,t)\cdot\hat{\mathbf{n}}d\mathbf{r}_1 = P_{rad}^{real}(r,t) + P_{rad}^{pseudo}(r,t) \quad (24)$$

Except point sources, it is generally difficult to determine the real radiative power flux and the pseudo radiative power flux. However, at large r, we may assume that the radiative energy in the very thin spherical shell $[r - cdt, r]$ will pass through S_r approximately at the light velocity c, then we have $P_{rad}^{real}(r,t)dt \approx \oint_{S_r}\int_r^{r+cdt} w_{rad}(r,t)dr\,dS$, which leads to

$$P_{rad}^{real}(r,t) \approx c\oint_{S_r}\left(-\mathbf{D}\cdot\frac{\partial\mathbf{A}}{\partial t}\right)dS \quad (25)$$

The pseudo power flux on one observation surface can be obtained by subtracting the real power flux from the Poynting power flux $P_{pv}(r,t)$. The pseudo power flux can be either positive or negative, but its amplitude generally decreases with the increase of the distance to the sources.

However, the real radiative power flux on an observation surface is always positive, and its amplitude remains almost unchanged at larger r, as shown in Fig.1(b).

3. Radiation of Harmonic Sources

For harmonic fields with time convention of $\exp(j\omega t)$, the radiation is assumed to last in time domain from $-\infty$ to $+\infty$, so the radiative energy is infinitely large. The Poynting theorem can be applied to describe the balance between the time averaged powers and the varying rate of the energies,

$$-\frac{1}{2}\int_{V_s}\mathbf{J}^*\cdot\mathbf{E}d\mathbf{r}_1 = 2j\omega\int_{V_a}\left[\frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* - \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^*\right]d\mathbf{r}_1 + \frac{1}{2}\oint_{S_a}\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{n}}dS \tag{26}$$

from which the time averaged radiative power at infinity can be evaluated with source distributions,

$$(P_{rad})_{av} = \mathrm{Re}\left\{\frac{1}{2}\oint_{S_\infty}\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{n}}\mathrm{dS}\right\} = -\mathrm{Re}\left\{\frac{1}{2}\int_{V_s}\mathbf{E}\cdot\mathbf{J}^*d\mathbf{r}_1\right\} \tag{27}$$

The same symbols are used for the corresponding phasors for the sake of simplicity. With the theory proposed here, the power balance can be established within any domain enclosed by an observation surface S_a containing the source region V_s ,

$$-\int_{V_s}\mathbf{J}^*\cdot\mathbf{E}\,d\mathbf{r}_1 = 2j\omega\int_{V_s}\left(\frac{1}{4}\rho^*\phi + \frac{1}{4}\mathbf{J}^*\cdot\mathbf{A}\right)d\mathbf{r}_1$$

$$+\oint_{S_a}\left[\frac{1}{2}\mathbf{E}\times\mathbf{H}^* - j\omega\left(\frac{1}{4}\mathbf{H}^*\times\mathbf{A} + \frac{1}{4}\mathbf{D}^*\phi\right)\right]\cdot\hat{\mathbf{n}}dS \tag{28}$$

The time averaged radiative power crossing the observation surface can be obtained using the principal radiative power flux vector $\boldsymbol{S}_{rad}$ or the source distributions,

$$(P_{rad})_{av} = \mathrm{Re}\oint_{S_a}(\mathbf{S}_{rad})_{av}\cdot\hat{\mathbf{n}}dS = -\mathrm{Re}\left\{\int_{V_s}\frac{1}{2}\mathbf{J}^*\cdot\mathbf{E}d\mathbf{r}_1\right\} \tag{29}$$

Note that the observation surface is not required to approach infinity for evaluating the radiative power with (29).

The time averaged reactive energy can be calculated with the fields and the vector potential,

$$(W_{react})_{av} = \mathrm{Re}\left\{\int_{V_\infty}\left(\frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* + \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* + \frac{1}{2}j\omega\mathbf{D}^*\cdot\mathbf{A}\right)\right\} \tag{30}$$

It is easy to verify that $(W_S)_{av} = 0$, so the time averaged reactive energy can be alternatively calculated using the source-potential products as follows

$$(W_{react})_{av} = \mathrm{Re}\left\{\int_{V_s}\left(\frac{1}{4}\rho\phi^* + \frac{1}{4}\mathbf{J}^*\cdot\mathbf{A}\right)d\mathbf{r}_1\right\} \tag{31}$$

Generally, (31) is much more efficient to use than (30). However, for point sources, (31) is not bounded. We may have to use (30) to evaluate the energy in the region excluding the point source area.

