A triply cubic polynomials approach for globally convergent algorithm in coefficient inverse problems

Quan-Fang Wang

Mechanical and Automation Engineering,
The Chinese University of Hong Kong, Shatin, N. T., Hong Kong

E-mail: quanfangwang@hotmail.com, qfwang@mae.cuhk.edu.hk

Abstract. In this paper, a triply cubic polynomials approach is proposed firstly for solving the globally convergent algorithm of coefficient inverse problems in three dimension. Using Taylor type expansion for three arguments to construct tripled cubic polynomials for the approximate solution as identifying coefficient function of parabolic initial-boundary problems, in which unknown coefficients appeared at nonlinear term. The presented computational approach is effective to execute in spatial 3D issues arising in real world. Further, the complete algorithm can be straightforwardly performed to lower or higher dimensions case.

1. Background

Worldwidely, current BQP (Bi-quadratic polynomials) approach is used to solve the globally convergent algorithm of coefficient inverse problems in the spatial dimension of two. Since it had been proposed in [1, 2], recognized and named as BQP approach in article [3], fortunately, it already had been adopted in numerous publications such as [4], [5], [6, 7, 8, 9], and so forth. Actually, the methodology is worked to recover the profile of realistic systems (parabolic, hyperbolic, elliptic type) in industrial, electric-magnetic, optical frequency sounding, engineer, medical tomography, physical as well as chemical fields. The question arising in here is for high dimension cases: what suitable approach for three dimension case? whether it possess enough smoothness to simply use BQP approach in three dimension? whether there is exact approach to construct the approximate solution in coefficients inverse problems of 3D or high dimensions? To find the answer, in this paper, we introduce the perfect methodology to make the completeness of globally convergent algorithm in three dimension case. Our idea is using Taylor type expansion for three arguments to construct tripled cubic polynomials for the approximate solution in finding coefficient of parabolic initial-boundary problems, in which unknown coefficient appeared at nonlinear term. Indeed, one can use the provided

triplet cubical polynomials in the sequential minimization algorithm in convexification (convex performance index) approach, it leads to get a stable approximate solution via minimizing finite sequences of strictly convex objective function. Through the posed triple cubic polynomials approach, the determination of unknown coefficient can be effectively solved for a broad class applications, which was restricted in 1D/2D case at [10], [11] and [12]. In addition, recent years reports on adaptive technique (cf. [13, 14, 15, 16]) are indirect method with its limitation.

The purpose of the article is to address the triple cubic polynomials approach for solving coefficient inverse problems in three dimension at the globally convergent algorithm, upon with previous bi-quadratic polynomials methodology given in [3] for two dimension case. The systematical theoretical proposals and computational procedure are feasible in 3-dimensional case. For verifying, numerical demonstration and laboratory experiments are desired as evidences and complementaries in our future works.

We divided this paper into 5 sections. Section 2 is to set the mathematical formulation of the coefficient inverse problem for nonlinear parabolic initial-boundary equation. Section 3 is to propose the triply cubic polynomials approach for globally convergent algorithm associated with parabolic coefficient inverse problems in three dimension case. In Section 4, the convergence and error estimate are proved for the proposed approach and criteria function. Section 5 is to supply the computational procedure for numerical experiments. Lastly, Section 6 contains the conclusions.

2. Preliminaries

In order to interpret the newly triple cubic polynomials approach, firstly, it needs to give the mathematical setting for nonlinear parabolic initial-boundary problems with unknown coefficient appeared in nonlinearity. Consider the Cauchy problem described in the form of nonlinear parabolic partial differential equation at $\mathbf{R}^3 \times (0, \infty)$ for $(\mathbf{x}, t)$, $\mathbf{x} = (x_1, x_2, x_3)$

