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A Special Case of Surface Integral Equations for Homogeneous Dielectric Bodies with Accurate Infinite Series Solution

Kang Zhang

Abstract—Being a special case of accurately solving the surface integral equations (SIEs) in three-dimensional electromagnetics, a circular symmetric magnetic loop current at the center of a homogeneous dielectric board is revisited. To any general SIE, for condition that the lateral faces equivalent currents are negligible, the equivalent magnetic currents on the board top and bottom planar faces are the same and the equivalent electric currents are of a minus sign to each other, thus, integrals over the closed surfaces can be separated into top face and bottom face plane integrals, both are further reduced into definite integrals in Hilbert space of radial functions, as the equivalent electric/magnetic currents and the free space Green's function are circular symmetric. The finalized coupled radial function integral equations are solved in terms of the complete set of orthogonal radial functions on a finite-length interval as basis-and-testing functions in the method of moments (MoM). Proof of the solution convergence is guaranteed with the mathematics of bounded linear operator on the countably infinite dimensional Hilbert spaces. Compared with arbitrary geometry MoM applied to solve the general SIEs for thin boards, computations present the accuracy and convergence of the proposed numerical solution to this special case.

Index Terms—current density, dielectric bodies, harmonic analysis, integral equations, moment methods.

I. INTRODUCTION

IN computational electromagnetics, finding the accurate solution to the surface integral equation (SIE) and the volume integral equation (VIE) is one of the fundamental problems on the *full-wave analysis* of three-dimensional electromagnetic (EM) phenomenon. The boundary value problem in physics has the unique solution; mathematically, a *satisfiable* EM integral equation gives a unique solution to any unique excitation. For example, free of interior resonances, the well-known combined field integral equation (CFIE) for the closed perfectly electrically conducting (PEC) surfaces only yields the trivial solution under null incidence. However, on the other hand, solving the integral equations numerically, the solution may not be guaranteed to converge to that exact one. It has been claimed

that “the nature of the method of moments (MoM) technique precludes proof of absolute convergence of the solution for a given problem.” [1], [2], “Because the theory of nonself-adjoint operators is not well understood, it is not possible, based on any known mathematically sound convergence criteria, to compare the approximation obtained in the method of moments to the exact solution. This difficulty is typical of convergence problems in electromagnetics.” [1], [3], “...especially when dealing with three-dimensional (3-D) objects, even worst, the convergence of the method (integral equation formulations) cannot be guaranteed due to the singular nature of the integral equation kernel.” [4] These insight comments have revealed the importance of knowing how the space of solutions is mapped to the space of given excitations.

Currently, at the end of the beginning on the long journey of devoting hard work to this difficult subject shading on the EM integral equations, there are two advantages making the success of solving EM fields in cylindrical decomposition with guaranteed convergence. First, for those unknown planar currents in the classical domains of circles and concentric annular rings on the polar plane, the closed formed Green's functions for the planar layered medium [5], [6] are available in the Cartesian spectral domain. Second, since the plane vector is a function of the polar radius (ρ) and the azimuthal angle (ϕ), the solution spaces of those planar currents can be spanned by the complete set of orthogonal cylindrical harmonics which are eigenmodes of scalar Helmholtz equation in cylindrical coordinates. This successful approach has applied to accurate analysis of annular-ring/circular disk microstrip antennas [7]–[9], and scattering by PEC circular disk against oblique incident plane waves [4], [10], etc.

Situations with accurately solving the circular symmetric unknown planar currents are simpler. For radial functions which are circular symmetric on the polar plane, the Hankel transform of order 0 is identical to the Fourier transform in two dimensions, and thus the free space Green's function can be manipulated into an order 0 Hankel integral form [6]. This essential together with the employment of the Hankel transform of order 1 of the electric/magnetic circular symmetric planar currents as the radial functions have analytically formulated the free space EM radiation of electric/magnetic circular symmetric planar currents, which has produced several canonical circular symmetric cylindrical EM structures such as annular slot on PEC screen [11], conducting circular disk against axially symmetric illumination [12], [13], and flanged open-ended coaxial line (see, e.g., [14], and the references

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therein), etc., found in both computational and practical researches. In these existing works, a single radial function integral equation is solved for unknown electric current over the finite radii circular disk [12], [13], or unknown magnetic current over the finite annular ring [11], [14], while the undetermined circular symmetric cylindrical EM fields in the planar layered penetrable dielectric regions are considered as transversely infinite, represented in the spectral domain of the radial direction by superposition of discrete or continuous eigenmodes transmitting along the longitudinal direction [14].

The aim of this paper is to present using the convergent numerical method to solve the SIEs for a finite-sized homogeneous dielectric board with a circular symmetric planar current inside. A sufficient condition that a *lossy* board with lateral faces equivalent currents *negligible* is emphasized, which finally leads to an infinite series solution that is of L^2 -norm convergence to the true radial distributions of equivalent electric/magnetic current densities on the board top and bottom planar faces. To this special case, reduction of the general SIEs for homogeneous dielectric bodies are handled based on utilizing the planar symmetry of the equivalent electric/magnetic currents at the top of a dielectric board to those at the bottom, and then borrowing from the above mentioned existing works the free space EM radiation of those radial-varying planar currents represented in the radial spectral domain. On the convergence of the proposed numerical solution, the accuracy of the infinite series solution to the finalized coupled radial function integral equations of this special case is guaranteed by the L^2 -norm convergent solution of the generalized simultaneous linear operator equations on Hilbert spaces of countably infinite dimensions, referring to the lemmas presented in the Appendix.

II. PLANAR SYMMETRY OF EQUIVALENT CURRENTS ON BOARD TOP AND BOTTOM PLANAR FACES

For homogeneous dielectric m filled free space, operators T_m and K_m are defined as

$$T_m \{ \bar{X}_s \} (\bar{r}) = -j\omega\mu_0 \iint_{S'} \bar{X}_s(\bar{r}') G_m(\bar{r}, \bar{r}') dS' + \iint_{S'} \frac{\nabla_s \cdot \bar{X}_s(\bar{r}')}{j\omega\epsilon_0\epsilon_m} \nabla G_m(\bar{r}, \bar{r}') dS', \quad (1)$$

$$K_m \{ \bar{X}_s \} (\bar{r}) = \iint_{S'} \bar{X}_s(\bar{r}') \times \nabla G_m(\bar{r}, \bar{r}') dS', \quad (2)$$

with $\bar{X}_s$ is the electric current $\bar{J}_s$ or the magnetic current $\bar{M}_s$ on surface S' , and the relative complex permittivity is defined as $\epsilon_m = \epsilon_m' - j\sigma_m / (2\pi f \epsilon_0)$, $\epsilon_m' \geq 1.0$, $\sigma_m \geq 0$ S/m, where $(\epsilon_0\epsilon_m', \mu_0, \sigma_m)$ are the permittivity, permeability, and conductivity with dielectric m . $G_m(\bar{r}, \bar{r}') = e^{-jk_0\sqrt{\epsilon_m}|\bar{r}-\bar{r}'|} / 4\pi|\bar{r}-\bar{r}'|$ denotes the free space Green's function, where $k_0 = 2\pi f \sqrt{\mu_0\epsilon_0}$ denotes the vacuum wavenumber. The reasonably selected square root of the relative complex permittivity is at the fourth quadrant of the complex plane, i.e., $\text{Re}(\sqrt{\epsilon_m}) \geq 0$ and $\text{Im}(\sqrt{\epsilon_m}) \leq 0$.

The electric current produced electric and magnetic fields are

$$\bar{E}^J = T_m(\bar{J}_s) \quad (3)$$

$$\bar{H}^J = -K_m(\bar{J}_s) \quad (4)$$

The magnetic current produced electric and magnetic fields are

$$\bar{E}^M = K_m(\bar{M}_s) \quad (5)$$

$$\bar{H}^M = \epsilon_0\epsilon_m / \mu_0 \cdot T_m(\bar{M}_s) \quad (6)$$

We consider the planar currents $\bar{X}_s$ on the plane $z' = z_0$ in dielectric m filled homogenous free space, as $\bar{X}_s = \bar{X}_s(x', y', z' = z_0)$. Denoting any specified field point in the upper half-space w.r.t. the plane $z' = z_0$ by coordinate $(x, y, z_0 + z_1)$ with $z_1 > 0$, the respective symmetric field point in the lower half-space is with coordinate $(x, y, z_0 - z_1)$. For the operators T_m and K_m , we have

Property 1:

$$T_m \{ \bar{X}_s \} (x, y, z_0 + z_1) \Big|_{\text{tan}} = T_m \{ \bar{X}_s \} (x, y, z_0 - z_1) \Big|_{\text{tan}} \quad (7)$$

Property 2:

$$K_m \{ \bar{X}_s \} (x, y, z_0 + z_1) \Big|_{\text{tan}} = -K_m \{ \bar{X}_s \} (x, y, z_0 - z_1) \Big|_{\text{tan}} \quad (8)$$

A homogenous dielectric board with planar currents inside is depicted in Fig. 1.

