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A proof of $P \neq NP$ (New symmetric encryption algorithm against any linear attacks and differential attacks)

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1 Abstract

P vs NP problem is the most important unresolved problem in the field of computational complexity. Its impact has penetrated into all aspects of algorithm design, especially in the field of cryptography. The security of cryptographic algorithms based on short keys depends on whether P is equal to NP. In fact, Shannon[1] strictly proved that the one-time-pad system meets unconditional security, but because the one-time-pad system requires the length of key to be at least the length of plaintext, how to transfer the key is a troublesome problem that restricts the use of the one-time-pad system in practice. Cryptography algorithms used in practice are all based on short key, and the security of the short key mechanism is ultimately based on "one-way" assumption, that is, it is assumed that a one-way function exists. In fact, the existence of one-way function can directly lead to the important conclusion $P \neq NP$.

In this paper, we originally constructed a short-key block cipher algorithm. The core feature of this algorithm is that for any block, when a plaintext-ciphertext pair is known, any key in the key space can satisfy the plaintext-ciphertext pair, that is, for each block, the plaintext-ciphertext pair and the key are independence, and the independence between blocks is also easy to construct. This feature is completely

different from all existing short-key cipher algorithms.

Based on the above feature, we construct a problem and theoretically prove that the problem satisfies the properties of one-way functions, thereby solving the problem of the existence of one-way functions, that is, directly proving that $P \neq NP$.

2 Introduction

Cryptography is one of the most important applications in the field of communication and computer science. In recent years, with the application of commerce, enterprises, banks and other departments, cryptography has been developed rapidly. Especially after Shannon put forward the mathematical analysis of security in "Communication theory of secrecy systems" [1], various design tools for cipher algorithms and corresponding attack tools have been developed one after another. Among them, the most common attack methods include linear attacks and differential attack.

Linear attack was first proposed by M. Matsui [2], this is an attack method that is currently applicable to almost all block encryption algorithms. Kaliski BS [3] proposed a multi-linear attack based on the linear attack, but the multi-linear attack has many limitations. And the Biryukov A [4] and Chao, J.Y [5] and others further improved the framework of multi-linear attacks, thus making linear attacks a larger application.

The differential attack method was first proposed by Eli Biham [6]. BIHAM E [7] extended it to a more powerful attack method. TSUNOO [8] further constructed

multiple attack methods. These attack methods have extremely high skill in the attack process, which is worthy in-depth study.

In this paper, we first designed a new encoding algorithm, which we named Eagle . The feature of the Eagle algorithm is that the output of the algorithm with the same input and the same parameters is completely random.

Based on the Eagle encoding algorithm, we designed a new block symmetric cipher algorithm. Based on this cipher algorithm, for any block of plaintext-ciphertext pairs, for any key in the key space, a suitable encryption method can be found. That is to say, there is no specific mathematical relationship between the plaintext, key, and ciphertext in each block, showing a completely randomly property. It can also be understood that for any plaintext, encrypted with the same key every time, the ciphertext obtained is not uniquely determined, but completely randomly in the possible ciphertext space. And this feature makes the cipher algorithm can resistant all forms of linear attacks and differential attacks.

At the end of this paper, we further construct a new cipher system. Under this cipher system, if any plaintext-ciphertext pair is known, if an attacker wants to guess the possible correct key, he cannot do it in polynomial time. We have proved theoretically that this kind of problem satisfies the properties of one-way functions, that is, theoretically prove that one-way functions exist, so that $P \neq NP$.

3 Introduction to Eagle encoding algorithm

We first introduce two common bit operations. XOR denoted as $\oplus$. Do left cycle

shift of D by n bits which can be denoted as D^{+n} , for example $(1001101)^{+2} = (0110110)$.

We select two L -bits parameters w_0 and w_1 , have odd number of different bits.

For example 11010011 and 10100101 have 5 bits (5 is odd) different.

$$\begin{array}{r} 11010011 \\ 10100101 \\ \hline x\ x\ x\ \ x\ x \end{array}$$

We set the initial state of L -bit as S_0 , we choose one parameter in w_0 or w_1 , without loss of generality, assume that we choose w_0 , then we define the following calculation

$$S_1 = w_0 \oplus (S_0 \oplus S_0^{+1}) \quad (3.1)$$

From (3.1), we can easily know

$$S_0 \oplus S_0^{+1} = S_1 \oplus w_0 \quad (3.2)$$

If we only know S_1 , we don't know whether w_0 or w_1 we used in (3.1), we can confirm it through a simple trial-and-error. For example, we guess the parameter w_1 was used in (3.1), we need to find a certain number S_x to Satisfy

$$S_x \oplus S_x^{+1} = S_1 \oplus w_1 \quad (3.3)$$

In fact, since w_0 and w_1 have odd number of different bits, such S_x does not exist. See Theorem 1 for details.

