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A new kind of locked circuit: the Quasi-Periodic Locked Loop (Q-PLL)

Diego Luis González, Lorenzo Grassi, and Alberto U. G. Maurizi

Abstract—A new nonlinear circuit with frequency locking capability in the case of a generic quasi-periodic input, is presented. Due to this capability the circuit is called a Quasi-Periodic Locked Loop (Q-PLL). The locked frequency is parametrically selected from among those prescribed by the theory of resonances in dynamical systems. In particular, the locked frequency forms a three-frequency resonance with the frequencies of the quasi-periodic input. The circuit is able to lock also in case of deterministic perturbation (additional frequency components) and stochastic perturbation (wide-band noise). The circuit is closely related to the pitch perception of complex sound in humans and, as such, can be considered a bio-inspired device. From the point of view of applications, it may be considered as an extension of the Phase Locked Loop (PLL) with the additional ability of locking simultaneously to more than one frequency. Due to the dynamical and structural robustness of the locked states, the Q-PLL represents a tangible advance for the development of specific applications, for example, in medicine (hearing aids, and cochlear implants), in robotics (artificial senses), and in industrial and consumer electronics (improvement of speech intelligibility, pitch-based processing, etc.).

Index Terms—Quasi-periodic signals, Frequency locking, Pitch perception

I. INTRODUCTION

PHASE-Locked Loop (PLL) circuits are widely used in electronics. PLL circuits can be used as frequency multipliers or dividers, tracking generators, or clock recovery circuits. Their main technological applications derive from a capacity to recover very weak periodic signals, which would otherwise be lost in background noise, and the detection and/or synthesis of frequency and phase-controlled signals. Notwithstanding their ubiquitous presence in modern electronics, PLLs have a basic limitation in their locking capability: they can lock only to one periodic signal at a time. PLLs fail to lock when the input is perturbed by additional deterministic frequencies of competing amplitudes. However, theoretically, it is possible to have a system locked simultaneously to two or more periodic signals which are usually mutually incommensurate.

In fact, the PLL in its locked regime should be considered as a synchronised oscillator [1]–[4]. Usually the working regime is with 1/1 synchronisation, in which forcing and response frequencies are the same. However, it is not unusual, depending on the kind of PLL and its operating regime, to have

locked states corresponding to other synchronised conditions such as, for example, 1/n, where the loop is locked to a sub-harmonic of the forcing signal. The locked responses are indeed resonances of a system with two frequencies (two-frequency resonances).

Dynamical systems theory provides a framework for systems with n -frequency resonances. It has been demonstrated [5] that locking in the case of 3-frequency resonances is possible, *i.e.*, the response synchronises with two independent forcing frequencies.

Moreover, it has also been shown that three-frequency resonances are related to a relevant aspect of neural dynamics, namely the pitch perception of complex sounds [6], [7]. This aspect is relevant for applications because it opens up the possibility of technological developments in various fields including medicine, artificial senses, and robotics.

Our main objective here is to develop, on the basis of a biologically oriented paradigm, a loop circuit able to lock simultaneously to two independent frequencies. Such a circuit would extend the capabilities of the well-known phase-locked loop (PLL) and overcome the structural stability limitation of previously studied dynamical systems (coupled nonlinear oscillators, coupled phase-locked loops and coupled circle maps [5], [8], [9]). This new non-linear circuit is termed a Quasi-Periodic Locked Loop (Q-PLL, [10], [11]) due to its main property, that is, its ability to stably lock on a quasi-periodic input given by two independent frequencies. The circuit is interesting for applications envisaging real time responses of the auditory system, and also shows promise for other electronic applications representing a significant step forward with respect to standard PLL circuits.

First, the properties of three-frequency resonances from a dynamical systems point of view will be recalled and their implications for pitch perception outlined. The structural stability problems of known implementations will be discussed. Then, the circuit architecture will be presented and the strategy of the implementation for obtaining robust three-frequency resonant responses will be explained. After this, the response of the Q-PLL circuit in parameter space will be studied and its deterministic (signal perturbations) and structural (parameter perturbations) stability properties analysed. Finally, several implementations and possible future developments and applications will be discussed.

II. THREE-FREQUENCY RESONANCES AND PITCH PERCEPTION

Synchronisation was discovered by Christiaan Huygens in 1665 [12]. He found that the pendulums of two clocks, fixed

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on the same mounting, after some time swung synchronously. Moreover, Huygens found that the phases of clock oscillations showed an asymptotic tendency towards a certain fixed value of the phase difference. Because of this tendency, synchronisation is also known as phase locking.

Synchronisation is a well-known universal phenomenon occurring widely in nature, from heartbeats to the movement of celestial bodies [13], [14]. The most common case, well described by the Huygens clocks, is when the forcing and response frequencies are equal. This is called 1/1 synchronisation and corresponds, for example, to the case of orbital and rotational synchronised frequencies characterising the movement of our moon and responsible of the fact that we see always the same face.

Synchronisation can appear in dynamical systems with more than one external periodic forcing. Consider the general case of a nonlinear system forced by n independent external forces. For $n = 1$, a periodic response corresponds to a synchronised, or locked, state. As said above, this resonance can differ from simple 1/1 synchronisation. In fact, when the quotient between the proper frequency f_0 of the unperturbed system and the frequency f_1 of the external forcing, approach a rational number, *i.e.*, $f_0/f_1 \simeq p/q$ with $p, q \in \mathbb{N}$, a solution characterised by a sub-harmonic of the forcing frequency, *i.e.*, f_1/q can arise. When the response locks exactly to the q sub-harmonic, then $f_0/f_1 = p/q$ and thus $pf_0 - qf_1 = 0$. For $p = q = 1$, it reduces to 1/1 synchronisation which implies $f_0 = f_1$. This last relationship is a special case of the more general case represented by

$$\sum_{k=0}^{N-1} m_k f_k = 0 \quad (1)$$

where m_k are integers. A system which satisfies Equation (1) non-trivially, *i.e.*, $m_k \neq 0$ for all values of k , is said to be N -resonant (the frequencies form an N -resonance).