$$\begin{cases} \frac{\partial u(\mathbf{x}, t)}{\partial t} - \nabla^2 u(\mathbf{x}, t) + k(\mathbf{x})u(\mathbf{x}, t) = -s(\mathbf{x}, t), \\ \frac{\partial u(\mathbf{x}, t)}{\partial \eta} \Big|_{t=0} + au(\mathbf{x}, t)|_{t=0} = u_0(\mathbf{x}), \end{cases} \quad (2.1)$$

where $k(\mathbf{x}) \geq 0$ is an unknown coefficient (usually, it is considered as $k(\mathbf{x}, \mathbf{u})$ without lost of generality and nonlinearity. In this paper, we take $k(\mathbf{x})$. To stress the new approach, the measurement and noise of k is neglected at setting the issue), a is a constant and $\frac{\partial}{\partial \eta}$ represents the unit outer normal derivative in the Robin initial condition. Let

$$\mathbf{R}_+^3 = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 > 0 \text{ and } x_2 > 0 \text{ and } x_3 > 0\}$$

and S denotes the plane $S = \{(x_2, x_3) \mid x_1 = 0\}$. Assume that the source function $s(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(t) \geq 0$ is compactly supported in $\mathbf{R}^3$. Let $\Omega \subset \mathbf{R}_+^3$ be a cubic domain defined by

$$\Omega = \{\mathbf{x} \in \mathbf{R}_+^3 \mid x_1 \in (0, L), x_2 \in (0, L), x_3 \in (0, L)\}.$$

The support of $s(\mathbf{x}, t)$ belongs to the set $\mathbf{R}_+^3 \setminus \Omega$. Using the convexification approach (cf. [11]) to denote the Laplace transform of $u(\mathbf{x}, t)$ by

$$w(\mathbf{x}, s) = \int_0^\infty e^{-st} u(\mathbf{x}, t) dt,$$

where the parameter $s > 0$. Then applying Laplace transform to the Cauchy problem (2.1) to obtain

$$\begin{cases} [s + k(\mathbf{x})]w - \nabla^2 w = S_0(\mathbf{x}, s), \\ \lim_{|\mathbf{x}| \rightarrow \infty} w = 0, \\ \frac{\partial w(\mathbf{x}, s)}{\partial \eta} + aw(\mathbf{x}, s) = w_0(\mathbf{x}, s), \quad \mathbf{x} \in S. \end{cases} \quad (2.2)$$

In here, $\frac{\partial w}{\partial \eta}$ is the outer normal derivative of w at the boundary. The $w_0(\mathbf{x}, s)$ and $S_0(\mathbf{x}, s)$ denote Laplace transforms of u_0 , and the source function $s(x, t)$, respectively. That is

$$\begin{aligned} w_0(\mathbf{x}, 0) &= \mathcal{L}(u_0) = \int_0^\infty e^{-st} u_0(\mathbf{x}) dt, \\ S_0(\mathbf{x}, s) &= \mathcal{L}(s(\mathbf{x}, t)) = \int_0^\infty e^{-st} \delta(\mathbf{x} - \mathbf{x}_0) \delta(t) dt = \delta(\mathbf{x} - \mathbf{x}_0), \end{aligned}$$

where $\mathcal{L}$ represents the Laplace transform operator. For simplicity, assume that $S_0(\mathbf{x}, s) = \delta(\mathbf{x} - \mathbf{x}_0)$, where $\mathbf{x}_0 = (0, x_2, L - l_0) \in \mathbf{R}_+^3 \setminus \Omega$ is a fixed source position, and l_0 is a small positive number. For the positivity of the function $w(\mathbf{x}, s)$, refer Lemma 1 and Lemma 2 of [11].