[Theorem 1] We arbitrarily choose two L -bit parameters w_0 and w_1 which have odd number of different bits, for arbitrary S_0 , we set $S_1 = w_0 \oplus (S_0 \oplus S_0^{+1})$, then there doesn't exists S_x satisfy $S_x \oplus S_x^{+1} = S_1 \oplus w_1$.

Proof: Firstly, by definition we get

$$S_1 \oplus w_1 = w_0 \oplus (S_0 \oplus S_0^{+1}) \oplus w_1 = (S_0 \oplus S_0^{+1}) \oplus (w_0 \oplus w_1),$$

Where w_0 and w_1 have odd number of different bits, so there are odd number of 1 in the bit string of $w_0 \oplus w_1$.

Suppose there have S_x satisfy $S_x \oplus S_x^{+1} = S_1 \oplus w_1$, then

$$S_x \oplus S_x^{+1} = (S_0 \oplus S_0^{+1}) \oplus (w_0 \oplus w_1), \text{ by simple calculation we get}$$

$(S_0 \oplus S_x) \oplus (S_0 \oplus S_x)^{+1} = w_0 \oplus w_1$, we set $S_y = S_0 \oplus S_x$, then there are odd number of 1 in the bit string of $S_y \oplus S_y^{+1}$, without of generality, we suppose that the bits with 1 are $l_1, l_2, \dots, l_u$ (u is odd). Compare to the first bit of S_y , the $l_1 + 1$ bit of S_y is different from the first bit of S_y , the $l_2 + 1$ bit of S_y is the same with the first bit of S_y , the $l_3 + 1$ bit of S_y is different from the first bit of S_y , the $l_4 + 1$ bit of S_y is the same with the first bit of S_y , and so on, since u is odd, the $l_u + 1$ bit of S_y is different from the first bit of S_y , by the definition of cycle shift, we conclude that the first bit of S_y is different from the first bit of S_y , this is contradictory.

So there doesn't exists S_x satisfy $S_x \oplus S_x^{+1} = S_1 \oplus w_1$.

Let's go back to the discussion just now, after a trial-and-error, we can accurately confirm which one (w_0 or w_1) we just used in (3.1).

Now we suppose that there is a binary sequence $M = b_1 b_2 \dots b_L$ with length L . Start with S_0 , read each bit of M from left to right sequentially, when the bit b_i ($1 \leq i \leq L$) is 0, we set $S_i = w_0 \oplus (S_{i-1} \oplus S_{i-1}^{+1})$, when the bit b_i is 1, we set

$$S_i = w_1 \oplus (S_{i-1} \oplus S_{i-1}^{+1}).$$

According to the above calculation, for every S_i , we can find the only w_0 or w_1

such that there exists S_{i-1} satisfy $S_{i-1} \oplus S_{i-1}^{+1} = S_i \oplus w_x$. According to the properties of XOR and cyclic shift, we can easily know that there are only two S_{i-1} that satisfy $S_{i-1} \oplus S_{i-1}^{+1} = S_i \oplus w_x$, and the two S_{i-1} that with each bit different. As long as we know any bit of S_{i-1} , S_{i-1} can be uniquely determined. So we only need to save any bit of S_i , by S_L , we can completely restore the original state S_0 and the binary sequence M .

Based on the above discussion, we can construct a complete Eagle encoding and decoding algorithm. The entire algorithm consists of three processes: generating parameters, encoding, and decoding.

[Parameter generation]

Firstly we choose two L -bit parameters w_0 and w_1 which have odd number of different bits, then we choose L -bit initial state S_0 .

[Encoding]

For input data M , we record $M[i] (1 \leq i \leq L)$ as the i -th bit of M , $M[i] \in \{0,1\}$, L is the length of M , the encoding process is as follows.

[E1] Execute E2 to E4 with i from 1 to L .

[E2] If $M[i] = 0$, set $S_i = w_0 \oplus (S_{i-1} \oplus S_{i-1}^{+1})$.

[E3] If $M[i] = 1$, set $S_i = w_1 \oplus (S_{i-1} \oplus S_{i-1}^{+1})$.

[E4] Set the last bit of S_i as the i -th bit of C , $C[i] = S_i[L]$.

[E5] Use (C, S_L) as the output.