Analog phase-locked loop circuits represent an electronic realisation of a 2-resonant system showing responses locked to the forcing frequency, *i.e.*, the case with $N = 2$ and $|m_k| = 1$ in Equation 1.

The next case in increasing dimensionality ($N = 3$) corresponds to quasi-periodic forcing, *i.e.*, a system forced with two independent external frequencies f_1 and f_2 . Quasi-periodically forced systems exhibit a rich variety of dynamical behaviour [15]. These range from periodic behaviour, to 2- and 3-frequency quasi-periodic responses, to strange non-chaotic, and chaotic attractors. The Ruelle-Takens-Newhouse theorem establishes that quasi-periodic attractors with 3 incommensurate frequencies are on the border of chaos. These quasi-periodic attractors imply that Equation (1) with $N = 3$ has only the trivial solution ($m_k = 0$ for all k values). The theorem implies that in the immediate neighbourhood of these responses, chaotic attractors must be present. As a consequence, quasi-periodic attractors with three-frequencies would be unstable in parameter space.

However, when Equation (1) has non-trivial solutions for $N = 3$, the three frequencies satisfy a resonant condition, *i.e.*, are not independent and, therefore, the Ruelle-Takens-

Fig. 1. Schematic representation of the relationship between tones and pitch for different stimuli. Musical sounds are complex tones consisting of a lowest frequency, called *fundamental* (f_0 here) and integer multiples of it, called *overtones* or *harmonics*. Panel a) reports the complex tone time series and power spectrum of the first 10 harmonics of a violin whose pitch coincides with the fundamental. In panel b) the fundamental and the first few higher components of the spectrum are removed. The pitch does not change with respect to a): it remains that of the (missing) fundamental. Panel c) displays the effect of uniformly shifting the harmonics of panel b) by a quantity Δf . Although the difference combination tones remain unchanged, the pitch shifts by a quantity ΔP proportional to Δf . Panel d) reports the relationship between the frequency shift and the perceived pitch according to [6] superimposed on data from the pitch shift experiment in [16].

Newhouse theorem does not apply. Thus, the system is expected to have resonant responses with stability regions of non-zero size in parameter space. This has already been shown to hold in several nonlinear quasi-periodically forced systems [8].

Although in the study of nonlinear dynamical systems the characterisation of deterministic chaos has attracted a great deal of attention, more regular dynamics are found to be important for modelling many real processes. In particular, the study of 3-frequency resonances was found to be particularly important for modelling the auditory perception of complex sounds [6], [7].

The pitch of a complex sound is a psychoacoustic quantity, *i.e.*, a subjective placement of the sound along an inner template that goes from bass to high tones, as in a musical scale. If a correspondence between pitch and the frequency

spectrum of the stimulus is sought, for the majority of musical sounds, *i.e.*, those composed of a fundamental and a series of harmonics, the task is relatively easy: the pitch can be well described by the fundamental component of the sound (see Figure 1a). However in some cases, the pitch is not trivially connected to the lowest frequency of the stimulus. In fact, even when a few of the lowest components of a musical sound are removed, the perceived pitch still corresponds to that of the (removed) fundamental (see Figure 1b). This phenomenon is called *missing fundamental* or *residue*. In order to explain this behaviour, a static nonlinear theory was formulated by Helmholtz [17]: the difference tone (one of the possible combination tones), produced within the ear, plays the role of the missing fundamental: the difference between two adjacent partials equals the fundamental component. As an example, in Figure 1b, $f_5 - f_4 \equiv 5f_0 - 4f_0 = f_0$. Nevertheless, *pitch shift* experiments, consisting of the observation that the pitch shifts when all the frequency components are shifted by the same amount (see Figure 1c where it is shown that the pitch is also shifted by the amount ΔP) [16], demonstrated the fallacy of this view (Figure 1c). This fact was interpreted historically as a failure of the nonlinear approach [18]–[20]. However, more recently, a nonlinear dynamical theory [6], [7] (as opposed to the static view of Helmholtz) was successful in explaining the key feature of pitch shift (Figure 1d). This new theory is based on the concept of dynamical resonances. The main results of the theory and its application to pitch perception modelling are briefly outlined here below (see [6], [7] for a more detailed description).

When a generic dynamical system characterised by an intrinsic frequency f_0 is forced by two external independent frequencies f_1 and f_2 , a web of three-frequency resonances is generated in its parameter space. The general procedure for finding the possible resonances starts with the continued fraction development of the quotient of the external frequencies, $r = f_1/f_2$ (see, *e.g.*, [21]):

$$r = [a_0; a_1, \dots, a_n, \dots]. \quad (2)$$

The development is finite if the quotient is rational and infinite otherwise. Successive truncations up to a given finite order of the development give the so-called approximants of the ratio,

$$r_n = \frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]. \quad (3)$$

For any approximant a main three-frequency resonance exists which is given by:

$$f_{0_n} = \frac{f_1 + f_2}{p_n + q_n} \quad (4)$$

The hierarchy of the frequencies f_{0_n} characterising the web of three-frequency resonances is described by a generalised Farey sum operation. The usual Farey sum operation on rational numbers, $(p/q) \oplus (r/s) = (p+r)/(q+s)$, describes the hierarchy of the web of synchronised responses in periodically forced oscillators (two-frequency resonances). Such hierarchical organisation is known as the *Devil's staircase*.