Fig. 1. Planar currents at the mid-plane of a homogenous dielectric board: (a) Front view; (b) Top view.

As in Fig. 1, according to the equivalence principle [15], for condition that the board lateral faces equivalent currents are negligible, with '+' and '-' denoting the positive and the negative side of the board surface S_1 respectively, we

denote $\pm \bar{J}_{s1}^{\pm}(x', y')$ and $\pm \bar{J}_{s1}^{\pm}(x', y')$ as the equivalent electric current $\pm \bar{J}_{s1}^{\pm}(x', y', z' = d)$ on the board top planar face $z' = d$ and $\pm \bar{J}_{s1}^{\pm}(x', y', z' = -d)$ on the bottom planar face $z' = -d$, respectively. Similarly, we denote $\pm \bar{M}_{s1}^{\pm}(x', y')$ and $\pm \bar{M}_{s1}^{\pm}(x', y')$ as the equivalent magnetic current $\pm \bar{M}_{s1}^{\pm}(x', y', z' = d)$ on the board top planar face and $\pm \bar{M}_{s1}^{\pm}(x', y', z' = -d)$ on the bottom planar face, respectively.

With the superscript 'I' denoting the interior volume of S_1 , and with $\bar{E}_{inc}^I$ and $\bar{H}_{inc}^I$ denoting the respective incident electric and magnetic fields toward the negative side of S_1 produced by the given current sources at the mid-plane of the board, on the top planar face, for any negative side field point (x, y, d^-) , and the respective positive side field point (x, y, d^+) , following the tangent continuity of the total electric field, we have

$$\begin{aligned} & \left[T_1 \left\{ -\bar{J}_{s1}^+ \right\} + K_1 \left\{ -\bar{M}_{s1}^+ \right\} + T_1 \left\{ -\bar{J}_{s1}^- \right\} + K_1 \left\{ -\bar{M}_{s1}^- \right\} + \bar{E}_{inc}^I \right] (x, y, d^-) \Big|_{tan} \quad (9) \\ & = \left[T_0 \left\{ \bar{J}_{s1}^+ \right\} + K_0 \left\{ \bar{M}_{s1}^+ \right\} + T_0 \left\{ \bar{J}_{s1}^- \right\} + K_0 \left\{ \bar{M}_{s1}^- \right\} \right] (x, y, d^+) \Big|_{tan} \end{aligned}$$

and following the tangent continuity of the total magnetic field, we have

$$\begin{aligned} & \left[-K_1 \left\{ -\bar{J}_{s1}^+ \right\} + \frac{\epsilon_0 \epsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{s1}^+ \right\} - K_1 \left\{ -\bar{J}_{s1}^- \right\} + \frac{\epsilon_0 \epsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{s1}^- \right\} + \bar{H}_{inc}^I \right] (x, y, d^-) \Big|_{tan} \quad (10) \\ & = \left[-K_0 \left\{ \bar{J}_{s1}^+ \right\} + \epsilon_0 / \mu_0 \cdot T_0 \left\{ \bar{M}_{s1}^+ \right\} - K_0 \left\{ \bar{J}_{s1}^- \right\} + \epsilon_0 / \mu_0 \cdot T_0 \left\{ \bar{M}_{s1}^- \right\} \right] (x, y, d^+) \Big|_{tan} \end{aligned}$$

To the four unknown equivalent currents $\bar{J}_{s1}^+$, $\bar{M}_{s1}^+$, $\bar{J}_{s1}^-$, and $\bar{M}_{s1}^-$, equations (9) and (10) are the PMCHWT (Poggio-Miller-Chang-Harrington-Wu-Tsai) integral equations [16]-[18] hold on the top planar face. If we respectively replace the top planar face negative side field point (x, y, d^-) in the left hand side (LHS) of (9) and (10) with its symmetric bottom planar face negative side field point $(x, y, -d^-)$, and respectively replace the top planar face positive side field point (x, y, d^+) in the right hand side (RHS) of (9) and (10) with its symmetric bottom planar face positive side field point $(x, y, -d^+)$, we have the following integral equations

$$\begin{aligned} & \left[T_1 \left\{ -\bar{J}_{s1}^+ \right\} + K_1 \left\{ -\bar{M}_{s1}^+ \right\} + T_1 \left\{ -\bar{J}_{s1}^- \right\} + K_1 \left\{ -\bar{M}_{s1}^- \right\} + \bar{E}_{inc}^I \right] (x, y, -d^-) \Big|_{tan} \quad (11) \\ & = \left[T_0 \left\{ \bar{J}_{s1}^+ \right\} + K_0 \left\{ \bar{M}_{s1}^+ \right\} + T_0 \left\{ \bar{J}_{s1}^- \right\} + K_0 \left\{ \bar{M}_{s1}^- \right\} \right] (x, y, -d^+) \Big|_{tan} \end{aligned}$$

$$\begin{aligned} & \left[-K_1 \left\{ -\bar{J}_{s1}^+ \right\} + \frac{\epsilon_0 \epsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{s1}^+ \right\} - K_1 \left\{ -\bar{J}_{s1}^- \right\} + \frac{\epsilon_0 \epsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{s1}^- \right\} + \bar{H}_{inc}^I \right] (x, y, -d^-) \Big|_{tan} \quad (12) \\ & = \left[-K_0 \left\{ \bar{J}_{s1}^+ \right\} + \epsilon_0 / \mu_0 \cdot T_0 \left\{ \bar{M}_{s1}^+ \right\} - K_0 \left\{ \bar{J}_{s1}^- \right\} + \epsilon_0 / \mu_0 \cdot T_0 \left\{ \bar{M}_{s1}^- \right\} \right] (x, y, -d^+) \Big|_{tan} \end{aligned}$$

Equations (11) and (12) express the tangent continuity of the total electric field and magnetic field across the bottom planar face, respectively, and they are the PMCHWT integral equations hold on the bottom planar face.

For that only the electric currents exist on the mid-plane of the board $z' = 0$, referring to Property 1 and Property 2 (see (7) and (8)), it allows we have the incident electric and magnetic fields as

$$\bar{E}_{inc}^I(x, y, d) \Big|_{tan} = \bar{E}_{inc}^I(x, y, -d) \Big|_{tan} \quad (13)$$

$$\bar{H}_{inc}^I(x, y, d) \Big|_{tan} = -\bar{H}_{inc}^I(x, y, -d) \Big|_{tan} \quad (14)$$

If we replace $\bar{J}_s^-$ with $\bar{J}_s^+$, and replace $\bar{M}_s^-$ with $-\bar{M}_s^+$, referring to Property 1 and Property 2, it is easy to verify that for any field point (x, y, d) on the top planar face, once the PMCHWT integral equations (9) and (10) hold, there must be the PMCHWT integral equations (11) and (12) hold at the respective field point $(x, y, -d)$ that is planar symmetrically on the bottom face, and vice versa. As the solution of the PMCHWT integral equations for homogeneous dielectric bodies uniquely exists, this symmetry as

$$\bar{J}_{s1}^- = \bar{J}_{s1}^+ \quad (15)$$

$$\bar{M}_{s1}^- = -\bar{M}_{s1}^+ \quad (16)$$

is indeed that the unique solution behaves.

On the duality [15], for that only the magnetic currents exist on the mid-plane of the board $z' = 0$, it is concluded that we have the incident electric and magnetic fields as

$$\bar{E}_{inc}^I(x, y, d) \Big|_{tan} = -\bar{E}_{inc}^I(x, y, -d) \Big|_{tan} \quad (17)$$

$$\bar{H}_{inc}^I(x, y, d) \Big|_{tan} = \bar{H}_{inc}^I(x, y, -d) \Big|_{tan} \quad (18)$$

while the symmetry of the equivalent currents as

$$\bar{J}_{s1}^- = -\bar{J}_{s1}^+ \quad (19)$$

$$\bar{M}_{s1}^- = \bar{M}_{s1}^+ \quad (20)$$

Therefore, with the symmetry in (15) and (16) for only the electric current sources exist on the mid-plane of the board, or (19) and (20) for only the magnetic current sources exist, the unique solution can be determined only we select the PMCHWT integral equations (9) and (10) hold on the top planar face, or we select the bottom ones (11) and (12).

III. CONVERGENT METHOD OF MOMENTS SOLUTION OF INTEGRAL EQUATIONS ON HILBERT SPACES OF RADIAL FUNCTIONS

In the following, the nonnegative real number k_ρ denotes the radial wavenumber, and the complex number k_z denotes the longitudinal wavenumber, with complex relative permittivity ϵ_r . We have k_ρ and k_z following the square-sum-constraint identity

$$k_z^2 + k_\rho^2 = k_0^2 \epsilon_r \quad (21)$$

with k_z is at the fourth quadrant of the complex plane, i.e., $\text{Re}(k_z) \geq 0$ and $\text{Im}(k_z) \leq 0$.