[Decoding]

The output (C, S_L) of the above encoding process is used as the input of the

decoding process, the decoding process is as follows.

[D1] Execute D2 to D4 with i from L to 1.

[D2] Do trial-and-error testing with $S_i \oplus w_0$ or $S_i \oplus w_1$, find the unique w_x ($x=0$ or $x=1$) satisfy $S_x \oplus S_x^{+1} = S_i \oplus w_x$.

[D3] After D2, use x as the i -th bit of M , $M[i] = x$.

[D4] For the two possible S_x satisfy $S_x \oplus S_x^{+1} = S_i \oplus w_x$ in D2, we set the one which the last bit is equal to $C[i]$ as S_{i-1} .

[D5] Use M as the output.

It is not difficult to find that the above encoding process and decoding process are correct, that is (C, S_L) generated by encoding from M can be completely restored through the decoding process. In addition, the encoding process is sequential encoding in the order of M 's bits, and the decoding process is sequential decoding in the reverse order of C 's bits.

$$S_0 \xrightarrow{m_1} S_1 \xrightarrow{m_2} S_2 \longrightarrow \dots \xrightarrow{m_L} S_L$$

$$S_0 \xleftarrow{c_1} S_1 \xleftarrow{c_2} S_2 \longleftarrow \dots \xleftarrow{c_L} S_L$$

We also noticed the fact that in the above encoding and decoding process, all inputs and outputs do not need to appear S_0 , This means that the selection of S_0 will not affect the correctness of the encoding process and decoding process. The arbitrary of S_0 will bring the uncertainty of the encoded output.

For the convenience of the discussion in the following chapters, here we briefly analyze the effect of uncertainty of S_0 on the encoded output. Given the

parameters w_0 and w_1 that have odd number of different bits, for a certain input M of L bits, since S_0 is arbitrarily selected, it is obvious that C is uncertain, but is the final state S_L necessarily uncertain?

In fact, the answer is no. In some cases, such as $L = 2^u$ (that is, the parameter length is the power of 2), the final state S_L is determined for different choices of S_0 . The final state variable S_L which is the output of the encoding process is only related to the input M and has nothing to do with the choice of the initial state S_0 . See Theorem 2 for details.

[Theorem 2] In Eagle encoding, given the parameters w_0 and w_1 that have L bits with different odd bits, for a certain L bit input M , if $L = 2^u$ is satisfied, then for any initial state S_0 , after the Eagle encoding process, the final state S_L is only related to the input M , and is unrelated with the choice of the initial state S_0 .

Proof: We represent M as binary stream $x_1x_2...x_L$, which $x_i \in \{0,1\}$, $1 \leq i \leq L$.

We execute the Eagle encoding process to M from x_1 to x_L as follows.

$$S_1 = w_{x_1} \oplus (S_0 \oplus S_0^{+1}) = f_1(w_{x_1}) \oplus (S_0 \oplus S_0^{+1})$$

$$S_2 = w_{x_2} \oplus (S_1 \oplus S_1^{+1}) = f_2(w_{x_1}, w_{x_2}) \oplus (S_0 \oplus S_0^{+2})$$

$$S_3 = w_{x_3} \oplus (S_2 \oplus S_2^{+1}) = f_3(w_{x_1}, w_{x_2}, w_{x_3}) \oplus (S_0 \oplus S_0^{+1} \oplus S_0^{+2} \oplus S_0^{+3})$$

$$S_4 = w_{x_4} \oplus (S_3 \oplus S_3^{+1}) = f_4(w_{x_1}, w_{x_2}, w_{x_3}, w_{x_4}) \oplus (S_0 \oplus S_0^{+4})$$

It is not difficult to find that for any $m = 2^v$, $S_m = f_m(w_{x_1}, ..., w_{x_m}) \oplus (S_0 \oplus S_0^{+m})$

holds, this can be proved by a simple mathematical induction. In fact, the conclusion is correct for $v = 1$. We assume that the conclusion is correct for $v - 1$,

we have $S_{m/2} = f_{m/2}(w_{x_1}, ..., w_{x_{m/2}}) \oplus (S_0 \oplus S_0^{+m/2})$, since $S_{m/2}$ to S_m must do

calculations with $m/2$ steps, we have

$$S_m = f_m(w_{x_1}, \dots, w_{x_m}) \oplus (S_0 \oplus S_0^{+m/2}) \oplus (S_0^{+m/2} \oplus S_0^{+m}) = f_m(w_{x_1}, \dots, w_{x_m}) \oplus (S_0 \oplus S_0^{+m})$$

Since $L = 2^u$, we have $S_L = f_L(w_{x_1}, \dots, w_{x_L}) \oplus (S_0 \oplus S_0^{+L})$, where $f_i(\dots)$ is irrelevant with S_0 , by definition of cycle shift, we have $S_0 = S_0^{+L}$, so $S_L = f_L(w_{x_1}, \dots, w_{x_L})$ is irrelevant with S_0 .