Analogously, for three-frequency resonances, there is a hierarchical organisation given by the generalised Farey sum

Fig. 2. Sketch of the first few levels of the generalised Farey sum hierarchy.

operation [8] defined as follows: for a given convergent $r_n = p_n/q_n \equiv (p_n, q_n)$, define an interval of frequencies $[f_1/p_n; f_2/q_n]$. Then, the first step of the generalised Farey sum is defined as

$$\frac{f_1}{p_n} \oplus \frac{f_2}{q_n} = \frac{f_1 + f_2}{p_n + q_n}. \quad (5)$$

The operation can be iterated between adjacent, taking into account that two frequencies f_x and f_y are considered as adjacent if $s f_x - r f_y = q f_1 - p f_2$. At variance with the usual Farey sum, the generalised Farey sum acts on quotients between (mixed) real and integer numbers. In this way, a hierarchical organisation of resonances (similar to that corresponding to periodically forced oscillators) is formed in generic quasi-periodically forced systems [22]. The first steps of this hierarchy, based on the generalised Farey sum, are sketched in Figure 2. The fundamental fact here, is that the pitch of complex sounds, including the phenomenon of missing fundamental, can be described in terms of three-frequency resonances. In particular, the pitch elicited by a quasi-periodic stimulus corresponds to a main three-frequency resonance. In the simplest case of a two frequencies stimulus, the pitch is represented by a three-frequency resonance obtained as the generalised Farey sum between the sub-harmonics f_1/p and f_2/q where p/q is an approximant of the stimulus frequency ratio. Such a description is in excellent agreement with experimental psycho-acoustical data explaining, with high precision, the phenomenon of pitch shift (see Figure 1c and d).

Having a robust dynamical model for the prediction of the pitch of complex sounds may be very useful in some applications. However, from the practical point of view, a major drawback appears: the stability intervals corresponding to the main three-frequency resonances appear to be very narrow in all the dynamical systems studied so far: 1) nonlinear coupled oscillators with piecewise analytical solutions, 2) coupled PLLs [9], 3) quasi-periodically forced circle-maps [5], [8]. This is due to the fact that the web of resonances existing in the narrow interval determined by f_1/p and f_2/q is very complex (in principle, infinite resonances, which correspond to the successive levels of the generalised Farey sum operation, can coexist). This is similar to the case of the infinite steps of the Devil's staircase in periodically forced systems.

In order to tackle this stability problem, a system which can stably produce these main three-frequency resonances over

Fig. 3. Implementation of Q-PLL in Simulink®. The input signal $F(t)$ enters the S/H module whose output $F'(t)$ is multiplied by $F(t)$ itself and then integrated over time (low-pass filtered) to give $x'(t)$ which, in turn, determines the VCO frequency. The output of the system $x(t)$ is the locked signal which stems from the VCO. The sign function between the VCO output and the S-H trigger input, is used to transform a sine wave to a square wave in order to trigger the S/H.

wide intervals of parameter space is required. This implies that some kind of dynamic control must be implemented in order to enhance the existence intervals of the main resonances. The following Section describes an electronic circuit, based on a sampling locked loop control, displaying very large stability intervals for the main three-frequency resonances, and which can lock not only on periodic but also on quasi-periodic inputs.

III. CIRCUIT DESCRIPTION

The Q-PLL circuit robustly implements three-frequency resonator dynamics. The main idea underlying the circuit design is to use a sub-Nyquist sampling taking into account that the desired resonances are below the Nyquist frequency of the forcing. Sub-sampling strategies has been used previously in the development of locked electronic circuits, but mainly with the aim of improving phase noise performance and/or reducing spur in Phase Locked Loops (see, *e.g.*, [23]–[25]). Within this framework, a feedback system is developed which locks stably to one of the possible three-frequency resonances.

A. Functional blocks model

In Figure 3 a possible realisation of the circuit in terms of a Simulink® model is shown. It also represents the general functional scheme of the system. The quasi-periodic input signal $F(t)$ is sent to a triggered sampler (Sample and Hold, S/H). The sampling frequency of the S/H is determined by the output frequency of a Voltage Controlled Oscillator (VCO). The sampled signal $F'(t)$ is then multiplied ($\times$) by the non-sampled one and, subsequently sent to a time integrator ($1/s$). Then, the integrated signal $x'(t)$ is sent to the VCO to determine the VCO frequency. The output of the VCO, once transformed into a square wave, is sent to the trigger port of the S/H. The response of the system is the output signal of the VCO $x(t)$.

Formally, the output of the VCO can be expressed by

$$x(t) = A \cos(2\pi f_Q t + \varphi(t)) \quad (6)$$

where the (time varying) phase $\varphi(t)$ depends on the history of the VCO input $x'(t)$:

$$\varphi(t) = \varphi_0 + 2\pi G \int_0^t d\tau x'(\tau). \quad (7)$$

Parameters of the VCO are: A , the amplitude; f_Q , the quiescent frequency (free oscillation for zero input or, mathematically, the initial frequency); φ_0 , the initial phase; G , the sensitivity, *i.e.*, the rate of change of VCO frequency with the input amplitude.

