A Generic Algebraic Proof of the Unified Power Conservative Equivalent Circuit Theorem

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Abstract—This paper presents a generic algebraic proof of a recently published theorem [4], on the power conservative equivalent circuit for linear DC networks formed by time-invariant resistors and independent voltage and current sources. As the cited publication states, the internal losses of any network have two components: one variable and dependent on the internal resistances of the actual circuit and the power transferred to the pair of accessible terminals; and the other constant and dependent only on the internal voltage and current sources and the resistances of the actual network. It is also noted that the traditional Thévenin and Norton equivalent circuits are particular cases of the proposed equivalent circuit.

Index Terms—DC networks, Thévenin, Norton, power conservative, DC network theorem, equivalent circuit.

I. INTRODUCTION

THE Thevenin [1] and Norton [2] equivalent circuits are important to the theory of DC networks and are commonly used in the analysis of electrical circuits, since they provide simplicity in many practical applications.

However, it is well known that these approaches are limited to representing the phenomena that occur in the actual circuit, since they can only be used to determine voltage and current at a pair of accessible terminals of the network. Therefore, these traditional equivalent circuits are not appropriate for determining the efficiency and power of the actual network.

In [3-4] it is demonstrated, using simplified analysis and case studies, that any linear and bilateral DC network formed by time-invariant resistors and independent voltage and current sources, with a pair of accessible terminals ab , can be replaced by the equivalent circuit shown in Fig. 1. In this paper a generic algebraic proof of the proposed theorem is presented.

II. STATEMENT OF THE UNIFIED POWER CONSERVATIVE EQUIVALENT CIRCUIT THEOREM

As stated in [4], *Theorem: Any linear DC network consisting of independent voltage and current sources, and resistors, with two accessible terminals (Fig. 1(a)) can be replaced by an equivalent circuit with a DC voltage source V_{T1} , a DC current source I_{N1} and three resistors R_X , R_y and R_T (Fig. 1(b)). The voltage V_{T1} is measured at the open terminals, with all internal current sources of the network set equal to zero. The current I_{N1} is the current through a short circuit applied to the external terminals, with all network internal voltage sources set equal to zero. The resistance R_X associated in parallel with the voltage source V_{T1} is given by the equation $R_X = \frac{V_{T1}^2}{P_X}$, where P_X is the power supplied by the internal*

voltage sources and dissipated internally in the circuit with the external terminals open and all internal current sources set equal to zero. The resistance R_y associated in series with the current source I_{N1} is given by $R_y = \frac{P_y}{I_{N1}^2}$, where P_y is the power supplied by the internal current sources and dissipated in the network, with a short circuit applied to the external terminals and all internal voltage sources set equal to zero. The resistance R_T is that measured at the terminals with all internal voltage replaced by a short circuit and all internal current sources replaced by an open circuit.

Fig. 1. The power conservative unified equivalent circuit of DC networks consisting of independent voltage and current sources, and resistors.

III. GENERIC ALGEBRAIC PROOF OF THE THEOREM

Fig. 2 shows a reciprocal and symmetric DC network formed by two independent ideal voltage sources V_1 and V_2 , two independent ideal current sources I_1 and I_2 , and a pair of accessible terminals ab to which an external voltage source V_o is connected. The resistors of the network are in the box denoted by R .

The equations that describe the network in terms of inverse hybrid parameters (or g-parameters) [2] are

$$I_1 = g_{11}V_1 + g_{12}V_2 + g_{1p}I_p + g_{1q}I_q + g_{1o}V_o \quad (1)$$

$$I_2 = g_{21}V_1 + g_{22}V_2 + g_{2p}I_p + g_{2q}I_q + g_{2o}V_o \quad (2)$$

$$V_p = g_{p1}V_1 + g_{p2}V_2 + g_{pp}I_p + g_{pq}I_q + g_{po}V_o \quad (3)$$

$$V_q = g_{q1}V_1 + g_{q2}V_2 + g_{qp}I_p + g_{qq}I_q + g_{qo}V_o \quad (4)$$

Fig. 2. DC network formed by resistors, independent voltage and current sources, and a pair of accessible terminals ab .

Fig. 3. The network with the terminals ab open.