4. Numerical Examples

4.1. Hertzian Dipole

A Hertzian dipole is put at the origin, as shown in Fig. 2(a). The moment of the dipole is assumed to be $ql\cos\omega t$, the scalar potential and the vector potential of which can be readily derived from the Hertzian potential $\Pi = (ql/4\pi r)\cos(\omega t - kr)$ [35], [38],

$$\mathbf{A} = -\frac{\omega\mu_0 ql}{4\pi r}\sin(\omega t - kr)\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right) \tag{32}$$

$$\varphi = \frac{\omega^2\mu_0 ql}{4\pi}\cos\theta\left[\frac{1}{k^2r^2}\cos(\omega t - kr) - \frac{1}{kr}\sin(\omega t - kr)\right] \tag{33}$$

The Hertzian dipole is a point source. Its total reactive energy is infinite. A common strategy is to evaluate the energies with respect to the fields and potentials in the whole space excluding a small sphere with radius a. For example, evaluate $W_S(t)$ with (8) instead of (14). All energies have been evaluated with the proposed formulae and are provided in the supplementary material.

Since the Hertzian dipole is a point source, its fields propagate radially and cross all concentric spherical observation surfaces with light velocity. The real radiative power flux on the spherical surface S_a can be exactly evaluated with (25), which is $P_{rad}^{real}(a,t) = 2\omega\alpha_0[1+\cos 2(\omega t - ka)]$, where $\alpha_0 = (\omega q l)^2 \mu_0 k/(24\pi)$. The amplitude of the real radiative power flux does not depend on the radius of the observation surface. The Poynting power flux $P_{pv}(a,t)$ on the same surface can be calculated with (24). Subtracting the real radiative power flux from it, we get the pseudo power flux,

$$P_{rad}^{pseudo}(a,t) = 2\omega\alpha_0\left[\left(\frac{2}{ka} - \frac{1}{k^3a^3}\right)\sin 2(\omega t - ka) - \frac{2}{k^2a^2}\cos 2(\omega t - ka)\right] \quad (34)$$

which apparently decreases with the distance to the dipole.

The principal radiative power flux at the observation surface is $2\omega\alpha_0$, a constant independent of the radius of the sphere. It is equal to the time averaged value of the real radiative power flux. The calculated Q factor of the dipole is exactly in agreement with the result shown in [39].

The well-established equivalent circuit model proposed by Chu [36] for Hertzian dipole is shown in Fig. 2(b). Assume that the current in the radiation resistor at the interface S_a with radius $r = a$ is $i_R = 4\omega\alpha_0\cos(\omega t - ka)$. The energies stored in the capacitor and the inductor can be easily derived using circuit theory. It has been verified that, both in time domain and in frequency domain, the reactive magnetic energy exactly equals to the energy stored in the inductor in the equivalent circuit, and the reactive electric energy exactly equals to the energy stored in the capacitor in the equivalent circuit. Detailed results can be found in the supplementary material.

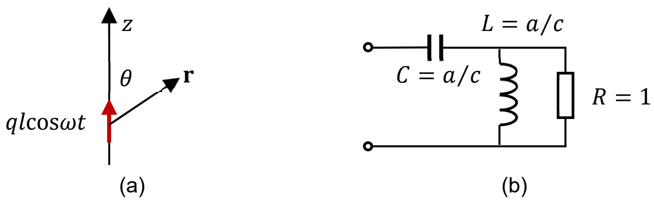


Fig. 2. Hertzian dipole. (a) Coordinate system, (b) Equivalent circuit model.

4.2. Solenoidal Loop

The solenoidal surface current on a ring is described by $\mathbf{J}_s(\mathbf{r},t) = I(t)\hat{\boldsymbol{\varphi}}$ [A/m], as shown in Fig. 3(a). The inner and outer radius of the ring is 0.08m and 0.1m, respectively. The temporal function is a modulated Gaussian pulse, $I(t) = e^{-\gamma^2}\sin\omega t$ for $0 \le t \le T$. $\omega = 2\pi \times 10^{10}$, $\gamma = 2\sqrt{5}(t - 0.5T)/T$, and $T = 1$ns. Two spherical surfaces with radius of 0.2m and 10m are chosen as the observation surfaces, with their centers coinciding with that of the source. They are labeled by sphere-1 and sphere-2, respectively. The principal radiative energy passing through sphere-1 and sphere-2 are calculated with integration of $P_{rad}^{pri}(t)$, as expressed in (22). $W_{pv}(t)$ is the integration of the Poynting power flux passing through the observation surface, i.e., $W_{pv}(t) = \int_0^t P_{pv}(r,\tau)d\tau$.