The input of the VCO is given by

$$x'(t) = \int_0^t d\tau F(\tau) F'(\tau) \quad (8)$$

where $F(t)$ is the forcing and it is assumed that $F(t) = 0$ for $t \leq 0$. The Sample-and-Hold function F' is defined as

$$F'(t) = F(t_i), \text{ for } t_i \leq t < t_{i+1} \quad (9)$$

where the discrete time series t_i is given by the solution of

$$\begin{cases} x(t_i) = 0 \\ \dot{x}(t_i) > 0 \end{cases} \quad (10)$$

or, in words, the input signal is sampled at any cycle of the VCO when its output crosses zero while rising (after the squaring sign function).

By substituting Equation (8) into Equation (7), we have

$$\varphi(t) = \varphi_0 + 2\pi G \int_0^t d\tau \int_0^\tau d\sigma F(\sigma) F'(\sigma). \quad (11)$$

Thus, the instantaneous frequency of the VCO is given by

$$f(t) = 2\pi f_Q + \dot{\varphi}(t) = 2\pi f_Q + 2\pi G \int_0^t d\tau F(\tau) F'(\tau). \quad (12)$$

The sampled signal $F'(t)$ makes this equation, analytically speaking, rather intractable. Features of the model will be studied mainly by means of numerical simulations.

Figure 4 reports, as an example, the power spectrum of the Q-PLL output (continuous line) along with that of the input signal (dashed line) for an input $f_1 = 200$ Hz and $f_2 = \phi f_1$ where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. This irrational number ensures that the frequency ratio is as far as possible from a rational ratio (quasi-periodic forcing). As described in the caption, this is an example of generation of a main three-frequency resonance with quasi-periodic forcing. In this case, the irrational forcing ratio is approximated by the rational convergent $2/3$.

B. Locking mechanism analysis

In the general case, the sampling frequency $f_s \equiv f_{VCO}$, including its initial value f_Q , is below the Nyquist frequency of the input. It is well known from sampling theory (see, *e.g.*, [26]), that sampling a sinusoidal signal of frequency f_1 with a sampling frequency $f_s < 2f_1$ produces a spectrum with aliasing components. The sequence f_{a_k} of aliases (on the positive frequency axis) is based on the quantity $\Delta_1 \equiv |f_1 - \hat{k}_1 f_s|$ ($\hat{k}_1$ being the nearest integer to f_1/f_s):

$$f_{a_k} = \{\Delta_1, \mp\Delta_1 + k f_s; k \in \mathbb{N}\}. \quad (13)$$

Fig. 4. Example of locking for quasi-periodic input: $f_1 = 200$ Hz, $f_1/f_2 = \phi$, the golden ratio, and for VCO parameters: $f_Q = 100$ Hz and $G = 2000$ V/Hz (the result is independent of the initial phase φ_0). The only surviving frequency in the output signal is the “residue” frequency corresponding to $f_0 \simeq 104.72$ Hz, according to three-frequency resonance theory as outlined in Section II.

Fig. 5. Sub-Nyquist sampling of an input signal composed of the sum of two sinusoids of frequencies $f_1 = 200$ Hz and $f_2 = \phi f_1 \simeq 323.61$ Hz and resulting alias components. Upper panel: $f_s = 103$ Hz coinciding with the main resonance. $\Delta_1 \neq \Delta_2$; both aliases sequences, based on Δ_1 and Δ_2 , respectively, have a component that coincides with one of the input frequencies (f_1 or f_2). Lower panel: $f_s = (f_1 + f_2)/(2+3)$. $\Delta_1 = \Delta_2$; the two aliases sequences coincide; the resulting spectra contain both the input frequencies. The fact that in the sampled signal there are components with the same frequencies than the input signal is responsible for producing a DC value in the product output.

As a direct consequence, for $k = \hat{k}_1$, it results that either $\Delta_1 + \hat{k}_1 f_s = f_1$ or $-\Delta_1 + \hat{k}_1 f_s = f_1$, in other words, the sampled signal contains the input frequency.

If the signal to be sampled (input signal) is a superposition of two sinusoids of frequencies f_1 and f_2 satisfying the condition $f_s < 2 \min\{f_1, f_2\}$, the power spectrum of the sampled signal, for linearity, is the superposition of two power spectra containing aliasing components based on Δ_1 and $\Delta_2 = |f_2 - \hat{k}_2 f_s|$. These results are qualitatively summarised in Figure 5.

Thus, according to the sub-Nyquist sampling properties just

Fig. 6. Scheme of the sampling sub-system implemented in Q-PLL. The input signal is the sum of two sinusoidal components of frequencies f_1 and f_2 . Sampling is triggered in S/H when a sine wave of independent frequency f_s crosses the zero from negative values. The signals to be analysed for the Q-PLL locking mechanism are the sampling output $F'(t)$ and the product of the input and the sampled signal F_x .

recalled, the sampled signal contains components at frequencies f_1 and f_2 coinciding with the input components. Taking the product ($\times$) of the two signals (input and sampled) produces a signal F_x with a non-null zero-frequency component $\mathcal{F}(0)$ of the power spectrum. This Direct Current (DC) term is the sum of the contribution resulting from the zero-frequency components, $\mathcal{F}_1(0)$ and $\mathcal{F}_2(0)$, of the two input signals.

Furthermore, along with the DC component, the two lowest alias frequencies Δ_1 and Δ_2 (see Figure 5) constitute the low frequency part of the spectrum which represent the major oscillatory components after the low-pass filtering produced by the integrator. The analysis of these quantities as a function of the sampling frequency allows a deeper insight into the locking mechanism as shown below.

Consider the open sub-system, *i.e.*, without feedback, shown in Figure 6, where the triggering signal (input to the S/H module) is replaced by a controlled frequency f_s . The Fourier transform of the product signal F_x is computed and the zero-frequency component $\mathcal{F}(0)$ (or DC component) is considered. In principle, this quantity is a function of two variables: the sampling frequency and its initial phase. Its calculation is performed over a range of f_s covering the convergence frequency intervals, with a step of 0.1 Hz, and for 32 values of the initial phase φ_s covering the full range $[0 : 2\pi]$. Results are reported in Figures 7–9.