From theorem 2, for any parameters w_0 and w_1 with length $L = 2^u$, for any initial state S_0 , execute Eagle encoding on M to obtain S_L which is irrelevant with S_0 . In order to facilitate the description in the following chapters, we introduce the symbol ξ and ζ .

$\xi_{w_0, w_1} : (S_0, M) \Rightarrow (S_1, C)$: use the key w_0, w_1 to execute Eagle encoding ([E1]-[E5]) on initial state S_0 and input M to obtain C and S_1 .

$\zeta_{w_0, w_1} : (S_1, C) \Rightarrow (S_0, M)$: use the key w_0, w_1 to execute Eagle decoding ([D1]-[D5]) on C and S_1 to obtain S_0 and M .

In all the following chapters of this paper, we assume the length is a power of 2.

4 Eagle encryption algorithm

The core idea of Eagle encryption algorithm comes from the Eagle encoding process. If we use the parameters w_0 and w_1 in the Eagle encoding process as encryption keys, the process of encoding *Input* can be regarded as the process of encrypting plaintext *Input*. Output $(Output, S_n)$ can be used as ciphertext. In fact, we can introduce uncertainty into the initial state S_0 without affecting the

correctness of the decoding process. We will see later that uncertainty allows us to design a more secure encryption system.

Next, we will introduce the Eagle encryption algorithm in detail. The entire Eagle encryption algorithm is divided into three processes: key generator, encryption process, and decryption process.

4.1 Eagle key generator

First, the choice of the key is completely random, and the key needs to be shared between the encryptor and the decryptor. Since w_0 and w_1 must have odd number of different bits, there are only 2^{2L-1} effective keys with bits length of $2L$, one bit will be lost. That is, in the Eagle encryption algorithm, the number of bits for the key is always an odd number.

We randomly generate a number with bits length of $2L$. We take the first L bits as w_0 . When the next L bits are different from w_0 by an odd number of "bits", then we directly take the next L bits as w_1 ; when the next L bits are different from w_0 by an even number of "bits", we set the next L bits as w_1 with the last bit inverted.

4.2 Eagle encryption process

For the $2L-1$ -bit key w_0 and w_1 , for the plaintext M , we construct Eagle encryption processes as follows:

[M1] The plaintext M is grouped by L bits, and the last group with less than L bits are randomly filled into L bits. The total number of groups is assumed to be T , the grouped plaintext M is recorded as $M_1, M_2, \dots, M_T$.

[M2] We randomly generate two numbers as the initial state S_0 and the random group M_{T+1} inserted into $M_1, M_2, \dots, M_T$.

[M3] Calculate from the first group to the $T+1$ group. For the first group M_1 , Start state is S_0 , after the encoding of [E1]-[E5], the state becomes S_1 , and the encoded data is C_1 (Each bit of C_1 is composed of the last bit of all intermediate states), which are recorded as $\xi_{w_0, w_1} : (S_0, M_1) \Rightarrow (S_1, C_1)$. then we reset the state S_1 as $S_1 = S_1 \oplus C_1$. For the second group M_2 , the state is S_1 , after the encoding of [E1]-[E5], the state becomes S_2 , and the encoded data is C_2 , which are recorded as $\xi_{w_0, w_1} : (S_1, M_2) \Rightarrow (S_2, C_2)$, then reset $S_2 = S_2 \oplus C_2$. After $T+1$ groups, the final state is S_{T+1} , and the encoded data of each group is $C_1, C_2, \dots, C_{T+1}$.

[M4] Output $C_1 C_2 \dots C_{T+1} S_{T+1}$ as ciphertext.

4.3 Eagle decryption process

With the same key w_0 and w_1 , for ciphertext $C_1 C_2 \dots C_{T+1} S_{T+1}$, the Eagle decryption processes are as follows:

[C1] Starting from the last group $T+1$ to the first group, do decryption in sequence. For the $T+1$ group, we reset $S_{T+1} = S_{T+1} \oplus C_{T+1}$ and use S_{T+1} and C_{T+1} to execute the decoding operations [D1]-[D5] introduced in Chapter 2 to obtain state S_T and decoded data M_{T+1} , which are recorded as $\zeta_{w_0, w_1} : (S_{T+1}, C_{T+1}) \Rightarrow (S_T, M_{T+1})$. For the T group, we reset $S_T = S_T \oplus C_T$ and use states S_T and C_T to execute the decoding operations [D1]-[D5] to obtain state S_{T-1} and decoded data M_T , which are recorded as $\zeta_{w_0, w_1} : (S_T, C_T) \Rightarrow (S_{T-1}, M_T)$. After execute all the processes until the first group, the decoded data

corresponding to each group is $M_{T+1}, M_T, \dots, M_1$ respectively.