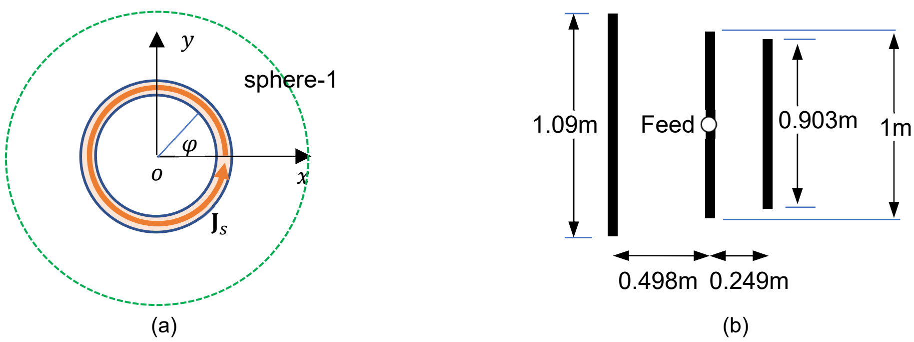
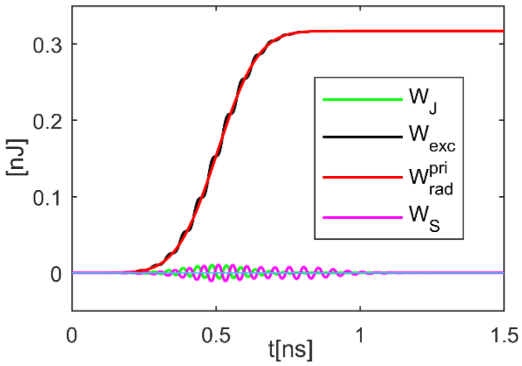


Fig. 3. Radiator examples. (a) Solenoidal loop current, inner and outer radius of the ring is 0.08m and 0.1m. (b) Thin plate Yagi antenna with 3 PEC plates.

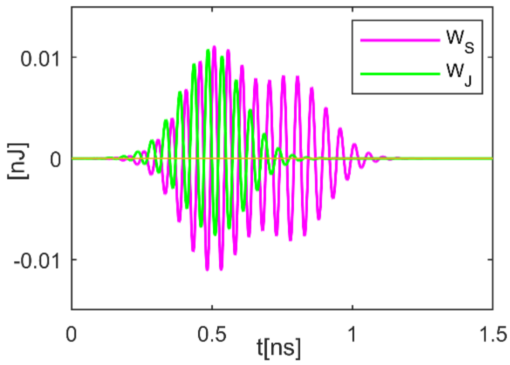
The excitation energy, the principal radiative energy, the macroscopic Schott energy, and the reactive energy evaluated with the proposed method are shown in Fig. 4(a). In this case, the current is solenoidal and its corresponding charge is zero. The reactive energy includes the contribution from the current alone, and is denoted as W_J in the figures. W_J oscillates with the source and admits negative values periodically. It is acceptable because the reactive energy is dependent on the potentials, which are values relative to their reference zero points. When the current varies and changes its direction periodically, the retarded vector potential in the source region lags behind and may point in direction opposite to that of the current, causing negative values. We consider that in this situation the loop current does not radiate energy into the surrounding space. Instead, it absorbs energy that it had radiated earlier. An equivalent explanation is, when the energy is negative, the inductance associated with the loop current is negative, as explained in the supplementary material. Note that the macroscopic Schott energy in the charged particle theory may also be negative [20], [40]. It is plotted in Fig.4(a) as well, and is zoomed-in in Fig.4(b) together with W_J. It can be seen that the macroscopic Schott energy oscillates like W_J but continues to exist for about 0.33ns after the source has disappeared at 1ns.