To gain an idea of how the locking mechanism acts, it is important to observe that a necessary condition for the output of the Q-PLL circuit in Figure 3 to lock to a fixed value, is that the input to the VCO be constant. Therefore, the input signal to the integrator must vanish. A necessary condition for this is that the DC component of F_x vanishes. A remarkable fact is that the zeros of the sum of two sine functions of two incommensurate angular frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$, respectively, are periodic according to the

Fig. 7. Projection of the DC component (zero frequency of the Fourier transform $\mathcal{F}(0)$ in arbitrary units) along the φ_s -axis (continuous black line) in the range of the convergence intervals for the first three approximants. It can be observed that the value of $\mathcal{F}(0)$ alternates its sign moving between adjacent convergents. The shaded regions represent the area bounded by f_1/p and f_2/q . It is worth noting that the ordering along the f_s axis of f_1/p and f_2/q alternates between adjacent convergents. Frequency of the main three-frequency resonance $f_0 = \frac{f_1 + f_2}{p + q}$ is reported for each convergence interval.

Fig. 8. Projection along the φ_s -axis of the DC component ($\mathcal{F}(0)$ in arbitrary units) of the product of input and sampled (F_x) signals over a range of frequencies around f_0 for convergent $(p_k, q_k) = (2, 3)$. Input frequencies are $f_1 = 200\text{Hz}$ and $f_2 = \phi f_1 \simeq 323.61\text{Hz}$. Shaded area shows the range between f_1/p and f_2/q . Vertical line at $f_s = f_0$ is the projection of the variation of $\mathcal{F}$ due to the change in phase φ_s which was observed only at that frequency. The dependence on this second variable is shown in the figure insert which displays, with the same units, the projection along the f_s -axis of $\mathcal{F}$ for $f_s = f_R$. It can be observed that for the rest of the plot, changes in $\mathcal{F}$ were observed to depend only upon f_s . Labels indicate the sign and the relationship between $\mathcal{F}_1$ and $\mathcal{F}_2$ for different intervals of f_s .

Fig. 9. Projection of the DC component (in arbitrary units) along the f_s -axis for $f_s = f_R$ and for convergent $(p, q) = (3, 5)$ as a function of the initial phase of the sampling signal. Input frequencies are $f_1 = 200\text{Hz}$ and $f_2 = \phi f_1 \simeq 323.61\text{Hz}$. It is interesting to observe that the response is cyclic and the number of cycles of the DC component in one cycle of the sampling initial phase, is exactly $p + q$ (compare to the insert in Figure 8).

prostaphaeresis formula

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2 \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right) \quad (14)$$

notably with periodicity based on the frequency sum $(\omega_1 + \omega_2)/2$ (along with another based on difference). This means that zeros are also periodic with period $(p_n + q_n)/(f_1 + f_2)$ which is the result of the generalised Farey's sum (Equation 5).

Figure 7 displays the projection of $\text{DC}(f_s, \varphi_s)$ along the φ_s axis. The function is displayed, over a wide range of f_s that cover the first three approximants of the frequency input ratio. The shaded areas enclose the intervals defined by $[f_1/p_k, f_2/q_k]$, for $(p_k, q_k) = (1, 2), (2, 3), (3, 5)$, with p_k/q_k being the k -th convergent of f_1/f_2 , and $k = 1, 2, 3$. The vertical lines represent the variations in $\mathcal{F}(0)$, caused by sampling signal phase change, which the system exhibits at exactly $f_s = f_{0k}$. To closely analyse this behaviour, Figure 8 displays a zoom of Figure 7 around the convergence region of the second approximant $(p_k, q_k) = (2, 3)$. Consider the values of the two components of $\mathcal{F}(0)$, namely $\mathcal{F}_1$ and $\mathcal{F}_2$: they are in some relationship with f_s . Outside the shaded region in Figure 8, for $f_s < f_1/p$ they are both positive while they are both negative for $f_s > f_2/q$. Inside the shaded region, their sign is opposite and, for a given $f_s = \hat{f}$, they equal each other in magnitude making $\mathcal{F}(0)$ vanish. However, Figure 8 shows that $\hat{f}$, although very close, does not correspond to the main resonance and, therefore, the locking mechanism must rely on some additional features of the system.

In fact, $\mathcal{F}(0)$ is, in general, a function of both the frequency f_s and the phase φ_s of the sampling signal, and Figure 8 should be viewed as a projection along the φ_s axis. However, it can be observed that over the range represented in Figure 8, except for $f_s = f_0$, $\mathcal{F}(0)$ does not depend on φ_s . Nevertheless, a strong dependence appears for $f_s = f_0$ as shown by the thick vertical line in Figure 8. It is known from Section II that in this

case $\Delta_1 = \Delta_2$ and the two alias frequency sequences coincide. The plot inserted in Figure 8 represents the section of $\mathcal{F}(0)$ along the φ_s axis at fixed $f_s = f_0$ and displays explicitly this dependence. It is important to observe that the variations of $\mathcal{F}(0)$ due to φ_s are such that $\mathcal{F}(0)$ crosses the zero value, thus providing a mechanism of convergence towards the main resonance frequency f_R .