[C2] Output $M_1 M_2 \dots M_T$ as the plaintext.

Obviously, the above decryption processes can get the correct plaintext.

In addition, for the plaintext with the L bit, in the encryption or decryption process, it is encoded or decoded bit by bit, and the encoding or decoding process of each bit is a certain and direct calculation process, so the encryption process and the decryption process have computational complexity $O(L)$

5 Linear attack analysis to Eagle encryption algorithm

Linear attack is a very effective attack method against the DES algorithm proposed by M. Matsui [2] at the European Cryptographic Conference in 1993. Later, scholars quickly discovered that the linear attacks are applicable to almost all block encryption algorithms, and linear attacks have become the main attacks for block encryption algorithms. Various new attacks based on linear attacks are constantly being proposed.

The core idea of linear attack is to take the nonlinear transformation in the cryptographic algorithm, such as the linear approximation of the S-box, and then extend the linear approximation to the linear approximation of the round function, and then connect these linear approximations to obtain the entire cryptographic algorithm a linear approximation, and finally a large number of known plaintext-ciphertext pairs encrypted with the same key are used to exhaustively obtain the plaintext and even the key.

We have noticed that the reason why linear attacks have become an effective attack for block encryption algorithms is that when the key is known, there is a certain implicit linear relationship between the ciphertext and the plaintext. By analyzing the known plaintext-ciphertext pairs, some effective linear relations can be obtained, and some bits of the key can be guessed.

In the Eagle encryption processes, for a certain group, suppose that the initial state at the beginning of the group is S_{i-1} , the plaintext of the group is M_i , the keys are w_0 and w_1 , after the [E1]-[E5], we obtain the new state S_i and the encoding result C_i . Only C_i is included in the ciphertext, and S_i is not included in the ciphertext, that is, S_i is invisible to the decryption party and thus invisible to the attacker.

At the beginning of each group, S_{i-1} can be regarded as completely random, this is because from the first group, S_0 is completely random, after the encoding process for the first group, C_1 is completely random, the state $S_1 \oplus C_1$ of the second group is also completely random. By analogy, the initial state S_{i-1} at the beginning of each group is completely random, and the ciphertext C_i of the group is completely random.

That is to say, for a certain key and plaintext, the ciphertext is completely random, which have no mathematical relationship between the plaintext or the key.

For an attacker, under the condition that a pair of plaintext M_i and ciphertext C_i of any group is known, for any key in the key space, a corresponding encryption or decryption method can be found to meet the conditions. For example, if an attacker chooses a specific key w_0 and w_1 , he can choose S_{i-1} randomly, and then

use the key pair S_{i-1}' and M_i to execute the encoding process of [E1]-[E5] to get the state S_i . According to the conclusion of [Theorem 2], the arbitrariness of the attacker's choice of S_{i-1}' does not affect the certainty of S_i . Then he can execute the decoding process of [D1]-[D5] according to S_i and C_i to get S_{i-1} . This means that by any key he can find a suitable S_{i-1} that satisfies the condition for a known plaintext M_i -ciphertext C_i pair, and the attacker cannot get any valid information from the pair, which can be denoted as

$\Pr(W = (w_0, w_1) | (M = m_i, C = c_i)) = 1 / (2^{2L-1})$. This can also be summarized as the following theorem:

[Theorem 3] For any L bit M_1 and C_1 , for any $2L-1$ bit $K = (w_0, w_1)$, there must be unique S_0 and S_1 , so that $\xi_{w_0, w_1} : (S_0, M_1) \Rightarrow (S_1, C_1)$ holds.

The independence of the plaintext-ciphertext pair with the key makes it impossible for any linear attacker to establish an effective relationship.

6 Differential attack analysis to Eagle encryption algorithm

Differential attack was proposed by Biham and Shamir [6] in 1990, it is a chosen-plaintext attack. Its core idea is to obtain key information by analyzing specific plaintext and ciphertext differences.