The energies passing through sphere-1 are shown in Fig. 4(c). The smallest and the largest distance between the source and sphere-1 are respectively 0.1m and 0.3m. The fields reach the observation surface at about 0.33ns. All fields, hence all radiative energy, should have passed through sphere-1 at 2ns. Therefore, the principal radiative energy evaluated at *t*=2ns must equal to the total radiative evaluated at the source region.

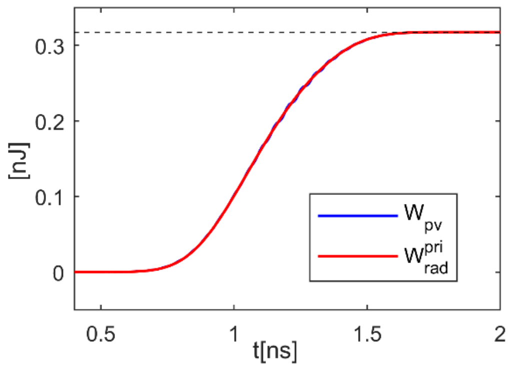
The excitation power (denoted by EJ), the principal radiative power flux and the time varying rate of the reactive energy are shown in Fig. 4(d). The power fluxes on sphere-1 and sphere-2 are shown in Fig.4(e) and (f), respectively. $P_{rad0}^{pri}(t)$ varies smoothly and remains positive, while the Poynting power flux contains ripples coming from $P_{rad}^{pseudo}(r,t)$, which gradually decreases with the propagation distance.

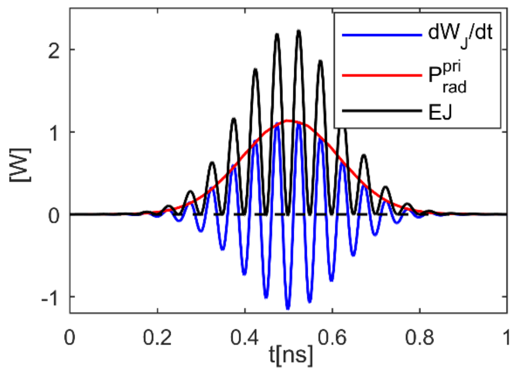


(a)

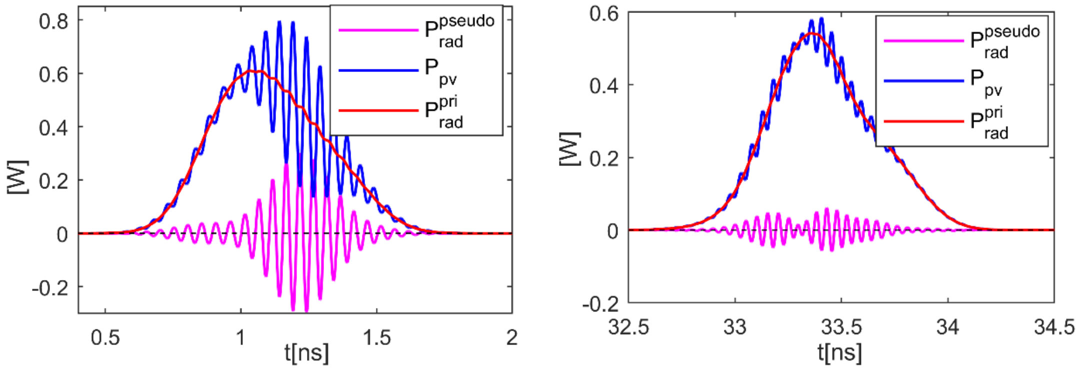


(b)





(c) (d)



(e) (f)

Fig.4. Loop current. (a) The energies evaluated in the source region. (b) The zoom-in figure for W_J and W_S. (c) The energies passing through sphere-1. (d) The powers evaluated in the source region. (e) The powers passing sphere-1. (f) The powers crossing sphere-2.

4.3. Thin Plate Yagi Antenna

The geometrical structure and parameters of the Yagi antenna is shown in Fig. 3(b). It consists of 3 PEC plates with zero thickness: a dipole in the middle, a reflector in the left and a director in the right. The width of all plates is 2mm. The dipole is fed at its center with a Delta-gap voltage source of $V_{feed}(t) = 1.0\sin(\omega t)\,[V],\ t \geq 0.\ \ \omega = 2\pi \times 1.5 \times 10^8$, which corresponds to 150MHz.