It is remarkable that this very special phase dependence happens exactly at $f_s = f_0$, which underlines the strong connection between three-frequency resonances and the properties of the signal F_x and, in turn, with the sub-Nyquist-sampled signal F' . Another remarkable feature is that the periodicity of $\mathcal{F}(0)$ as a function of φ_s is exactly $p + q$. The insert in Figure 8 corresponds to the convergent $(p_k, q_k) = (2, 3)$ and the function presents 5 cycles. As another example of this feature, Figure 9 shows the case corresponding to the convergent $(p_k, q_k) = (3, 5)$ for which the periodicity of the phase response results to be 8.

Consider now the closed circuit (Figure 3). As shown before, the DC component (the sum of the contributions from the two input frequencies) is positive for $f_s < f_1/p$ and negative for $f_s > f_2/q$ (white regions in Figure 8). This ensure that, if the system is outside the shaded region, it moves into it. Once inside the shaded region, if $f_Q < \min(\hat{f}, f_0)$ (see Figure 8), the resulting DC is positive and forces the system to increase the sampling frequency f_s (in the opposite case, if $f_Q > \max(\hat{f}, f_0)$ f_s decreases). When f_s crosses f_0 , a phase-locking mechanism is activated and the system adjusts the phase of the sampling signal to find one of the zeroes of the DC component (Figures 8 and 9). This corresponds to a relative phase which cancels the two low frequency components ($\Delta_1 = \Delta_2$) of the aliases, ensuring a zero input to the integrator. When far from the resonance, the two low frequency components ($\Delta_1 \neq \Delta_2$) provide a perturbation which leads the system to cross $f_s = f_0$ and then to switch to the phase-locking mechanism described above. The parameter acting as an amplifier of the perturbation is the VCO sensitivity G which, if too small does not allow the system to move far enough from the DC = 0 solution and cross the resonance value f_0 . On the other hand, if G is too large the circuit should become unstable.

C. Analog implementation

The Q-PLL system, above described in terms of functional blocks (see Figure 3), was also implemented as an electronic analog circuit for demonstrative purposes and in order to test the working concept in a real electronic circuit. The schematic of the circuit is shown in Figure 10. The input signal is sent to the input of a sample-and-hold (S&H) circuit (LF398) and to one of the inputs of a four-quadrants analog multiplier (AD633). The other input of the multiplier is fed by the output of the S&H (sampled signal). The negative edges of the square wave output of a Voltage Controlled Oscillator (internal VCO section of a PLL CD4046 circuit) determine the sample-times of the S&H. The square wave output of the VCO is conditioned for producing appropriate impulses to drive the sampling of the input signal; this is obtained by a mono-stable

Fig. 10. Analog electronics implementation of the system described in Figure 3.

configuration of a 555 circuit fed by the passive derivative (R-C) of the VCO output. This step is necessary to produce pulses short enough for sampling purposes. Finally, the output of the multiplier is sent to an analog inverting integrator configured by means of an operational amplifier (LF356). The output of the integrator is further inverted with an inverting amplifier (LF356) in order to obtain a non-inverted integrated signal and for conditioning the amplitude of such integrated signal. The output of the inverter amplifier is sent to the input of the VCO making the circuit a closed loop (compare with Figure 3). The time constant of the integrator is defined by the input resistance (33k) and the feedback capacitance ($1.5\mu\text{F}$) that gives $\tau \simeq 50\text{ms}$. Moreover, a large feedback resistor is used ($1.8\text{M}\Omega$) in parallel to the capacitor to avoid rapid discharge and so keeping the integrated value for a sufficiently long time. The analog inverting amplifier is fed by the output of the integrator through a variable resistor to implement, by means of signal amplification, a virtual variation of the VCO sensitivity. Both op amps were finely balanced to remove spurious offset voltages. The circuit require a positive and negative regulation and integrated circuits are uncoupled from the power supply through $0.1\mu\text{F}$ capacitors placed near the corresponding pins of the chips.

As a working example, Figure 11 reports the power spectrum of locking under the same conditions of Figure 4. Locking frequency average $\bar{f}_R$ measured by a frequency counter (with gate time equal to 0.1s), is extremely stable with oscillations smaller than 0.003% of $\bar{f}_R$.

IV. NUMERICAL AND ANALOG EXPERIMENTS: SETUP AND RESULTS

The system features were studied by means of numerical experiments. The implementation in terms of Simulink® blocks shown in Figure 3 is used. Furthermore, the analog circuit implementation (see Figure 10) is used to verify the main system features in a real electronic circuit.

As already shown in the example reported in Figure 4, the system is able to lock to a quasi-periodic input composed of two incommensurate sinusoidal components whose ratio, notably, is the “most irrational” number, *i.e.*, the golden ratio ϕ . The ability to lock to different convergents of the f_1/f_2 ratio over the entire range of the parameter f_Q , is investigated below.

Fig. 11. Example of locking of the analog circuit for the same input as in Figure 12, and for $f_Q = 92.85$ Hz. As for the mentioned figure, the result is independent of the initial phase φ_0 . Locking state corresponding to three-frequency resonances ($f_R = 104.721$ Hz) is extremely stable on the average (standard deviation of about 0.003 Hz when measured with a frequency counter with gate set to 0.1 s).

Before proceeding further, some considerations on the dependence of the system on G are in order. G provides the rate of change of the VCO frequency as a function of the integrated signal. Roughly speaking, it must depend on the expected frequency response which, in turn, depends on f_Q . This can be observed through examining the intervals of the DC component around the locking frequencies in Figure 7. As a rule of thumb, the smaller the interval extent, the smaller G must be in order to avoid jumps outside the given convergence basin and to maximise the existence regions of the main resonances. On this basis, appropriate values for $G \equiv \tilde{G}_{(p_k, q_k)}$ are selected for each of the k -th approximant (p_k, q_k) .