Definition 2.1. *For the system (2.1), given data $s(\mathbf{x}, t)$ on $w(\mathbf{x}, s)$, $\mathbf{x} \in S$ for a fixed source position $\mathbf{x}_0$ and sufficiently large T . The coefficient inverse problem (CIP) is to identify the unknown coefficient $k(\mathbf{x})$ in domain Ω .*

Remark 2.2. *Definition 2.1 provides the existence and uniqueness of solution $u(\mathbf{x}, t)$. For unknown coefficients $k(\mathbf{x})$, it provide the existence but it can not guarantee the uniqueness.*

Next, we intend to deduce equivalent equation without the unknown coefficient $k(\mathbf{x})$ from the nonlinear parabolic equation (2.1). Cite [3] for two dimension case, and refer Lemma 2 of [11] for 1D case. Actually, whether the source function $S_0(\mathbf{x}, t)$ exist or not, it is convenient to neglect this term in following deduces. One can complement

this term if necessary, the dealing is appeared in papers [10, 11, 12, 13, 14, 15, 16]. For $w > 0$ in Ω , set $v = \ln w$ and transform (2.2) as

$$\nabla^2 v + (\nabla v)^2 = s + k(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (2.3)$$

For the explanation of transformation method (see [3] for different selection), if introducing

$$f(\mathbf{x}, s) = v, \quad q(\mathbf{x}, s) = \frac{\partial f}{\partial s},$$

then the formulation (2.3) and Lemma 2 of [11] implies that

$$f(\mathbf{x}, s) = O(s^1), \quad q(\mathbf{x}, s) = O(1) \quad \text{as } s \rightarrow \infty.$$

The integral on (s, ∞) can be approximated as

$$f(\mathbf{x}, s) \approx - \int_s^{\bar{s}} q(\mathbf{x}, \nu) d\nu, \quad (2.4)$$

where $\bar{s}$ is a sufficiently large number (cf. [11]) for getting resultant value. Take $s > s_0 > 0$, for $\mathbf{x} \in \Omega$, $\Gamma = \partial\Omega$ and $s \in (s_0, \bar{s})$, using the integral on corresponding time variable ν , and substituting (2.4) to (2.3), the boundary problem (2.2) equivalent to the following nonlinear integro-differential system

$$\begin{cases} \int_s^{\bar{s}} \nabla^2 q(\mathbf{x}, \nu) d\nu + \left(\int_s^{\bar{s}} \nabla q(\mathbf{x}, \nu) d\nu \right)^2 - s = 0, \\ \frac{\partial q(\mathbf{x}, s)}{\partial \eta} + a q(\mathbf{x}, s) = q_0(\mathbf{x}, s), \quad \mathbf{x} \in \Gamma. \end{cases} \quad (2.5)$$

In here, q_0 correspond to w_0 . Obviously, $\mathbf{R}_+^3$ is open set, the normal derivative of (2.5) is valid on the entire domain. The boundary condition is transformed from (2.2), the following is details.

$$\begin{aligned} \text{Set } \frac{\partial v}{\partial \eta} &= \frac{\partial(\ln w)}{\partial \eta}; v = \ln w; v_0 = \ln w_0, & \text{then } \frac{\partial v}{\partial \eta} + a v &= v_0. \\ \text{Set } f(\mathbf{x}, s) &= v, & \text{then } \frac{\partial f(\mathbf{x}, s)}{\partial \eta} + a f(\mathbf{x}, s) &= f_0(\mathbf{x}, s). \\ \text{Set } \frac{\partial}{\partial \eta} \left(\frac{\partial f}{\partial s} \right); q(\mathbf{x}, s) &= \frac{\partial f}{\partial s}; q_0(\mathbf{x}, s) = \frac{\partial f_0}{\partial s}, & \text{then } \frac{\partial q(\mathbf{x}, s)}{\partial \eta} + a q(\mathbf{x}, s) &= q_0(\mathbf{x}, s). \end{aligned}$$

For numerical computation, q_0 can be calculated by using regularizing scheme of numerical differential on s . Above integral make the $k(\mathbf{x})$ disappeared, resultantly, (2.5) is the expected system without containing the unknown coefficient $k(\mathbf{x})$. Define the following set of functions

$$G(n') = \left\{ q(\mathbf{x}, s) \mid \frac{\partial q}{\partial \eta} + a q = q_0, \quad \mathbf{x} \in \Gamma, \quad \max_{s \in [s_0, \bar{s}]} \|q(\mathbf{x}, s)\|_{C^3(\Omega)} \leq n' \right\},$$

where $n' > 0$ is a given number.