Making $I_o = 0$ in (5) we find

$$I_o = g_{o1}V_1 + g_{o2}V_2 + g_{op}I_p + g_{oq}I_q + g_{oo}V_o \quad (5)$$

which may also be written in the matrix form

$$\begin{bmatrix} I_1 \\ I_2 \\ V_p \\ V_q \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{1p} & g_{1q} \\ g_{21} & g_{22} & g_{2p} & g_{2q} \\ g_{p1} & g_{p2} & g_{pp} & g_{pq} \\ g_{q1} & g_{q2} & g_{qp} & g_{qq} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} + \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} \cdot V_o \quad (6)$$

$$I_o = \begin{bmatrix} g_{o1} & g_{o2} & g_{op} & g_{oq} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} + g_{oo}V_o \quad (7)$$

For reciprocal and symmetric networks, $g_{21} = g_{12}$, $g_{pq} = g_{qp}$, $g_{o1} = g_{1o}$, $g_{o2} = g_{2o}$, $g_{p1} = -g_{1p}$, $g_{p2} = -g_{2p}$, $g_{q1} = -g_{1q}$, $g_{q2} = -g_{2q}$, $g_{po} = -g_{op}$, $g_{qo} = -g_{oq}$, $g_{po} = -g_{op}$ and $g_{qo} = -g_{oq}$.

Let G be defined as

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{1p} & g_{1q} \\ g_{21} & g_{22} & g_{2p} & g_{2q} \\ g_{p1} & g_{p2} & g_{pp} & g_{pq} \\ g_{q1} & g_{q2} & g_{qp} & g_{qq} \end{bmatrix} \quad (8)$$

The substitution of (8) in (6) leads to

$$\begin{bmatrix} I_1 \\ I_2 \\ V_p \\ V_q \end{bmatrix} = G \cdot \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} \cdot V_o \quad (9)$$

A. The Voltage At Terminals ab When They Are Open ($I_o = 0$)

Let us consider the condition in which the external voltage source is disconnected from the terminals ab . The current $I_o = 0$ and the voltage V_{ab} is the voltage of the Thévenin equivalent circuit, denoted by V_T . The representation of the network for this condition is shown in Fig. 3.

Making $I_o = 0$ in (5) we find

$$0 = g_{o1}V_1 + g_{o2}V_2 + g_{op}I_p + g_{oq}I_q + g_{oo}V_T \quad (10)$$

Hence,

$$V_T = -\frac{1}{g_{oo}} \begin{bmatrix} g_{o1} & g_{o2} & g_{op} & g_{oq} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} \quad (11)$$

Let us define V_{T1} and V_{T2} as

$$V_{T1} = -\frac{1}{g_{oo}} \begin{bmatrix} g_{o1} & g_{o2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (12)$$

$$V_{T2} = -\frac{1}{g_{oo}} \begin{bmatrix} g_{op} & g_{oq} \end{bmatrix} \begin{bmatrix} I_p \\ I_q \end{bmatrix} \quad (13)$$

The substitution of (12) and (13) in (11) leads to

$$V_T = V_{T1} + V_{T2} \quad (14)$$

which is a consequence of the superposition principle. The voltage V_{T1} is the open voltage at terminals ab when the current sources $I_p = I_q = 0$ while V_{T2} is the open voltage at the same terminals when the voltage sources $V_1 = V_2 = 0$, as shown in Fig. 4.

B. Output Current When $V_1 = V_2 = V_o = 0$

Fig. 5 shows the network when $V_1 = V_2 = V_o = 0$. The current I_o is the current source of the Norton equivalent circuit under this constraint, denoted by I_{N1} .

Making $V_1 = V_2 = V_o = 0$ in (7) we find

$$I_{N1} = g_{op}I_p + g_{oq}I_q \quad (15)$$

Substitution of (15) in (13) gives

$$V_{T2} = -\frac{1}{g_{oo}} I_{N1} \quad (16)$$

where V_{T2} is the voltage at the open terminals ab when $V_1 = V_2 = 0$, $I_p \neq 0$ and $I_q \neq 0$.

Fig. 4. The network with the terminals ab open. (a) when $I_p = I_q = 0$ and (b) when $V_1 = V_2 = 0$.

Fig. 5. The network when $V_1 = V_2 = V_o = 0$, $I_p \neq 0$ and $I_q \neq 0$.

C. The Resistance R_T of the Thévenin Equivalent Circuit

The resistance of the Thévenin equivalent circuit is the driving point resistance of the network at the terminals ab when all internal voltage sources are replaced by a short circuit and all internal current sources are replaced by an open circuit, as shown in Fig. 6.

Fig. 6. Measurement of the resistance R_T from the terminals ab .