Thus, by means of a trial-and-error procedure several values of $\tilde{G}_{(p_k, q_k)}$ were selected: $G_{(1,2)} = 8000$ Hz/V, $G_{(2,3)} = 2000$ Hz/V, $G_{(3,5)} = 1000$ Hz/V, $G_{(5,8)} = 500$ Hz/V, $G_{(8,13)} = 250$ Hz/V. Small variations of G around the optimal values defined above have no significant effect and, therefore, this parameter is not investigated further.

A. Q-PLL features for quasi-periodic input

1) *Numerical experiments:* Figure 12 reports the response frequency f_R of the Q-PLL as a function of f_Q . It can be observed that f_R locks exactly to f_{0k} , the main three-frequency resonance corresponding to the k -th convergent. The extent of the intervals over which the solutions are stable are very large, much larger than the $[f_1/p_k, f_2/q_k]$ intervals reported as continuous thin lines in Figure 12 (see also the shaded regions in Figure 7). It is worth noting that all other possible solutions corresponding to generalised mediants in the Devil's staircase-like portrait (see Figure 2), expected to exist between f_1/p_k and f_2/q_k , are suppressed. This is at variance with what has been found for some systems previously investigated: coupled phase-locked loops, coupled nonlinear oscillators with piece-wise analytic solution and a quasi-periodically forced circle map [5], [8], [9]. Within the convergence intervals the

Fig. 12. Response diagram as a function of f_Q for optimal values of $\tilde{G}_{(p,q)}$. Input frequencies are: $f_1 = 200$ Hz and $f_2 = f_1 \phi \simeq 323.61$ Hz, being ϕ the golden ratio. The frequency value corresponding to the first generalised mediant $(f_1 + f_2)/(p + q)$ (dotted lines) of the bounding values f_1/p and f_2/q for the analogous of the Devil's staircase (continuous thin lines) are shown (the latter not shown from $(p, q) \geq (5, 8)$). Solutions (thick lines) converge to the first mediant. The convergence intervals width decreases with $p + q$. Regions of f_Q for which f_R is not given, are regions of instability of the solution. Symbols represent the measurements from the analog circuit. Open circles are stable responses while open triangles are responses with different input phases for the same f_Q . They confirm the existence of regions in which the solution depends on the initial conditions, while for the stable regions, different input phases do not give rise to multi valued solutions. The errors associated with measurements, mainly caused by different sources of noise (see text), are small enough to be undetectable in this scale.

attractors are stable and, therefore, evolve to asymptotic orbits independently of the initial conditions, mainly the initial phase of the VCO.

The highest reported convergent is $(p_6, q_6) = (8, 13)$. Convergence intervals corresponding to higher convergents are too short to be represented fairly on the same scale. At the frontier of the reported stable solutions there are small regions in which the system is unstable where the responses may depend on the initial conditions. In these regions, however, other stable solutions may also occasionally appear (not reported here) which correspond to generalised mediants of the neighbouring (parent) solutions (see Figure 2).

Experiments to study the stability of the main resonances facing various perturbations of the input signals were also performed. First, the signal of Figure 4 was perturbed by an additive white Gaussian noise with an intensity of about -20 dBm. For the present system implementation and reported parameters, the system response remains stable for noise figures below or equal to this level. This can be appreciated as a peak corresponding to the expected resonance at 104.72 Hz in Figure 13. A thorough investigation about the stability to noise perturbation is beyond the scope of this paper but would undoubtedly be interesting for the design of electronics applications.

2) *Analog circuit experiments:* In the analog circuit (see schematic in Figure 10) the power supply (VDD and VSS) was set to ± 11 V.

The quiescent frequency of the VCO is varied through variable resistances on pins 11 and 12 of the CD4046. The

Fig. 13. Example of locking in response to a signal perturbed by noise. Input is as in Figure 4 with additive white Gaussian noise of -20dBm .

Fig. 14. Example of locking for one frequency input only (PLL-like behaviour) for $f_1 = 200\text{ Hz}$ and $f_Q = 90\text{ Hz}$. Output is the input frequency f_1 divided by $n = 2$. Q-PLL also produces harmonics of the response frequency of much lower amplitude.

ratio $R_2/R_1 \simeq 1$ was kept approximatively constant, obtaining a ratio $f_{\max}/f_{\min} \simeq 2$. In principle, $(f_{\max} - f_{\min})/(V_{DD} - V_{SS})$ is the VCO sensitivity. However, the gain of the second op amp will change it. In this experiment the gain is kept to a fixed value ($\simeq 25$) while the “nominal” sensitivity is varied keeping $f_{\max}/f_{\min}$ constant.

The input signal of the system is produced by a two-channel waveform generator (RIGOL D1022). Frequencies were selected to approximate ϕ with the maximum resolution of the function generator. This means that f_1 was set to 200.00000Hz and f_2 was set to 323.60680Hz . Signal amplitudes were set to 6V .

To test the dependence on initial conditions, some measurements were taken for a number of values of the phase φ_2 of the second component of the input while keeping f_Q constant.

The output frequency is measured using the frequency counter module of a Metex MS-9150. Data acquisition and processing were performed using an Analog Discovery 2 device. Experimental results are shown in Figure 12 along with the results of the numerical simulations.

B. PLL-like behaviour

If the Q-PLL is forced with a signal containing a frequency $f_1 = 200\text{ Hz}$ only, selecting f_Q from a suitable range around $f_Q = 200\text{ Hz}$ the system locks at $f_R = 200\text{ Hz}$ as would a normal PLL.