The essence of a differential attack is to track the "difference" of the plaintext pair, where the "difference" is defined by the attacker according to the target, which can be an exclusive XOR operation or other target values. For example, if you choose the plaintext M and the difference δ , the other plaintext is $M + \delta$. The attacker

mainly analyzes the possible keys by analyzing the difference between the ciphertext C and $C + \varepsilon$. For the Eagle encryption algorithm, suppose the differential attacker chooses two specific plaintexts M_1 and M_2 , their difference is δ , that is $M_2 = M_1 + \delta$, the corresponding ciphertexts are C_1 and C_2 , and the difference between the ciphertexts is ε , and That is $C_2 = C_1 + \varepsilon$. Since in the encryption processes of Eagle algorithm, C_1 and C_2 are completely random, it is completely uncertain whether the difference ε of the ciphertext is caused by randomness or the spread of the plaintext. Furthermore, $E_{w_0, w_1}(M_1)$ and $E_{w_0, w_1}(M_2)$ subject to the same probability distribution, which can be denoted as $\Pr(C_1 = c_1, C_2 = c_2 | (M_1 = m_1, M_2 = m_2)) = 1 / 2^{2L}$.

That is to say, for any specific plaintext M_1 and M_2 selected by the attacker, after being encrypted with the same key, the corresponding block ciphertexts C_1 and C_2 are completely random, and any possible value in the ciphertext space appears with equal probability. The attacker has no way to capture the propagation characteristics of the "difference" in the plaintext.

7 One-way function design

7.1 Introduction to one-way functions

Before constructing the one-way function, we briefly introduce the properties of one-way function and the relationships with the $P \neq NP$ problem.

[Definition 1] A function is a one-way function means that the function satisfies the following properties:

- a) For a given x , there exists a polynomial-time algorithm that output $f(x)$.
- b) Given y , it is difficult to find an x that satisfies $y = f(x)$, that is, there does not exists a polynomial-time algorithm that finding the x .

The NP-complete problem refers to a set of problems that are verifiable in polynomial-time algorithm. For all NP-complete problems, whether there exists algorithms that are solvable in polynomial-time, this is the P vs NP problem. If $P \neq NP$, then for some NP problems, there is no algorithm that is solvable in polynomial-time. If $P = NP$, then for all NP problems, there exists algorithms that are solvable in polynomial-time.

If one-way function exists, it means that there exists such an NP problem, which has no deterministic polynomial time solvable algorithm, that is, $P \neq NP$. This is a direct inference, which can be directly described as the following theorem, See [9] for details.

[Theorem 4] If one-way function exists, then $P \neq NP$.

We then introduce an additional simple algorithmic problem, which we describe as the following theorem.

[Theorem 5] For two sets selected completely independently, the number of elements is l_1, l_2 , then the average algorithm complexity of finding the common elements of the two sets (there may be only one common element at most) is at least $c * \min(O(l_1), O(l_2))$, where c is a certain constant.

This is because the remaining unvisited elements in the two sets will be visited at least once with equal probability before no common element is found.

7.2 Construction of one-way functions

Let us go back to the Eagle encryption algorithm mentioned in Chapter 4. Assuming that the key is $K = (w_0, w_1)$, with length $2L-1$ bit, the plaintext has n blocks, denoted as $M = (M_1, M_2, \dots, M_n)$ where each block with length L bit, and the encrypted ciphertext is $C = (C_1, C_2, \dots, C_n, C_{n+1}, S_{n+1})$.

Among them, the plaintext-ciphertext pair of the $i(1 \leq i \leq n)$ -th group is $M_i - C_i$, and the $n+1$ th group has only the ciphertext (C_{n+1}, S_{n+1}) known.

It is not difficult to see that since the last group has only ciphertext, any key is "right".

For the $i(1 \leq i \leq n)$ group, we might take the first group without loss of generality. Given the plaintext M_1 and the ciphertext C_1 , from Theorem 3, any key is also "right", so we can only "guess" the key from each group independently.

The plaintext-ciphertext pair between two discontinuous groups can also be regarded as completely independent, such as the plaintext or ciphertext between the i -th group and the $i+2$ th group has no relationship.

For two consecutive groups, the only connection between $M_i - C_i$ and $M_{i+1} - C_{i+1}$ lies in the initial state of the $i+1$ -th group equal to $S_i \oplus C_i$.

In fact, we can easily cut off the connection between the i -th group and the $i+1$ -th group. For example, change the initial state of the $i+1$ -th group to $S_i \oplus Y = S_{i+1}$, where Y is a random number that does not depend on any system variables. Then put Y into the increased group to encrypt, so that when the plaintext-ciphertext pairs of any multiple groups are known, the process of

"guessing" the key is independent for any grouping, and because Within each group, any key is "right", and the only way to "guess" the key is to use trial-and-error method.