Making $V_1 = V_2 = I_p = I_q = 0$ in expression (7) we find

$$R_T = \frac{V_o}{I_o} = -\frac{1}{g_{oo}} \quad (17)$$

D. Internal Power Dissipated in the Resistors of the Network

The power dissipated by the network resistors is

$$P = \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} G \begin{bmatrix} I_1 \\ I_2 \\ V_p \\ V_q \end{bmatrix} + V_o I_o \quad (18)$$

Substituting (9) in (18) we find

$$P = \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} G \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} + \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} V_o + V_o I_o \quad (19)$$

Since

$$\begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} G \begin{bmatrix} I_1 \\ I_2 \\ V_p \\ V_q \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} I_p & I_q \end{bmatrix} \begin{bmatrix} g_{pp} & g_{pq} \\ g_{pq} & g_{qq} \end{bmatrix} \begin{bmatrix} I_p \\ I_q \end{bmatrix} \quad (20)$$

the expression (19) becomes

$$P = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} I_p & I_q \end{bmatrix} \begin{bmatrix} g_{pp} & g_{pq} \\ g_{pq} & g_{qq} \end{bmatrix} \begin{bmatrix} I_p \\ I_q \end{bmatrix} + \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} V_o + V_o I_o \quad (21)$$

Let us define P_a and P_b as

$$P_a = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (22)$$

$$P_b = \begin{bmatrix} I_p & I_q \end{bmatrix} \begin{bmatrix} g_{pp} & g_{pq} \\ g_{pq} & g_{qq} \end{bmatrix} \begin{bmatrix} I_p \\ I_q \end{bmatrix} \quad (23)$$

Substitution of (22) and (23) in (21) gives

$$P = P_a + P_b + \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} V_o + V_o I_o \quad (24)$$

Substitution of (7) in (24) yields

$$P = P_a + P_b + \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} V_o + \begin{bmatrix} g_{1o} & g_{2o} & -g_{po} & -g_{qo} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} V_o + g_{oo} V_o^2 \quad (25)$$

We can observe that

$$\begin{aligned}
& \begin{bmatrix} V_1 & V_2 & I_p & I_q \end{bmatrix} \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{po} \\ g_{qo} \end{bmatrix} V_o \\
& + \begin{bmatrix} g_{1o} & g_{2o} & -g_{po} & -g_{qo} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_p \\ I_q \end{bmatrix} \\
& = 2 \begin{bmatrix} g_{1o} V_1 & g_{2o} V_2 \end{bmatrix}
\end{aligned} \quad (26)$$

and that

$$g_{1o} V_1 + g_{2o} V_2 = -g_{oo} V_{T1} \quad (27)$$

By substituting (26) and (27) in (25) we obtain

$$P = P_a + P_b - 2g_{oo} V_{T1} V_o + g_{oo} V_o^2 \quad (28)$$

which represents the total power delivered to the resistors of the network by the voltage sources V_1 , V_2 and V_o , and by the current sources I_p and I_q . Expression (28) may also be written in the form

$$P = P_a + P_b - 2g_{oo} V_{T1} V_o + g_{oo} V_o^2 + g_{oo} V_{T1}^2 - g_{oo} V_{T1}^2 \quad (29)$$

Since

$$g_{oo} V_o^2 - 2g_{oo} V_{T1} V_o + g_{oo} V_{T1}^2 = g_{oo} (V_o - V_{T1})^2 \quad (30)$$

we can write

$$P = P_a - g_{oo} V_{T1}^2 + P_b + g_{oo} (V_o - V_{T1})^2 \quad (31)$$

E. Internal Power Dissipated In The Resistors Of The Network When $I_p = I_q = I_o = 0$

Fig. 7 shows the network for the condition where $I_p = I_q = I_o = 0$, $V_1 \neq 0$ and $V_2 \neq 0$.

Fig. 7. The network when $I_p = I_q = I_o = 0$, $V_1 \neq 0$ and $V_2 \neq 0$.

The voltage at the terminals ab is equal to V_{T1} and is given by expression (12). Substituting (12) in (29) and making $P_b = 0$ and $V_o = V_{T1}$ we find the power dissipated by the internal

resistors of the network when $I_p = I_q = I_o = 0$ denoted, by P_x

$$P_x = P_a - g_{oo} V_{T1}^2 \quad (32)$$

Substituting (22) in (32) we find

$$P_x = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - g_{oo} V_{T1}^2 \quad (33)$$

From (12) and (33) the equivalent circuit with the resistor R_x shown in Fig. 8 is obtained, in which the power P_x is dissipated.