The first step is to calculate the surface current by solving the surface electric field integral equation (EFIE) with marching-on in time scheme (MOT) [41]. The plates are triangularly meshed, and the surface current is expanded with RWG basis functions [42]. There are 53, 59, and 50 RWGs on the left, the middle and the right plate, respectively. The Delt-gap voltage feeding is put on the common edge of the RWG in the middle of the dipole. The time step of the MOT is 2.67ps. Stable results of the surface current density and the surface charge density are available, together with their time derivatives.

The second step is to evaluate the energies and powers using the obtained surface currents and charges with the expressions we have proposed. When calculating the macroscopic Schott energy and the principal radiative energy, the integration interval of the innermost integral is dependent on the distance between two points. It may become much smaller than the time step and should be handled carefully to get satisfactory numerical accuracy. For $r_{ij} = 0$, we have to use the L'Hospital rule to find the limit of the innermost integral. A description on the numerical implementation can be found in the supplementary material.

The principal radiative power fluxes of the 3 plates are calculated separately with (23), but the integration domain is respectively replaced by those of the three plates. The results are shown in Fig. 5(a). The Coulomb-velocity energies are shown in Fig.5(b). As shown in the figures, the principal radiative power flux of the dipole is always positive. It radiates electromagnetic energy to the space from the beginning. However, the principal radiative power fluxes of the reflector and the director are negative at the beginning, which means that they absorb energy from the dipole at the transition stage. When the radiation enters the steady state, the reactive energies tend to become steady stable. Since the PEC reflector and the PEC director are passive elements, they do not radiate by themself, but only scatter away the electromagnetic energies they received from the dipole. Consequently, their principal radiative powers are zero at the steady state.

The total energies of the Yagi antenna (3 plates as a whole) are plotted in Fig. 5(c). After a short transition stage, the radiation approaches steady state. It can be seen that the macroscopic Schott energy gradually becomes an oscillation with approximately the same period of the excitation. It brings ripples to the total reactive energy and the total radiative energy. The total reactive energy tends to become steady and bounded, while the radiative energy increases approximately linearly. For a harmonic source that began radiating from $t = -\infty$, the radiative energy is naturally unbounded.

The principal radiative power flux and the Poynting power flux of the Yagi antenna are evaluated on a spherical observation surface with radius of 2m, the center of which locates at the feeding point of the antenna. The results are illustrated in Fig. 5(d).

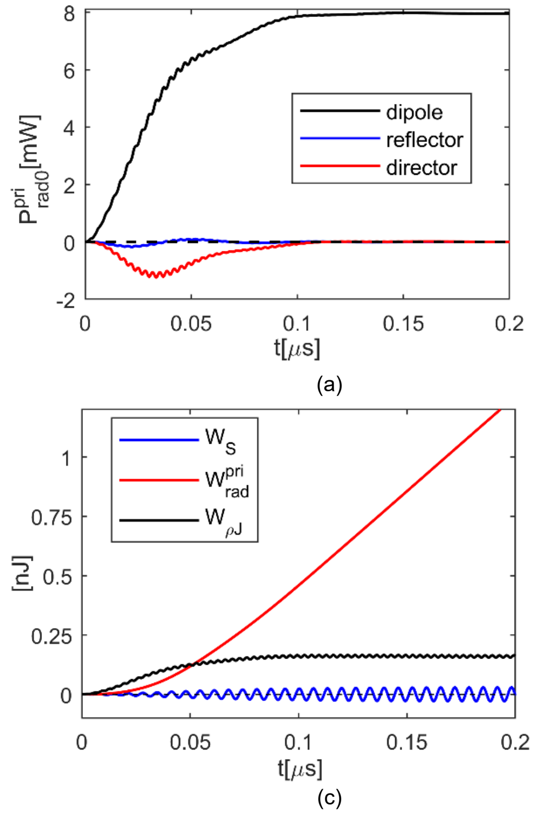


Fig. 5. Results of the Yagi antenna. (a)The principal radiative power flux passing through the surface of each plate. (b) The Coulomb-velocity energy of each plate. (c) The macroscopic Schott energy, principal radiative energy and the Coulomb-velocity energy of the antenna. (d) Electromagnetic powers passing through the observation surface.