Since the key is $2L-1$ bits, there are 2^{2L-1} possible keys in the entire key space. Obviously, the computational complexity of the trial-and-error method is exponential.

The above analysis is the core idea for us to construct a one-way function, and then we define the following encryption algorithm:

[Encryption Algorithm Q] For $2L-1$ bit key $K = (w_0, w_1)$, for two blocks of plaintext $M = (M_1, M_2)$, the encryption process is as follows:

[Q1]. Generate p L -bit random numbers completely independently, denoted as $X = (X_1, X_2, \dots, X_p)$, where $p > 1$.

[Q2]. Select three functions $f_1 : x \rightarrow y$, $f_2 : x \rightarrow y$, $f_3 : x \rightarrow y$, where x is a set composed of pL -bit binary data, and y_1, y_2, y_3 are composed of L -bit binary data. Calculate $Y_1 = f_1(X)$, $Y_2 = f_2(X)$, $Y_3 = f_3(X)$. Note that the selection of f_1, f_2, f_3 here are polynomial-time complexity.

[Q3]. Random numbers S_0 and M_3 are selected completely independently, and the following algorithm is executed:

$$\xi_{Y_2, Y_3} : (S_0, M_1) \Rightarrow (S_1, C_1)$$

$$\xi_{Y_2, Y_3} : (S_1 \oplus Y_1, M_2) \Rightarrow (S_2, C_2)$$

$$\xi_{Y_2, Y_3} : (S_2, M_3) \Rightarrow (S_3, C_3)$$

Output (C_1, C_2, C_3, S_3) as the first part of the ciphertext.

[Q4]. Use the Eagle encryption algorithm introduced in Chapter 4 to encrypt $X = (X_1, X_2, \dots, X_p)$ to obtain the second part of the ciphertext $(C'_1, C'_2, \dots, C'_p, C'_{p+1}, S'_{p+1})$.

Obviously, the algorithm Q is correct, because according to the ciphertext (C_1, C_2, C_3, S_3) and $(C'_1, C'_2, \dots, C'_p, C'_{p+1}, S'_{p+1})$, the plaintext (M_1, M_2) can be completely decrypted.

First use the Eagle encryption algorithm introduced in Chapter 4 to decrypt $(C'_1, C'_2, \dots, C'_p, C'_{p+1}, S'_{p+1})$ to get $X = (X_1, X_2, \dots, X_p)$, execute $Y_1 = f_1(X)$, $Y_2 = f_2(X)$, $Y_3 = f_3(X)$, and then execute the following algorithm:

$$\zeta_{Y_2, Y_3} : (S_3, C_3) \Rightarrow (S_2, M_3)$$

$$\zeta_{Y_2, Y_3} : (S_2 \oplus Y_1, C_2) \Rightarrow (S_1^*, M_2)$$

$$\zeta_{Y_2, Y_3} : (S_1^* \oplus Y_1, C_1) \Rightarrow (S_0, M_1)$$

Output the complete plaintext (M_1, M_2) .

The question we are going to ask is that in algorithm Q, if we know the plaintext (M_1, M_2) , what is the benefit for us to "guess" the key $K = (w_0, w_1)$.

We divide the Q algorithm into two parts, [Q1]-[Q2] are the parameter preparation stages, [Q3] is the first part of the algorithm, and [Q4] is the second part of the algorithm. Obviously these two parts are completely independent. The only connection is $Y_1 = f_1(X)$, $Y_2 = f_2(X)$, $Y_3 = f_3(X)$. Since f_1, f_2, f_3 can be regarded as compression functions, Unique Y_1, Y_2, Y_3 can be obtained for any X ; for Y_1, Y_2, Y_3 , there are $2^{(p-3)L-1}$ X satisfying $Y_1 = f_1(X), Y_2 = f_2(X), Y_3 = f_3(X)$.

From the decryption process, we can see that for the specified key $K = (w_0, w_1)$, a

unique X can be obtained, so that Y_1, Y_2, Y_3 are uniquely determined.

But without knowing the key $K = (w_0, w_1)$, the verification of Y_1 has also become a very "complex" problem.

This is because for the plaintext-ciphertext pair in the first part, for any Y_1 , there are at least 2^{L-1} possible Y_2, Y_3 that are "right".