Fig. 8. The network equivalent circuit when $I_p = I_q = I_o = 0$, $V_1 \neq 0$ and $V_2 \neq 0$.

The resistance R_x is determined by the expression

$$R_x = \frac{V_{T1}^2}{P_x} \quad (34)$$

F. Internal Power Dissipated In The Resistors Of The Network When $V_1 = V_2 = V_o = 0$

Fig. 5 shows the network for the condition where $V_1 = V_2 = V_o = 0$, $I_p \neq 0$ and $I_q \neq 0$, which implies that $V_{T1} = 0$ and $P_a = 0$.

Let us designate by P_y the power dissipated internally in the resistors of the network under this constraint.

Hence,

$$P_y = P_b \quad (35)$$

Substitution of (23) in (33) yields

$$P_y = \begin{bmatrix} I_p & I_q \end{bmatrix} \begin{bmatrix} g_{pp} & g_{pq} \\ g_{pq} & g_{qq} \end{bmatrix} \begin{bmatrix} I_p \\ I_q \end{bmatrix} \quad (36)$$

The resistor R_y , in which the power P_y is dissipated is given by

$$R_y = \frac{P_y}{I_{N1}^2} \quad (37)$$

The corresponding equivalent circuit is represented in Fig. 9.

Fig. 9. The equivalent circuit when $V_1 = V_2 = V_o = 0$, $I_p \neq 0$ and $I_q \neq 0$.

G. The Unified Equivalent Circuit

The substitution of (17), (35) and (36) in (31) leads to

$$P = P_x + P_y + \frac{(V_o - V_{T1})^2}{R_T} \quad (38)$$

Since $P_x = V_{T1}^2/R_x$ and $P_y = R_y I_{N1}^2$, the expression for the total power dissipated in the internal resistors of the network reduces to the form

$$P = \frac{V_{T1}^2}{R_x} + R_y I_{N1}^2 + \frac{(V_o - V_{T1})^2}{R_T} \quad (39)$$

The circuit that satisfies expressions (14), (16) and (39), by writing the Kirchhoff voltage and current laws along with the superposition principle is shown in Fig. 10.

Fig. 10. Power conservative equivalent circuit for the network formed by resistors and independent voltage and current sources.

The losses in R_x and R_y are not dependent on V_o , I_o or P_o . On the other hand, the losses in the resistor R_T are dependent on V_o . Therefore, the internal losses in the network resistors have two components, one of which is dependent on and the other independent of the power measured at terminals ab .

The theorem is not restricted to two independent voltage and current sources as shown in Fig. 2. The proof given above can easily be generalized to include any number of voltage and current sources.

IV. DERIVATION OF THÉVENIN AND NORTON EQUIVALENT CIRCUITS

Looking at the equivalent circuit shown in Fig. 10, we can conclude that the resistors R_x and R_y , responsible for the constant internal losses in the resistors of the actual network, do not affect the values of V_o , I_o or P_o . Therefore,

when internal losses can be neglected, these resistors can be excluded from the circuit, resulting in the non-conservative equivalent circuit shown in Fig. 11.

Fig. 11. Non-conservative equivalent circuit for the network formed by resistors and independent voltage and current sources.

The traditional Thévenin equivalent circuit can be derived from Fig. 11, as represented in Fig. 12(a). The equivalent resistance is R_T and the voltage source is determined by

$$V_T = V_{T1} + R_T I_{N1} \quad (40)$$

The traditional Norton's equivalent circuit can also be derived and is shown in Fig. 12(b), where $R_N = R_T$ and

$$I_N = I_{N1} + \frac{V_{T1}}{R_N} \quad (41)$$

Fig. 12. Traditional Thévenin equivalent circuit. (b) Traditional Norton equivalent circuit.

V. CONCLUSION

The paper has presented a generic algebraic proof of the recently introduced theorem on the unified power conservative equivalent circuit formed by time-invariant resistors, and independent voltage and current sources [3-4]. It is demonstrated that total power is the sum of the powers supplied simultaneously by the voltage sources with all current sources replaced by an open circuit, and the powers supplied by the current sources with all voltage sources replaced by a short circuit. One corollary to the proved theorem is that the internal losses have two components, one being constant and independent of the current or voltage at terminals ab , while the other is dependent on this voltage or current.

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