A. Free Space EM-Fields of a Circular Symmetric Planar Current

The circular symmetric currents on the plane $z' = z_0$ have loop (divergence-free) and star (curl-free) components as

$$\bar{J}_s(\rho', \varphi', z' = z_0) = J_{s\rho}(\rho', z' = z_0) \hat{\rho} + J_{s\varphi}(\rho', z' = z_0) \hat{\varphi} \quad (22)$$

$$\bar{M}_s(\rho', \varphi', z' = z_0) = M_{s\rho}(\rho', z' = z_0) \hat{\rho} + M_{s\varphi}(\rho', z' = z_0) \hat{\varphi} \quad (23)$$

In the homogeneous dielectric $(\epsilon_r, \epsilon_0, \mu_0)$ free space, for the radial-varying electric star current and loop current, by introducing the Hankel transform of order 1 defined as

$$\tilde{J}_{s\rho}(k_\rho, z_0) = \int_0^{+\infty} \rho' J_{s\rho}(\rho', z_0) \mathcal{J}_1(k_\rho \rho') d\rho' \quad (24)$$

$$\tilde{J}_{s\varphi}(k_\rho, z_0) = \int_0^{+\infty} \rho' J_{s\varphi}(\rho', z_0) \mathcal{J}_1(k_\rho \rho') d\rho' \quad (25)$$

where $J_1(\bullet)$ is the first-kind cylindrical Bessel function of order 1, the vector potentials of the respective electric star and loop currents are [6]

$$A_\rho^j(\rho, z) = -\frac{\mu_0}{2j} \int_0^{+\infty} k_\rho e^{-jk_z|z-z_0|} / k_z \tilde{J}_{s\rho}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho \quad (26)$$

$$A_\phi^j(\rho, z) = -\frac{\mu_0}{2j} \int_0^{+\infty} k_\rho e^{-jk_z|z-z_0|} / k_z \tilde{J}_{s\phi}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho \quad (27)$$

For the electric star current $J_{s\rho}(\rho', z'=z_0)\hat{\rho}$, by using the curl of the vector potential

$$\vec{H}(\rho, \phi, z)|_{J_{s\rho}} = -\frac{1}{\mu_0} \nabla \times [A_\rho^j(\rho, z)\hat{\rho}] = -\frac{1}{\mu_0} \frac{\partial A_\rho^j}{\partial z} \hat{\phi} \quad (28)$$

the magnetic field of the electric star current can be directly expressed as

$$H_\phi(\rho, z)|_{J_{s\rho}} = \begin{cases} -\frac{1}{2} \int_0^{+\infty} k_\rho e^{-jk_z(z-z_0)} \tilde{J}_{s\rho}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho & z > z_0 \\ \frac{1}{2} \int_0^{+\infty} k_\rho e^{jk_z(z-z_0)} \tilde{J}_{s\rho}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho & z < z_0 \end{cases} \quad (29)$$

For the electric loop current $J_{s\phi}(\rho', z'=z_0)\hat{\phi}$, the electric field can be directly expressed by the vector potential as

$$E_\phi(\rho, z)|_{J_{s\phi}} = -j\omega A_\phi^j(\rho, z) \quad (30)$$

$$= -\frac{1}{2} \omega \mu_0 \int_0^{+\infty} k_\rho e^{-jk_z|z-z_0|} / k_z \tilde{J}_{s\phi}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho$$

Following the principle of duality [15], through borrowing this above electric field expression of the electric loop current, the magnetic field of the magnetic loop current $M_{s\phi}(\rho', z_0)\hat{\phi}$ can be written as

$$H_\phi(\rho, z)|_{M_{s\phi}} = -\frac{1}{2} \omega \epsilon_0 \epsilon_r \int_0^{+\infty} k_\rho e^{-jk_z|z-z_0|} / k_z \tilde{M}_{s\phi}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho \quad (31)$$

with the Hankel transform of order 1 for the radial-varying magnetic loop current as

$$\tilde{M}_{s\phi}(k_\rho, z_0) = \int_0^{+\infty} \rho' M_{s\phi}(\rho', z_0) J_1(k_\rho \rho') d\rho' \quad (32)$$

Using (29) for the magnetic field of the electric star current and (31) for the magnetic field of the magnetic loop current, the respective electric fields of the two currents can be obtained by following the Faraday law of EM induction

$$\vec{E}(\rho, \phi, z) = \frac{1}{j\omega \epsilon_0 \epsilon_r} \nabla \times [\vec{H}(\rho, \phi, z)\hat{\phi}] \quad (33)$$

$$= \frac{1}{j\omega \epsilon_0 \epsilon_r} \left[-\frac{\partial}{\partial z} H_\phi \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \hat{z} \right]$$

Here for further usage we only list the radial electric field expressions, as follows

$$E_\rho(\rho, z)|_{J_{s\rho}} = -\frac{1}{2} \int_0^{+\infty} k_\rho \frac{k_z}{\omega \epsilon_0 \epsilon_r} e^{-jk_z|z-z_0|} \tilde{J}_{s\rho}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho \quad (34)$$

$$E_\rho(\rho, z)|_{M_{s\phi}} = \begin{cases} -\frac{1}{2} \int_0^{+\infty} k_\rho e^{-jk_z(z-z_0)} \tilde{M}_{s\phi}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho & z > z_0 \\ \frac{1}{2} \int_0^{+\infty} k_\rho e^{jk_z(z-z_0)} \tilde{M}_{s\phi}(k_\rho, z_0) J_1(k_\rho \rho) dk_\rho & z < z_0 \end{cases} \quad (35)$$

B. Coupled Radial Function Integral Equations for Finite-sized Planar Layered Cylindrical Geometry

Referring to Section III.A, the EM fields of the electric star current and the magnetic loop current are of TM mode (w.r.t. z-axis), while on the duality, the electric loop current and the magnetic star current produce TE mode EM-fields.

For a TM mode annular ring circular symmetric magnetic loop current at the center of the homogeneous dielectric board,

we denote the mid-plane of the board $z'=0$ by S_2 , as depicted in Fig. 2. This magnetic loop current source is defined as

$$\vec{M}_{S2} = M_{S2\phi}(\rho', z'=0)\hat{\phi} \quad a \leq \rho' \leq b \quad (36)$$

Fig. 2. Annular circular symmetric magnetic loop current at the center of a dielectric board: (a) Front view; (b) Top view.

In Fig. 2, for the dielectric board interior volume V_1 , the given magnetic loop current (36) produces TM mode incidence, thus, on both the negative sides of the top and the bottom planar faces, there only allow the TM mode equivalent currents which are at the top and the bottom circles of radius R_1 located coaxially with the annular ring of the magnetic planar source (36). By using the radial functions f_1 and g_1 to represent the respective equivalent electric and magnetic currents on the top planar face, as follows

$$\begin{cases} f_1(\rho') = J_{S1\rho}(\rho', z'=d) \\ g_1(\rho') = M_{S1\phi}(\rho', z'=d) \end{cases} \quad 0 \leq \rho' \leq R_1 \quad (37)$$

And referring to (19) and (20), we have the bottom planar face equivalent currents as

$$\begin{cases} J_{S1\rho}(\rho', z'=-d) = -f_1(\rho') \\ M_{S1\phi}(\rho', z'=-d) = g_1(\rho') \end{cases} \quad 0 \leq \rho' \leq R_1 \quad (38)$$

For the same convenience, we may use a radial function g_2 to represent the given annular magnetic loop current source (36) as

$$g_2(\rho') = M_{S_{2\varphi}}(\rho', z' = 0) \quad a \leq \rho' \leq b \quad (39)$$

In the following, the Hankel transform of order 1 of the radial functions f_1 , g_1 and g_2 are defined as

$$\begin{aligned} \tilde{f}_1(k_\rho) &= \int_0^{R_1} \rho' f_1(\rho') J_1(k_\rho \rho') d\rho' \\ \tilde{g}_1(k_\rho) &= \int_0^{R_1} \rho' g_1(\rho') J_1(k_\rho \rho') d\rho' \\ \tilde{g}_2(k_\rho) &= \int_a^b \rho' g_2(\rho') J_1(k_\rho \rho') d\rho' \end{aligned} \quad (40)$$