For the ciphertext of the second stage, only Y_1 is known, even though Y_2, Y_3 are known, there are at least $2^{(p-3)L-1}$ possible X that is "right". So the problem of verifying Y_1 becomes an exponentially-time problem. Therefore, it can be directly proved that the problem of "guessing" the key satisfies the property of a one-way function. See the following theorem and proof for details.

[Theorem 6] For a $2L-1$ bits key $K = (w_0, w_1)$, for two blocks of plaintext $M = (M_1, M_2)$, the ciphertext obtained after using the encryption algorithm Q are (C_1, C_2, C_3, S_3) and $(C'_1, C'_2, \dots, C'_p, C'_{p+1}, S'_{p+1})$, then under the condition that the plaintext $M = (M_1, M_2)$ is known, the problem of verifying Y_1 generated in [Q2] has no polynomial-time algorithms.

Proof: Observe the encryption algorithm Q, there are five independent variables K, X, Y_1, Y_2, Y_3 , [Q2] and [Q3] can be expressed as $F_1(X, Y_1, Y_2, Y_3) = 0$, [Q4] expressed as $F_2(K, X) = 0$.

Without knowing K and X , the problem of verifying $Y_1 = y_1$ is equivalent to the problem of finding X that satisfy $F_1(X, Y_1, Y_2, Y_3) = 0$ and $F_2(K, X) = 0$.

For the part [Q3], given $M = (M_1, M_2)$ and (C_1, C_2, C_3, S_3) , by the following calculation process:

$$\xi_{Y_2, Y_3} : (S_0, M_1) \Rightarrow (S_1, C_1)$$

$$\xi_{Y_2, Y_3} : (S_1 \oplus Y_1, M_2) \Rightarrow (S_2, C_2)$$

With C_1, C_2, M_2 fixed, for any combination of (Y_2, Y_3, Y_1) , a unique (S_0, M_1, S_2) can be derived. According to the simple combinatorics principle, it is not difficult to conclude that for any $y_1 \in Y_1$, there are at least 2^{L-1} possible $y_2, y_3 \in Y_2, Y_3$ that satisfy the condition. Therefore, by any algorithm, to verify $y_1 \in Y_1$, it is necessary to verify each of the possible $2^{(p-2)L-1}$ X .

For the part [Q4], under the condition that $(C_1', C_2', \dots, C_p', C_{p+1}', S_{p+1}')$ is known, since K is unknown, there are 2^{2L-1} possible X .

Since the process [Q4] is independent from [Q2][Q3], by theorem 5, the problem of finding X have work with exponent-complexity.

[Theorem 7] For a $2L-1$ -bits key $K = (w_0, w_1)$, for two blocks of plaintext $M = (M_1, M_2)$, the ciphertext obtained after using the encryption algorithm Q are (C_1, C_2, C_3, S_3) and $(C_1', C_2', \dots, C_p', C_{p+1}', S_{p+1}')$, then under the condition that the plaintext $M = (M_1, M_2)$ is known, The process of finding the key $K = (w_0, w_1)$ satisfies the properties of the one-way function.

Proof: First, for any key $K = (w_0, w_1)$, to determine whether it is correct, you only need to use K to decrypt the ciphertext, and then determine whether the obtained plaintext is M . Obviously, the computational complexity is $O(L)$, which satisfies 7.1.1.

Next, we need to prove condition 7.1.2, that is, there is no polynomial-time algorithm for the problem of "guessing" the key. Here we adopt the method of

proof by contradiction.

Assuming that there is a polynomial-time algorithm for the problem of "guessing the key", after the algorithm is executed, the ciphertext (C_1, C_2, C_3, S_3) and $(C'_1, C'_2, \dots, C'_p, C'_{p+1}, S'_{p+1})$ are decrypted to obtain X . The decryption only requires a complexity of $O(L)$. According to the assumption of [Q2], the calculation of $Y_1 = f_1(X)$ is also polynomial-time complexity. And this series of calculations are all polynomial-time complexity, which contradicts the conclusion of Theorem 6.

Therefore, the problem of "guessing" the key satisfies the properties of one-way functions.

According to Theorem 7, we find a problem that satisfies the properties of one-way functions, which proves the existence of one-way functions. According to Theorem 4, we also directly proved $P \neq NP$.

Finally, I will briefly introduce the selection of function $f_i (i = 1, 2, 3)$ in [Q2]. In the process of proving that f_i needs to satisfy the condition that for any Y_i , there are at least exponential possible X satisfying $Y_i = f_i(X)$. This is actually very easy to do. For example, if we choose $p = L$, take the i -th bit of each number of $X_1, X_2, \dots, X_L$ to satisfy the condition.

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