The directivity of the Yagi antenna inevitably varies in the transition stage. The normalized directivity patterns in the E-plane and the H-plane at 10ns, 20ns, and 40ns are depicted in Fig. 6. As can be seen, in the beginning stage, the passive reflector and the passive director absorb more electromagnetic energies than the energies they radiated. They perform like absorbers instead of radiators. The radiation of the Yagi antenna is mainly determined by the center dipole, so the pattern is like that of a single dipole. The antenna performs like a Yagi antenna only after the two passive plates have achieved a balance between their absorbed and scattered powers.

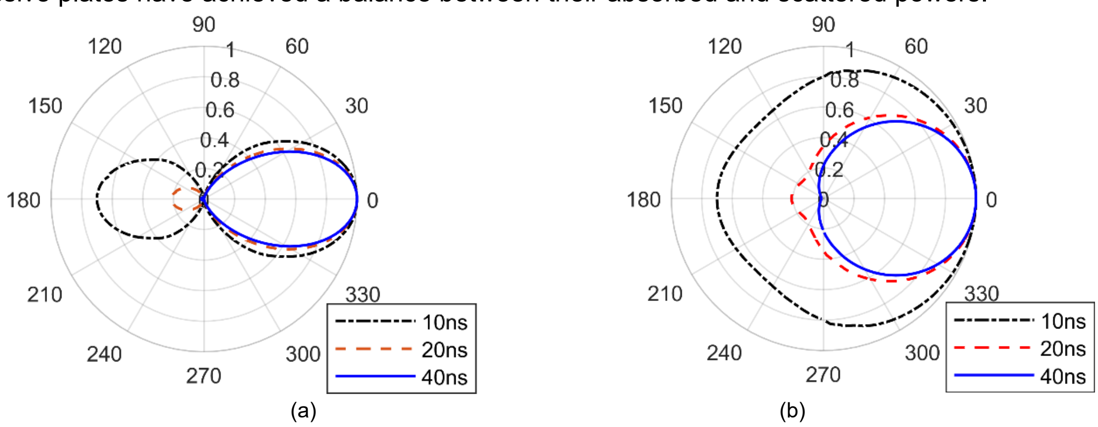


Fig. 6. Evolution of the radiation pattern. (a) E-plane. (b) H-plane.

5. Conclusions

Some issues concerning with the electromagnetic radiation of antennas have remained to be ambiguous or even controversial for decades long, especially the definitions for the reactive energy. The introduction of the macroscopic Schott energy makes it possible to separate the radiative energy and the reactive energy in a consistent formulation. As a result, a new form of power balance equation is derived from the Poynting relation. Analysis shows that the Poynting vector includes the contribution from the propagation of the radiative energy and the fluctuation of the reactive energy. The newly defined principal reactive energy $W_{rad}^{pri}(t)$ and its flux, the principal radiative power flux $P_{rad}^{pri}(t)$, can characterize the main property of the radiative energy. Furthermore, they can be numerically evaluated more efficiently.

As is pointed out in the introduction, the main problem of the issue is how to separate the radiative energy and the reactive energy. We divide the electromagnetic energy into three parts: the Coulomb-velocity energy $W_{\rho J}(t)$, the macroscopic Schott energy $W_S(t)$, and the radiative energy $W_{rad}(t)$. The explicit and accurate expressions for them are derived and verified with their temporal evolution properties. All the expressions are strictly derived from Maxwell equations with no approximations. They are no static limit as the potentials involved are all retarded ones.

The basic theory is for time varying pulse radiators. We also provided formulae for harmonic waves. Unlike Vandenbosch formulation in which the results obtained with time domain formulation sometimes may not agree with those obtained using the frequency domain formulation, in the theory proposed here, the results in time domain and frequency domain are completely in consistent because they are directly derived from the time domain Maxwell equations and the frequency domain Maxwell equations, respectively. The electromagnetic fields in frequency domain and time domain can be converted to each other with Fourier Transform.

The theory is completely different from the Carpenter formulation [28]. In Carpenter formulation, it was proposed to use the Coulomb-velocity energy alone as the total electromagnetic energy and to replace the Poynting Theorem with a new equation. The formulation, as well as the power flow vector $\phi\mathbf{J}$ by Slepian [43], was pointed out to be mathematically flawed by Dr. Endean [44]. In our theory, we combine the Coulomb-velocity energy and the macroscopic Schott energy together to form the reactive energy. The theory does not suffer from the mathematical flaw since there is no modification to the total electromagnetic energy and the Poynting Theorem.

Acknowledgements

The authors xxx.

Supplementary Material

This article has supplementary downloadable material available at xxx

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