With superscript 'I' denoting the interior volume of S_1 , we may select a magnetic field integral equation (MFIE) hold on the negative side of the top planar face, which can be written as the first radial function integral equation

$$J_{S_{1\rho}}(\rho, z = d^-) + H_{1\varphi}^I(\rho, z = d^-) + H_{2\varphi}^I(\rho, z = d^-) = 0 \quad 0 \leq \rho \leq R_1 \quad (41)$$

and represented by the linear operator equation as

$$L_{11}\{f_1\}(\rho) + L_{12}\{g_1\}(\rho) + L_{13}\{g_2\}(\rho) = 0 \quad (42)$$

where the linear operators

$$L_{11}\{f_1\}(\rho) = \int_0^{+\infty} k_\rho \frac{1 - e^{-jk_{1z} \cdot 2d}}{2} \tilde{f}_1(k_\rho) J_1(k_\rho \rho) dk_\rho \quad 0 \leq \rho \leq R_1 \quad (43)$$

$$L_{12}\{g_1\}(\rho) = \frac{\omega \varepsilon_0 \varepsilon_{r1}}{2} \int_0^{+\infty} k_\rho \frac{1 + e^{-jk_{1z} \cdot 2d}}{k_{1z}} \tilde{g}_1(k_\rho) J_1(k_\rho \rho) dk_\rho \quad 0 \leq \rho \leq R_1 \quad (44)$$

$$L_{13}\{g_2\}(\rho) = -\frac{\omega \varepsilon_0 \varepsilon_{r1}}{2} \int_0^{+\infty} k_\rho \frac{e^{-jk_{1z} \cdot d}}{k_{1z}} \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho \quad 0 \leq \rho \leq R_1 \quad (45)$$

Similarly, with the superscript 'O' denoting the exterior volume of S_1 , we may select a MFIE hold on the positive side of the top planar face, which can be written as the second radial function integral equation

$$J_{S_{1\rho}}(\rho, z = d^+) + H_{1\varphi}^O(\rho, z = d^+) = 0 \quad 0 \leq \rho \leq R_1 \quad (46)$$

and represented by the linear operator equation as

$$L_{21}\{f_1\}(\rho) + L_{22}\{g_1\}(\rho) = 0 \quad (47)$$

where the linear operators

$$L_{21}\{f_1\}(\rho) = \int_0^{+\infty} k_\rho \frac{1 + e^{-jk_{0z} \cdot 2d}}{2} \tilde{f}_1(k_\rho) J_1(k_\rho \rho) dk_\rho \quad 0 \leq \rho \leq R_1 \quad (48)$$

$$L_{22}\{g_1\}(\rho) = -\frac{\omega \varepsilon_0}{2} \int_0^{+\infty} k_\rho \frac{1 + e^{-jk_{0z} \cdot 2d}}{k_{0z}} \tilde{g}_1(k_\rho) J_1(k_\rho \rho) dk_\rho \quad 0 \leq \rho \leq R_1 \quad (49)$$

C. Boundedness for Linear Operators of Radial Function Integral Equations

If a complex valued real function $f(\rho)$ $\rho \in [0, +\infty)$ satisfies

$$\int_0^{+\infty} \rho |f(\rho)|^{1,2} d\rho < \infty \quad (50)$$

we say it is an integrable radial function. An integrable radial function is also a finite energy function. We denote the set of all the integrable radial functions by $F_{[0, +\infty)}$.

For any $f, h \in F_{[0, +\infty)}$, the inner products for f and h are defined by

$$\begin{aligned} \langle f, h \rangle &= \int_0^{+\infty} \rho f(\rho) \overline{h(\rho)} d\rho \\ \langle h, f \rangle &= \int_0^{+\infty} \rho h(\rho) \overline{f(\rho)} d\rho = \overline{\langle f, h \rangle} \end{aligned} \quad (51)$$

where $\bar{f}$ is the conjugate of f . The L^2 -norm of f is associated with the inner product as

$$\|f\|_2 = \sqrt{\langle f, f \rangle} \quad (52)$$

Thus, $F_{[0, +\infty)}$, the linear space of integrable radial functions, is a

Hilbert space.

In Section III.B, all the five linear operators L_{11} , L_{12} , L_{13} , L_{21} , and L_{22} are defined on the linear space of the integrable radial functions $F_{[0, +\infty)}$, and all of them are the composition of two linear operators as

$$L_{ij}(f) = C_{\rho_1, \rho_2} \circ L_{ij,0}(f) \quad 0 \leq \rho_1 < \rho_2 \quad (53)$$

The former linear operator C_{ρ_1, ρ_2} cuts out a section of radial function on a finite length interval $[\rho_1, \rho_2]$, which is defined as

$$C_{\rho_1, \rho_2}\{f\}(\rho) = f(\rho) \quad \rho \in [\rho_1, \rho_2] \quad (54)$$

and the latter linear operator $L_{ij,0}$ is a linear transform on $F_{[0, +\infty)}$, which is defined as

$$L_{ij,0}\{f\}(\rho) = \int_0^{+\infty} k_\rho \frac{A_{ij}(k_\rho)}{B_{ij}(k_\rho)} \left[\int_0^{+\infty} \rho' f(\rho') J_1(k_\rho \rho') d\rho' \right] J_1(k_\rho \rho) dk_\rho \quad (55)$$

where $A_{ij}(k_\rho)$ and $B_{ij}(k_\rho)$ are complex valued functions on $[0, +\infty)$.

Referring to Lemma 1 in the Appendix, the eigenfunctions and eigenvalues of $L_{ij,0}$ are $J_1(k_\rho \rho)$ $k_\rho, \rho \in [0, +\infty)$ and $A_{ij}(k_\rho)/B_{ij}(k_\rho)$. For any lossy dielectric board, linear operators $L_{ij,0}$ are bounded, this is because both $A_{ij}(k_\rho)$ and $B_{ij}(k_\rho)$ are nonzero and finite-valued.

With inequity $\|C_{\rho_1, \rho_2}(f)\|_2 \leq \|f\|_2$, we have $\|C_{\rho_1, \rho_2}\| = 1$. As the norm of the composition of linear operators does not exceed the multiplication of the norm of each linear operator, we have inequity $\|L_{ij}\| \leq \|C_{\rho_1, \rho_2}\| \cdot \|L_{ij,0}\|$. Thus, we have $\|L_{ij}\| \leq \|L_{ij,0}\|$ that all the five linear operators L_{11} , L_{12} , L_{13} , L_{21} , and L_{22} are bounded.

D. Method of Moments Solution Using Orthogonal Radial Functions

On the finite-length interval $[0, R_1]$, where R_1 is a positive real number, there are an infinite number of orthogonal real functions [19] as circular symmetric cylindrical harmonics

$$\Phi_{1,n}(\rho) = \frac{\sqrt{2} J_1(k_{1,n} \rho)}{R_1 |J_1(k_{1,n} R_1)|} \quad 0 \leq \rho \leq R_1 \quad (56)$$

$$\langle \Phi_{1,m}(\rho), \Phi_{1,n}(\rho) \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad m, n = 1, 2, \dots \quad (57)$$

with that the positive real roots of the transcendental equation $J_0(kR_1) = 0, k \geq 0$, where $J_0(\bullet)$ is the first-kind cylindrical Bessel function of order 0, are successively numbered from the smallest one to the large ones, as $k_{1,n}$ $n=1, 2, \dots$ respectively. These orthogonal radial functions $\{\Phi_{1,n} \quad n=1, 2, \dots\}$ form a normalized complete set that spans an integrable radial functions space $F_{[0, R_1]}$. As a subset of $F_{[0, +\infty)}$, $F_{[0, R_1]}$ is a countably infinite dimensional Hilbert space of integrable radial functions $f(\rho)$ on interval $[0, R_1]$ satisfies $f(0)=0$, as $\Phi_{1,n}(0)=0 \quad n=1, 2, \dots$.

For the problem with a given magnetic loop current of g_2 (see (39)), the unknown equivalent electric and magnetic currents as f_1 and g_1 (see (37) and (38)) are solved by the method of moments.

The basis functions for both f_1 and g_1 are selected from the set of $\{\Phi_{1,n} \mid n=1,2,\dots\}$, as the truncated $f_1(\rho')$ and the truncated $g_1(\rho')$ are respectively expressed by the linear combination of the first N_1 number of terms from this complete set of orthogonal radial functions on the finite-length interval $[0, R_1]$, as

$$\begin{cases} f_1^{(N_1)}(\rho') = \sum_{n=1}^{N_1} a_n \Phi_{1,n}(\rho') \\ g_1^{(N_1)}(\rho') = \sum_{n=1}^{N_1} b_n \Phi_{1,n}(\rho') \end{cases} \quad (58)$$

with the electric current coefficients $\{a_n \mid n=1,2,\dots,N_1\}$ in dimension of Ampere (A), and the magnetic current coefficients $\{b_n \mid n=1,2,\dots,N_1\}$ in dimension of Volt (V).