If now the quiescent frequency is set, for example, to $f_Q = 90\text{ Hz}$, the system locks to $f_R = 100\text{ Hz}$ (Figure 14). This frequency is the first sub-harmonic of the forcing frequency, $f_R = f_1/2$. In general, the system is able to lock to any sub-harmonic $f_R = f_1/n$ depending on the value of f_Q . Figure 15 shows the response diagram as a function of f_Q . Solutions are very stable and do not depend on the VCO sensitivity G (in Figure 15, $G = 1000\text{ Hz/V}$ is used). In this situation the Q-PLL behaves, in the limit, as a PLL with frequency-divider capability such as, for example, those in the sampling class of PLLs.

Fig. 15. Response diagram for one input frequency only (PLL-like behaviour) as a function of f_Q , for $f_1 = 200\text{ Hz}$. Depending on the VCO quiescent frequency various sub-harmonics of f_1 can be selected.

C. Q-PLL as a pitch detector

As the focus of this work is on the ability of the circuit to model the pitch of complex sounds, its ability to lock in the case of the sum of two sine waves with harmonic frequency ratio $f_1/f_2 = 2/3$ is analysed. In musical terms, this particular ratio corresponds to a perfect fifth interval. Such a ratio is used as a paradigmatic example being considered one of the most consonant musical intervals. It should be borne in mind that this interval represents a stimulus with a missing fundamental at $f_1/2 \equiv f_2/3$.

Figure 16 displays the input and response frequencies for the perfect fifth case, $f_1 = 200\text{ Hz}$ and $f_2 = 300\text{ Hz}$ with $f_Q = 90\text{ Hz}$. The system locks stably to the missing fundamental value $f_R = 100\text{ Hz}$.

Next, the more complex case corresponding to a pitch-shift is analysed. Figure 17, reports the result for a non-harmonic signal obtained by applying a pitch shift operator to the harmonic signal used in the perfect fifth example. A

Fig. 16. Example of locking for harmonic input: $f_1 = 200$ Hz, $f_2 = 300$ Hz and $f_Q = 90$ Hz. Harmonics of the response frequency are also produced with lower amplitude.

Fig. 17. Example of locking for inharmonic input ("pitch shift" of $\Delta f = 21$ Hz with respect to Figure 16): $f_1 = 221$ Hz, $f_2 = 321$ Hz and $f_Q = 90$ Hz. The only surviving frequency in the output signal is the "residue" frequency corresponding to $f_0 = (f_1/2 + f_2/3) = 108.4$ Hz, according to Equation (4)

small shift $\Delta f = 21$ Hz was selected to keep the ratio $2/3$ as the first approximant of the input frequency ratio. In fact, the continued fractions development of the ratio $221/321$ is:

$$[1, 2, 4, 1, 3, 5] \quad (15)$$

and the corresponding convergents: $1, 3/2, 13/9, \dots$

Figure 17 shows the power spectrum density of the input and output signals. The system locks exactly (within the numerical precision) to the first median of $f_1/2$ and $f_2/3$, i.e., $f_0 = 108.4$ Hz. Another remarkable feature is that, in virtue of the locking mechanism described in Section III, only the residue frequency survives.

Another interesting variation of the input signal, related to the possibility of using the Q-PLL as a model for auditory pitch recognition, is represented by a signal which is a harmonic complex with two-to-several components under pitch shift [7]. This stimulus is the basis of several psychoacoustics experiments (see, e.g., [16], [27]–[30]). Although the argument

Fig. 18. Example of locking in response to a signal comprised of seven shifted harmonics with amplitude decay as a function of the harmonic number. Amplitudes of the stimulus (components 2 to 7) are: 1.0, 0.8, 0.6, 0.3, 0.1, 0.3, roughly derived from a piano spectrum.

is very complex and stimuli can be very different, in order to provide an example, an input signal comprised of the first 7 harmonics of a sine wave of frequency 100 Hz, shifted by the same amount used in previous pitch-shift examples ($\Delta f = 21$ Hz), is fed into the Q-PLL. Figure 18 displays the spectrum of the input and the response signals. The response remains stable to $f_0 = (f_1 + f_2)/(2 + 3) = 108.4$ Hz, that corresponds to the observed pitch of the stimulus, under significant variations in the different magnitudes of the input components, provided that the amplitudes mostly decay towards high harmonic numbers (typical of musical sounds).

The resulting spectrum is more complex than that reported in Figure 17, displaying not only the locking peak, but also a number of lower power components. This happens because in this case the low frequency components resulting from the sub-Nyquist sampling (see Section III) do not mutually cancel as in the case of a stimulus comprising only two components of the same amplitude.

V. CONCLUSION

A new nonlinear circuit which can maintain a locked state when forced by two incommensurate frequencies has been presented. The circuit was developed in a biologically inspired paradigm within the framework of dynamical systems theory. The circuit features have been studied by means of numerical simulations and also verified in an analog electronic implementation. Locking states are characterised by the generation of a third frequency which forms a so-called three-frequency resonance together with the forcing frequencies. Moreover, it has been demonstrated that these resonances characterise, in turn, the pitch of complex sounds when the fundamental component is absent. Thanks to this property, the circuit can emulate one of the main psycho-acoustical parameters of auditory perception. Thus, it opens the way for the development of real-time auditory processors which can be utilised in a diversity of applications, from medicine (hearing aids, artificial cochlea), to robotics (artificial senses),

to industrial and consumer electronics (by including auditory-like processing in audio applications, as a natural extension of the universal PLL). This new circuit, being orbitally and structurally stable, thanks to a control strategy based on sample-and-hold feedback, supersedes former attempts (such as coupled PLLs) and prompts further study regarding optimisation of its features and performance in the light of future technological applications.

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