For testing functions, each individual from the same first N_1 number of orthogonal terms is used one-by-one to perform inner product with the first radial function integral equation (41) and the second one (46), respectively. All the coefficients of the resultant linear system of $2N_1$ equations are in the form of an inner product of one radial function $\Phi_{1,m}$ and the other radial function which is in the range of one of the five linear operators L_{11} , L_{12} , L_{13} , L_{21} , and L_{22} , written as

$$\langle \Phi_{1,m}, C_{0,R_1} \circ L_{ij,0} \{ \Phi_{1,n} \} \rangle = \langle \Phi_{1,m}, L_{ij,0} \{ \Phi_{1,n} \} \rangle \quad (59)$$

Following that the inner product preserves under orthogonal transform, the RHS of (59) is rewritten as

$$\langle \Phi_{1,m}, L_{ij,0} \{ \Phi_{1,n} \} \rangle = \langle \tilde{\Phi}_{1,m}, A_{ij} / B_{ij} \tilde{\Phi}_{1,n} \rangle \quad (60)$$

with the Hankel transform of order 1 of $\Phi_{1,n}$ in a closed form [19] as

$$\tilde{\Phi}_{1,n}(k_\rho) = \begin{cases} \frac{\sqrt{2}}{|J_1(k_{1,n}R_1)|} \frac{k_\rho J_1(k_{1,n}R_1) J_0(k_\rho R_1)}{k_{1,n}^2 - k_\rho^2} & k_\rho \neq k_{1,n} \\ \frac{\sqrt{2}}{2} R_1 |J_1(k_{1,n}R_1)| & k_\rho = k_{1,n} \end{cases} \quad (61)$$

To this end, coefficients of the method of moments linear system of $2N_1$ equations are formulated as a definite integral (w.r.t. k_ρ) for regular spectral functions of the Hankel transform of order 1, and all of them can be precisely and easily evaluated following ordinary quadrature rules. $\square$

For the above mentioned coupled radial function integral equations (41) and (46), the annular magnetic loop current source (see (39)) is practically produced by a coaxial waveguide opening, as depicted in Fig. 3.

Fig. 3. Coaxial waveguide terminated at the annular opening on the complementary PEC screen.

This refers to the equivalent problems as follows:

For an original problem that has given magnetic planar current sources at the mid-plane (see Fig. 4(a)), we can construct two equivalent problems.

The first one is characterized as (see Fig. 4(b)):

- 1) At mid-plane of the board, a whole PEC plate is placed.
- 2) The original magnetic planar currents are removed, while new magnetic planar currents which are *half* of the original ones are placed onto the same areas at the positive side ($z=0^+$) of the PEC plate.

Based on the first equivalent problem, a second one is characterized as (see Fig. 4(c)):

- 1) At the mid-plane of the board, a PEC screen complementary to the areas of given magnetic planar currents is placed, and then the original magnetic planar currents are removed.
- 2) The tangent electric field at the waveguide open end produces an equivalent magnetic planar current which is *half* of the original one at the positive side ($z=0^+$) of the respective aperture on the complementary PEC screen.

Their equivalence to the original problem can be verified by firstly applying the equivalence principle to the two-layer volume regions $0^+ \leq z \leq d^-$ and $z \geq d^+$ of the upper half-space respectively, and then using the mirror image principle [15] for the plane $z=0$.

Fig. 4. Two equivalent problems for magnetic planar currents at the mid-plane of a dielectric board: (a) Original problem; (b) Magnetic planar currents at PEC plate; (c) Waveguide apertures on the complementary PEC screen.

On the composition of the annular magnetic loop current source of the original problem (see (39)), for case that in coaxial waveguide the incident wave consists of TEM and TM_{0n} ($n=1,2, \dots$) modes only, the total tangent electric field at annular aperture on plane S_2 is circular symmetric and in radial direction, which can be written as $E_{2\rho}(\rho', z'=0^-)\hat{\rho}$. Following that annular magnetic loop current source is *twice* of the equivalent magnetic planar current at the positive side ($z=0^+$) of the annular aperture on plane S_2 , as

$$\begin{aligned} M_{S2\rho}(\rho', z'=0)\hat{\rho} \\ = 2[E_{2\rho}(\rho', z'=0^+)\hat{\rho} \times \hat{z}] \\ = 2[E_{2\rho}(\rho', z'=0^-)\hat{\rho} \times \hat{z}] \end{aligned} \quad (62)$$

$a \leq \rho' \leq b$

We obtain the total radial electric field in coaxial waveguide at annular aperture as

$$E_{2\rho}(\rho', z'=0^-) = -1/2 g_2(\rho') \quad a \leq \rho' \leq b \quad (63)$$

For positive real numbers $0 < a < b$, on the finite length interval $[a, b]$, there are an infinite number of orthogonal real functions [19] as another circular symmetric cylindrical harmonics

$$\begin{aligned} \Phi_{2,0}(\rho) &= \frac{1}{\rho \sqrt{\ln(b/a)}} \\ \Phi_{2,n}(\rho) &= \frac{\pi k_{2,n}}{\sqrt{2}} \cdot \frac{Y_0(k_{2,n}a)J_1(k_{2,n}\rho) - J_0(k_{2,n}a)Y_1(k_{2,n}\rho)}{\sqrt{J_0^2(k_{2,n}a)/J_0^2(k_{2,n}b) - 1}} \end{aligned} \quad (64)$$

$n=1,2,\dots$
 $a \leq \rho \leq b$

$$\langle \Phi_{2,m}(\rho), \Phi_{2,n}(\rho) \rangle = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases} \quad m,n=0,1,2,\dots \quad (65)$$

where $Y_{0,1}(\bullet)$ is the second-kind cylindrical Bessel function of order 0,1. The positive real roots of the transcendental equation $Y_0(ka)J_0(kb) = J_0(ka)Y_0(kb)$ $k \geq 0$ are successively numbered from the smallest one to the large ones, as $k_{2,n}$ $n=1,2, \dots$, respectively. The complete set of normalized orthogonal radial functions $\{\Phi_{2,n} \quad n=0,1,2,\dots\}$ spans an integrable radial functions space $F_{[a,b]}$, which is a *countably infinite dimensional* Hilbert space of integrable radial functions on interval $[a, b]$, as a subset of $F_{[0,+\infty)}$.

For a given TEM and TM_{0n} ($n=1,2, \dots$) modes incident wave in coaxial waveguide, following the tangent continuity condition at the coaxial opening for the magnetic field, by letting $H_{\text{reflec},\varphi}(\rho, z)\hat{\varphi}$ denote the azimuthal magnetic field of the reflection wave, which can be determined by (63) and the incident azimuthal magnetic field $H_{\text{inc},\varphi}(\rho, z)\hat{\varphi}$, we have the third radial function integral equation for the azimuthal magnetic field at the annular aperture as

$$\begin{aligned} H_{1\varphi}^I(\rho, z=0^+) + H_{2\varphi}^I(\rho, z=0^+) \\ = H_{\text{reflec},\varphi}(\rho, z=0^-) + H_{\text{inc},\varphi}(\rho, z=0^-) \end{aligned} \quad a \leq \rho \leq b \quad (66)$$

and represented by the linear operator equation as

$$L_{31}\{f_1\}(\rho) + L_{32}\{g_1\}(\rho) + \left(-\frac{1}{2}L_{\Lambda} + L_{33}\right)\{g_2\}(\rho) = h(\rho) \quad (67)$$

where the linear operators

$$L_{31}\{f_1\}(\rho) = -\int_0^{+\infty} k_{\rho} e^{-jk_{1z}d} \tilde{f}_1(k_{\rho}) J_1(k_{\rho}\rho) dk_{\rho} \quad a \leq \rho \leq b \quad (68)$$

$$L_{32}\{g_1\}(\rho) = \omega \epsilon_0 \epsilon_{r1} \int_0^{+\infty} k_{\rho} e^{-jk_{1z}d} / k_{1z} \tilde{g}_1(k_{\rho}) J_1(k_{\rho}\rho) dk_{\rho} \quad a \leq \rho \leq b \quad (69)$$

$$L_{33}\{g_2\}(\rho) = -\frac{\omega \epsilon_0 \epsilon_{r1}}{2} \int_0^{+\infty} k_{\rho} / k_{1z} \tilde{g}_2(k_{\rho}) J_1(k_{\rho}\rho) dk_{\rho} \quad a \leq \rho \leq b \quad (70)$$

and the linear operator L_{Λ} is a diagonal transform on $F_{[a,b]}$ as

$$L_{\Lambda}\{g_2\}(\rho) = \int_a^b \rho' g_2(\rho') \Lambda(\rho, \rho') d\rho' \quad a \leq \rho \leq b \quad (71)$$

by using

$$\Lambda(\rho, \rho') = \sum_{n=0}^{\infty} \frac{\omega \epsilon_0 \epsilon_{r2}}{k_{2z}(n)} \Phi_{2,n}(\rho) \Phi_{2,n}(\rho') \quad a \leq \rho, \rho' \leq b \quad (72)$$

where ϵ_{r2} denotes relative complex permittivity of coaxial filling, both the TEM mode longitudinal wavenumber $k_{2z}(0) = k_0 \sqrt{\epsilon_{r2}}$ and the TM_{0n} ($n=1,2, \dots$) modes longitudinal wavenumber $k_{2z}(n) = \sqrt{k_0^2 \epsilon_{r2} - k_{2,n}^2}$ are at the fourth quadrant of the complex plane. The specified excitation for integral equation (66) follows the given incident radial electric field $E_{\text{inc},\rho}(\rho, z)\hat{\rho}$ in coaxial waveguide, as

$$h(\rho) = 2L_{\Lambda}\{E_{\text{inc},\rho}(\rho', z'=0^-)\}(\rho) \quad a \leq \rho \leq b \quad (73)$$

To this end, we have the coupled radial function integral equations (41)-(46)-(66) for the coaxial waveguide excited equivalent problem (see Fig. 3).

On the other viewpoint, we may let the half-thickness dielectric board in Fig. 3 expand to the transverse infinity. Thus we can use planar stratified spectral representation [14] for

solving this two-layer structure. Denoting a linear operator by

$$L_{33}^* \{g_2\}(\rho) = -\frac{\omega \varepsilon_0 \varepsilon_{r1}}{2} \int_0^{+\infty} k_\rho / k_{1z} \tilde{S}(k_\rho) \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho \quad (74)$$

using a complex valued function defined as

$$\tilde{S}(k_\rho) = \frac{1 - e^{-jk_{1z} 2d} (\varepsilon_{r1} k_{0z} - k_{1z}) / (\varepsilon_{r1} k_{0z} + k_{1z})}{1 + e^{-jk_{1z} 2d} (\varepsilon_{r1} k_{0z} - k_{1z}) / (\varepsilon_{r1} k_{0z} + k_{1z})} \quad (75)$$

On the annular magnetic loop current source (see (39)) as an unknown, an integral equation is expressed as

$$\left(-\frac{1}{2} L_\Lambda + L_{33}^* \right) \{g_2\}(\rho) = h(\rho) \quad (76)$$

On the boundedness of linear operators of radial function integral equations (67) and (76), that the dielectric board is *lossy* is also a sufficient condition. As same as the analysis in Section III.C, we can immediately draw a conclusion that both the linear operators L_{31} , L_{32} , as well as L_{33} and L_{33}^* are *bounded*. Besides, the coaxial waveguide TEM and TM_{0n} ($n=1,2, \dots$) modes wave admittances $\omega \varepsilon_0 \varepsilon_{r2} / k_{2n}(n)$ $n=0,1,2, \dots$ are square summable, so that L_Λ is a bounded linear operator, and because the norm of the summation of linear operators does not exceed the summation of the norms of each linear operator, we have both linear operators $-1/2 L_\Lambda + L_{33}$ and $-1/2 L_\Lambda + L_{33}^*$ are *bounded*. $\square$

The coupled radial function integral equations (41)-(46)-(66) and radial function integral equation (76) are equivalent on solving the unknown annular magnetic loop current source $g_2(\rho')$ $a \leq \rho' \leq b$.

The method of moments for operating on the radial function integral equations (66) and (76) are identical. The first $1+N_2$ number of terms from the complete set of orthogonal radial functions $\{\Phi_{2,n} \ n=0,1,2, \dots\}$ are selected as basis functions to expand the truncated $g_2(\rho')$, and also as testing functions to perform inner product with (66) and (76). All the coefficients of the resultant linear system of $1+N_2$ number of equations are also formulated as a definite integral (w.r.t. k_ρ) for regular spectral functions of the Hankel transform of order 1, similar to the inner product in (60). The Hankel transform of order 1 for $\Phi_{2,n}(\rho')$ $n=0,1,2, \dots$ as $\tilde{\Phi}_{2,n}(k_\rho)$ are in the closed form [19], as

$$\begin{aligned} \tilde{\Phi}_{2,0}(k_\rho) &= \frac{J_0(k_\rho a) - J_0(k_\rho b)}{k_\rho \sqrt{\ln(b/a)}} \\ \tilde{\Phi}_{2,n}(k_\rho) &= \begin{cases} \frac{\sqrt{2} k_\rho}{k_{2,n}^2 - k_\rho^2} \cdot \frac{J_0(k_{2,n} a) J_0(k_\rho b) - J_0(k_{2,n} b) J_0(k_\rho a)}{J_0(k_{2,n} a) \sqrt{J_0^2(k_{2,n} a) / J_0^2(k_{2,n} b) - 1}} & k_\rho \neq k_{2,n} \\ \frac{\sqrt{2}}{2} \cdot \frac{b J_0(k_{2,n} a) J_1(k_{2,n} b) - a J_0(k_{2,n} b) J_1(k_{2,n} a)}{J_0(k_{2,n} b) \sqrt{J_0^2(k_{2,n} a) / J_0^2(k_{2,n} b) - 1}} & k_\rho = k_{2,n} \end{cases} \\ n &= 1, 2, \dots \end{aligned} \quad (77)$$

$\square$

To the TEM and TM_{0n} ($n=1,2, \dots$) modes in coaxial waveguide, let T_n and Γ_n both in dimension of Volt (V) represent the respective incident and reflection electric field coefficients w.r.t. the negative side of the annular aperture ($z=0^-$). The incident electric field coefficients are specified from the given incident radial electric field $E_{inc,\rho}(\rho, z=0^-) \hat{\rho}$ as

$$T_n = \langle \Phi_{2,n}(\rho), E_{inc,\rho}(\rho, z=0^-) \rangle \quad n=0,1,2, \dots \quad (78)$$

With the truncated $g_2(\rho')$ solved by method of moments as

$$g_2^{(N_2)}(\rho') = \sum_{n=0}^{N_2} c_n \Phi_{2,n}(\rho') \quad (79)$$

where the magnetic current coefficients $\{c_n \mid n=0,1,2, \dots, N_2\}$ are in dimension of Volt (V), the unknown reflection electric field coefficients of TEM and TM_{0n} ($n=1,2, \dots$) modes are obtained as

$$\Gamma_n = -\frac{1}{2} \langle \Phi_{2,n}, g_2 \rangle - T_n = -\frac{1}{2} c_n - T_n \quad n=0,1,2, \dots \quad (80)$$

which are more convenient candidates in contrast to the board surface equivalent currents on the observation of solution convergence.

E. Convergence of the Infinite Series Solution

Lemma 1, 2, 3 in the Appendix assure that in Section III.D the numerical solution proposed for the coupled radial function integral equations (41)-(46)-(66) as well as the radial function integral equation (76) are both L^2 -norm convergent to the exact solution. Firstly, in physics, those two integral equations both behave *satisfiability*. A null EM incidence in coaxial waveguide produces a null EM field at the annular aperture as well as in the two-layer upper half-space, and this is vice versa. Any of those two integral equations always give a unique solution to any given unique incidence. Secondly, in mathematics, both the solution space and the excitation space for both those two integral equations are countably infinite dimensional Hilbert spaces. The dielectric board is *lossy* is a sufficient condition that, not only all linear operators of those two integral equations are bounded, but also a limit for those resultant infinite number of consecutive finite truncated linear systems of equations does exist, which is a bounded infinite dimensional linear system of equations equivalent to the respective original integral equation. This is based upon the convergence of those respective infinite number of bounded linear operators on the consecutive subspaces.

Let X and Y be countably infinite dimensional Hilbert spaces of finite energy functions. $X^{(N)}$ and $Y^{(N)}$ are N -dimensional subspace of X and Y , respectively. Projections are defined as $P_X^{(N)}: X \mapsto X^{(N)}$ and $P_Y^{(N)}: Y \mapsto Y^{(N)}$. For any bounded linear operator $L: X \mapsto Y$, an infinite number of bounded linear operators $\{L^{(N)} = P_Y^{(N)} \circ L \circ P_X^{(N)}: X \mapsto Y\}$ which are also $N \times N$ matrices converge to L , as a limit exists

$$\lim_{N \rightarrow \infty} L^{(N)} = L \quad (81)$$

IV. NUMERICAL RESULTS

To observe the convergence of solution, the numerical results of the TEM mode reflection coefficient (see (80), with reflections normalized to the incidence for the radial electric field) can be chosen as representative, see Table I.(a)-(c). Typically, given a TEM mode incidence in coaxial waveguide, with gradually increased orthogonal expansions for the three unknown radial functions f_1 , g_1 and g_2 , the coupled radial function integral equations (41)-(46)-(66) as well as the radial

function integral equation (76) are solved respectively by using the numerical solution proposed in Section III.D. It can be observed that all the numerical results in various cases from low to high complex permittivity and from very thin to thick board behave excellent convergence against the respective expansion number N_1 , N_1 and $1+N_2$. In the computation, the coaxial waveguide with lossless dielectric filling ($\epsilon_{r2}=2.54$) has inner radii (a) and outer radii (b) of 2.00 and 7.55mm, respectively. The frequency of the time-harmonic EM field is 500MHz. The cylindrical board solved by the coupled radial function integral equations (41)-(46)-(66) is of radius $R_1=20.0$ mm, that the lateral faces equivalent currents can be neglected. By referring to Lemma 3, with integer N_1 adequate for approximating f_1 and g_1 , and integer $1+N_2$ adequate for approximating g_2 , the resultant linear system of $N_1+N_1+1+N_2$ number of equations is directly solved by the Gauss elimination that the solution is indeed of a very tiny relative residual error, this simultaneously allows of the solution does closely approach to the true solution of (41)-(46)-(66).

TABLE I
CONVERGENCE OF SOLVED TEM-MODE REFLECTION COEFFICIENTS
(A) Half-thickness (d) = 0.0000001 mm

N_1	$1+N_2$	Low Complex Permittivity ^a		High Complex Permittivity ^b	
		(41)-(46)-(66)	(76)	(41)-(46)-(66)	(76)
5	1	0.9999773 358.1383°	0.9999988 358.1800°	0.9761946 354.7598°	0.9999987 358.1800°
17	1	0.9999948 358.1711°		0.9950334 357.4728°	
34	1	0.9999982 358.1779°		0.9989685 358.0330°	
5	26	0.9999706 358.1996°	0.9999990 358.2615°	0.9852695 355.9788°	0.9999988 358.2615°
17	26	0.9999913 358.2458°		0.9993855 357.9210°	
34	26	0.9999966 358.2561°		0.9998101 358.1631°	

^a relative permittivity =2.2, conductivity=0.001 S/m;

^b relative permittivity =100.0, conductivity=1.1111 S/m;

(B) Half-thickness (d) = 0.25 mm

N_1	$1+N_2$	Low Complex Permittivity ^a		High Complex Permittivity ^b	
		(41)-(46)-(66)	(76)	(41)-(46)-(66)	(76)
5	1	0.9998547 357.8635°	0.9998645 357.8811°	0.8741509 338.4413°	0.8838344 340.1059°
17	1	0.9998633 357.8795°		0.8830164 339.9666°	
34	1	0.9998646 357.8809°		0.8838990 340.1166°	
5	21	0.9998525 357.9400°	0.9998659 357.9670°	0.8781618 339.1281°	0.8850300 340.3740°
17	21	0.9998655 357.9648°		0.8842257 340.2309°	
34	21	0.9998664 357.9667°		0.8850498 340.3765°	
5	26	0.9998527 357.9406°	0.9998659 357.9676°	0.8783216 339.1566°	0.8851423 340.3941°
17	26	0.9998657 357.9654°		0.8843548 340.2540°	
34	26	0.9998664 357.9673°		0.8851655 340.3972°	

^a relative permittivity =2.2, conductivity=0.001 S/m;

^b relative permittivity =100.0, conductivity=1.1111 S/m;

(C) Half-thickness (d) = 4.00 mm

N_1	$1+N_2$	Low Complex Permittivity ^a		High Complex Permittivity ^b	
		(41)-(46)-(66)	(76)	(41)-(46)-(66)	(76)
5	1	0.9990063 356.2496°	0.9990078 356.2508°	0.6923497 246.0029°	0.6923460 246.0040°
17	1	0.9990072 356.2496°		0.6923498 246.0029°	
34	1	0.9990072 356.2496°		0.6923499 246.0029°	
5	21	0.9991140 356.5115°	0.9991148 356.5125°	0.6804774 256.2546°	0.6804739 256.2586°
17	21	0.9991137 356.5115°		0.6804762 256.2583°	
34	21	0.9991140 356.5115°		0.6804763 256.2582°	
5	26	0.9991146 356.5127°	0.9991154 356.5137°	0.6803994 256.3650°	0.6803957 256.3690°
17	26	0.9991145 356.5128°		0.6803979 256.3687°	
34	26	0.9991146 356.5128°		0.6803979 256.3687°	

^a relative permittivity =2.2, conductivity=0.001 S/m;

^b relative permittivity =100.0, conductivity=1.1111 S/m;

As a comparative study, the PMCHWT integral equations (9) and (10) hold on the top planar face and the radial function integral equation (66) are selected to form coupled integral equations which are solved by an arbitrary geometry MoM solution. For example, the Rao-Wilton-Glisson (RWG) functions [20] on triangular meshes of the dielectric board top and bottom planar faces are employed to expand each of the equivalent electric and magnetic currents, and to test the respective equations (9) and (10). For the sake of integrity, the details of the solution can be summarized as follows:

- 1) An identical triangular mesh is applied to the top and bottom planar faces. Every common side shared by two adjacent triangles on the mesh is assigned with a RWG function $\vec{f}_n^{\text{RWG}}(x', y')$, the number of RWG functions N_{RWG} is equal to the total number of the triangle inner sides.
- 2) On both the top and bottom planar faces of dielectric board, the equivalent electric and magnetic currents are discretized by the same set of RWG functions as

$$\begin{aligned}
 \vec{J}_{s1}^+(x', y') &= \sum_{n=1}^{N_{\text{RWG}}} I_n \vec{f}_n^{\text{RWG}}(x', y') \\
 \vec{J}_{s1}^-(x', y') &= \sum_{n=1}^{N_{\text{RWG}}} (-I_n) \vec{f}_n^{\text{RWG}}(x', y') \\
 \vec{M}_{s1}^+(x', y') &= \sum_{n=1}^{N_{\text{RWG}}} V_n \vec{f}_n^{\text{RWG}}(x', y') \\
 \vec{M}_{s1}^-(x', y') &= \sum_{n=1}^{N_{\text{RWG}}} V_n \vec{f}_n^{\text{RWG}}(x', y')
 \end{aligned} \tag{82}$$

with coefficients I_n in dimension of A/m and V_n in dimension of V/m.

- 3) To the radial function integral equation at annular aperture (66), all the azimuthal magnetic fields are tested at the annular radius on positive x -axis, and the first term of the LHS, which is the illuminating azimuthal magnetic field at the aperture radiated by the equivalent currents on the negative side of the dielectric board planar faces, is generally replaced by the following term

$$\begin{aligned}
H_{1y}^I(x, y=0, z=0^+) & \quad a \leq x \leq b \\
&= \left[-K_1 \left\{ -\bar{J}_{S1}^+ \right\} + \frac{\varepsilon_0 \varepsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{S1}^+ \right\} \right] (x, y=0, z=0^+) \Big|_y \\
&+ \left[-K_1 \left\{ -\bar{J}_{S1}^- \right\} + \frac{\varepsilon_0 \varepsilon_{r1}}{\mu_0} T_1 \left\{ -\bar{M}_{S1}^- \right\} \right] (x, y=0, z=0^+) \Big|_y
\end{aligned} \quad (83)$$

then, equation (66) is rewritten as

$$\begin{aligned}
H_{1y}^I(x, y=0, z=0^+) &+ H_{2\varphi}^I(x, y=0, z=0^+) \\
&= H_{\text{reflec}, \varphi}(x, y=0, z=0^-) + H_{\text{inc}, \varphi}(x, y=0, z=0^-) \quad a \leq x \leq b
\end{aligned} \quad (84)$$

Finally, the definite integral on the $[a, b]$ section of the positive x -axis is precisely evaluated by quadrature rules as the following

$$\begin{aligned}
&\int_a^b x \Phi_{2,n}(x) H_{1y}^I(x, y=0, z=0^+) dx \\
&= \sum_{x_i \in [a,b]} w_i x_i \Phi_{2,n}(x_i) H_{1y}^I(x_i, y=0, z=0^+) \quad n=0,1,\dots,N_2
\end{aligned} \quad (85)$$

- 4) On the excitation terms in the PMCHWT equations (9) and (10), the tangent incident EM-fields at field point $(x, y, z=d^-)$ on the negative side of dielectric board top planar face follow that

$$\begin{aligned}
\bar{E}_{\text{inc},x}^I(x, y, d^-) &= -\frac{x}{2\rho} \int_0^{+\infty} k_\rho e^{-jk_{1z}d} \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho \\
\bar{E}_{\text{inc},y}^I(x, y, d^-) &= -\frac{y}{2\rho} \int_0^{+\infty} k_\rho e^{-jk_{1z}d} \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho \\
H_{\text{inc},x}^I(x, y, d^-) &= \frac{y}{2\rho} \cdot \omega \varepsilon_0 \varepsilon_{r1} \int_0^{+\infty} k_\rho \frac{e^{-jk_{1z}d}}{k_{1z}} \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho \\
H_{\text{inc},y}^I(x, y, d^-) &= -\frac{x}{2\rho} \cdot \omega \varepsilon_0 \varepsilon_{r1} \int_0^{+\infty} k_\rho \frac{e^{-jk_{1z}d}}{k_{1z}} \tilde{g}_2(k_\rho) J_1(k_\rho \rho) dk_\rho
\end{aligned} \quad (86)$$

- 5) The detailed formulations for all the entries of MoM matrix resulted from the PMCHWT integral equations can be found in [21]-[24]. All the double surface-integrals related to the interactions between the equivalent electric currents and magnetic currents are evaluated with adequate precision. The outer integral is evaluated through the Gaussian quadrature with moderate integration points on triangle domain [25], while for dealing with the inner integral, cases that a specified field point locates within the distance of $0.8/|k_1|$ (k_1 is the dielectric complex wavenumber of the board) from a source triangle are treated as near singularity, so that singularity extraction techniques in the classic paper [26], [27] and a remedy for considering the near-singularities when the specified field point locates at the positive extension of one side of the source triangle [28] are employed.

Now redo the numerical examples above in this Section by solving the coupled integral equations (9)-(10)-(84) using arbitrary geometry MoM. The solved square dielectric board whose lateral faces equivalent currents can be neglected is of the size $2R_1 \times 2R_1$ ($40.0 \times 40.0 \text{ mm}^2$), as depicted in Fig. 5.

Figure 5. The $2R_1 \times 2R_1$ square dielectric board with an annular circular symmetric magnetic loop current located at its center: (a) Front view; (b) Top view.

The dielectric board solved by this arbitrary geometry MoM is of relative permittivity 2.2 and conductivity 0.001S/m. This is a case of low complex permittivity, and by referring to the converged infinite series solution in this Section, the reflection higher modes $\text{TM}_{0n}(n=1,2,\dots)$ can be neglected and only the principal TEM mode reflection in coaxial waveguide is considered, as $1+N_2=1$, this reduces amount of computations for the excitation terms in the PMCHWT equations (9) and (10). In Table II, the arbitrary geometry MoM solution is compared with the convergent infinite series solution. The square top planar face as well as the bottom one is respectively meshed into adequate number of 648 congruent isosceles right triangles which form totally 936 free RWG functions. The resultant linear system of equations is simply diagonally preconditioned and then iteratively solved by a TFQMR solver, with zero initial guess and 1.0×10^{-3} relative residual error. To the cases of thick board, it is in expectation that the two solutions agree well, while with the board thickness gradually decreasing, increased error of the arbitrary geometry MoM solution is observed.

TABLE II
TEM MODE REFLECTION COEFFICIENTS WITH A DECREASING THICKNESS

$2d$ (mm)	Arbitrary Geometry MoM		Convergent MoM: $N_1=34, 1+N_2=1$	
	Iteration Number	(9)-(10)-(84)	(41)-(46)-(66)	(76)
20.0	54	0.9988579 356.0010°	0.9988736 356.0266°	0.9988759 356.0266°
8.00	84	0.9989187 356.0211°	0.9990072 356.2496°	0.9990078 356.2508°
2.00	202	0.9991654 356.1155°	0.9995390 357.2165°	0.9995389 357.2169°
1.50	276	0.9992118 355.0609°	0.9996347 357.4061°	0.9996356 357.4064°

V. CONCLUSION

Accurately solving the surface integral equations for finite-sized homogeneous dielectric board excited by an embedded circular symmetric magnetic loop current is discussed in this paper. The board lateral faces equivalent currents are negligible reduces the surface integral equations into coupled radial function integral equations which are simultaneous linear operator equations on countably infinite dimensional Hilbert spaces. The solution proved to be L^2 -norm convergent under condition that all linear operators are bounded. On the rate of convergence, the numerical results show that the selected complete set of orthogonal radial functions on a finite-length interval $[0, R_1]$ as circular symmetric cylindrical harmonics constructs rapidly convergent expansions of the unknown radial-varying electric and magnetic equivalent currents, especially for the case that the exciting annular circular symmetric magnetic loop current has a dominant component of the coaxial TEM principal mode, and therefore, a small-sized truncated linear system of equations can produce accurate solution.

APPENDIX

In this Appendix, we present three lemmas [29] related to bounded linear operator on Hilbert spaces.

Let L be a linear transform on the space of integrable radial functions. On the eigenfunction and eigenvalue of L , we have **Lemma 1** holds.

Lemma 1. Let $L(f)=h$ be a linear operator equation, the domain and range of L are both the space of integrable radial functions. The Hankel transform of order 1 for the solution f and the excitation h are denoted by $\tilde{f}$ and $\tilde{h}$, respectively.

Proposition-1:

A nonzero-complex-valued function $\tilde{A}(k_\rho)$ $k_\rho \geq 0$ exists to satisfy $\tilde{f}(k_\rho) = \tilde{A}(k_\rho)\tilde{h}(k_\rho)$ for any excitation $\tilde{h}$ and the corresponding solution $\tilde{f}$.

Proposition-2:

$J_1(k_\rho\rho)$ $k_\rho, \rho \in [0, +\infty)$ and $1/\tilde{A}(k_\rho)$ are eigenfunctions and eigenvalues of L , i.e., $L[J_1(k_\rho\rho)] = J_1(k_\rho\rho)/\tilde{A}(k_\rho)$.

Then, *Proposition-2* is a necessary and sufficient condition to *Proposition-1*. $\square$

Proof of Lemma 1. Substitute the order 1 Hankel integral expression of f and h into the left hand side (LHS) and the right hand side (RHS) of $L(f)=h$, respectively. We have

$$\text{LHS} = L\left[\int_0^{+\infty} k_\rho \tilde{f}(k_\rho) J_1(k_\rho\rho) dk_\rho\right] = \int_0^{+\infty} k_\rho \tilde{f}(k_\rho) L[J_1(k_\rho\rho)] dk_\rho \quad (87)$$

$$\text{RHS} = \int_0^{+\infty} k_\rho \tilde{h}(k_\rho) J_1(k_\rho\rho) dk_\rho \quad (88)$$

Substitute $\tilde{f}(k_\rho) = \tilde{A}(k_\rho)\tilde{h}(k_\rho)$ into the LHS, then

$$\text{LHS} = \int_0^{+\infty} k_\rho \tilde{A}(k_\rho)\tilde{h}(k_\rho) L[J_1(k_\rho\rho)] dk_\rho \quad (89)$$

Subtract the LHS with the RHS, we have

$$\int_0^{+\infty} k_\rho \tilde{h}(k_\rho) \{\tilde{A}(k_\rho) L[J_1(k_\rho\rho)] - J_1(k_\rho\rho)\} dk_\rho = 0 \quad (90)$$

The above definite integral always equals to zero for any excitation h , this obviously allows we have

$$\tilde{A}(k_\rho) L[J_1(k_\rho\rho)] = J_1(k_\rho\rho) \quad (91)$$

to be a necessary and sufficient condition to Proposition-1. $\square$

For the countably infinite dimensional Hilbert space of finite energy functions, on the convergence of the direct sum decomposition for any function in it, we have **Lemma 2** holds.

Lemma 2. Let H be a countably infinite dimensional Hilbert space of finite energy functions, which is spanned by a normalized complete set of orthogonal functions $\{\Phi_n \mid n=1, 2, \dots, N_1, \dots\}$.

Then, for any $f \in H$, $f^{(N_1)} = \sum_{n=1}^{N_1} \langle \Phi_n, f \rangle \Phi_n$ is L^2 -norm convergence

to f with $N_1 \rightarrow \infty$. $\square$

Proof of Lemma 2 can be found in *Functional Analysis* Tutorials. $\square$

For a simultaneous N number of linear operator equations with N unknowns which is defined on the metric spaces of countably infinite dimensions, does a numerical solution converge to its exact solution? **Lemma 3** provides a sufficient condition.

Lemma 3. For positive integer $N \geq 1$, let

$$\sum_{j=1}^N L_{ij}(f_j) = h_i \quad i=1, 2, \dots, N \quad (92)$$

be a simultaneous N number of linear operator equations with N unknowns, which is defined on $2N$ number of countably infinite dimensional Hilbert spaces of finite energy functions.

We provide that

Proposition-1:

To any given unique excitation $\{h_i^* \in \mathfrak{R}_i \mid i=1, 2, \dots, N\}$, a unique solution $\{f_j^* \in \mathfrak{S}_j \mid j=1, 2, \dots, N\}$ exists, where all the range $\mathfrak{R}_i$ and domain $\mathfrak{S}_j$ ($1 \leq i, j \leq N$) are countably infinite dimensional Hilbert spaces of finite energy functions.

Proposition-2:

All linear operators $L_{ij} : \mathfrak{S}_j \mapsto \mathfrak{R}_i$, $1 \leq i, j \leq N$ are bounded.

Proposition-3:

On forming the linear system of equations, a normalized complete set of orthogonal functions $\{\Phi_{j,n} \mid n=1, 2, \dots, N_1, \dots\}$ of each domain $\mathfrak{S}_j$ is selected as the corresponding expansion basis, and a normalized complete set of orthogonal functions $\{\Pi_{i,m} \mid m=1, 2, \dots, N_1, \dots\}$ of each range $\mathfrak{R}_i$ is selected for

performing inner product with the $No.i$ linear operator equation, respectively.

Then, the numerical solution is L^2 -norm convergence to the exact solution with $N_i \rightarrow \infty$. $\square$

Proof of Lemma 3 can be made by referring to the **Banach Inverse Operator Theorem** and **Lemma 2**. A Proof of **Lemma 3** is available from the author upon request. $\square$

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