Theorem 2.3. *There exists at most one solution $q(\mathbf{x}, s)$ to problem (2.5) such that*

$$\max_{s \in [s_0, \bar{s}]} \|q(\mathbf{x}, s)\|_{C^3(\Omega)} < \infty.$$

The proof of Theorem 2.3 can be obtained using the result of [11].

3. Triply cubic polynomials approach

3.1. Integro-differential vectors system

Suppose a regular domain $\Omega = [0, L] \times [0, L] \times [0, L]$ for $L > 0$. Divide Ω by the partition of $N = n \times n \times n$ cubic lattices with n number intervals at each direction of $x_i, i = 1, 2, 3$. Set $\Omega_1 = \{ (x_1, x_2, x_3) \mid -\mathbf{R} \leq x_1, x_2, x_3 \leq \mathbf{R} \}$ for fixed real number $\mathbf{R} > 0$. Assume that q_0 is observed on $\Gamma = \partial\Omega_1 \approx \Omega \cap S$. Take $\{\phi_k(x_1, x_2, x_3)\}_{n=1}^{\mathbf{n}} \subset C^3(\bar{\Omega}_1)$ as a set of cubic basis functions that approximate $q(\mathbf{x}, s)$ and its $\mathbf{x}$ -derivatives up to the third order, i.e.,

$$D_{\mathbf{x}}^{\alpha} q(\mathbf{x}, s) \approx D_{\mathbf{x}}^{\alpha} \sum_{n=1}^{\mathbf{n}} \eta_n(s) \phi_n(x_1, x_2, x_3), \quad (\mathbf{x}, s) \in \Omega \times [s_0, \bar{s}], \quad |\alpha| \leq 3, \quad (3.1)$$

where $\eta_n(s), n = 1, 2, \dots, \mathbf{n}$ is the expansion coefficients depended on s . Then from Theorem 2.3, for $q \in G(n')$, there is a number $\mathbf{n}(\varepsilon, n')$ can be chosen corresponding to ε, n' for sufficiently small $\varepsilon > 0$ and $\eta_k(s)$, such that

$$\max_{s \in (s_0, \bar{s})} \left\| q(\mathbf{x}, s) - \sum_{n=1}^{\mathbf{n}(\varepsilon, n')} \eta_n(s) \phi_n(x_1, x_2, x_3) \right\|_{C^3(\Omega)} < \varepsilon.$$

With the results in [3] on two dimension parabolic coefficient inverse problems for space variables (x_2, x_3) and time variable s , suppose

$$q(\mathbf{x}, s) \approx \sum_{n=1}^{\mathbf{n}} \eta_n(s) \phi_n(x_1, x_2, x_3), \quad (\mathbf{x}, s) \in \Omega \times [s_0, \bar{s}].$$

Then, one can deduced that

$$\nabla^2 q(\mathbf{x}, s) = \sum_{n=1}^{\mathbf{n}} \eta_n(s) \frac{\partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_1^2} + \sum_{n=1}^{\mathbf{n}} \eta_n(s) \frac{\partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_2^2} + \sum_{n=1}^{\mathbf{n}} \eta_n(s) \frac{\partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_3^2}.$$

That is $\nabla^2 q(\mathbf{x}, s) = \sum_{n=1}^{\mathbf{n}} \eta_n(s) \nabla^2 \phi_n(x_1, x_2, x_3)$. Substituting (3.1) into (2.5), therefore, the initial-boundary parabolic problem (2.5) imply that

$$\begin{cases} \int_s^{\bar{s}} \sum_{n=1}^{\mathbf{n}} \eta_n(\nu) \nabla^2 \phi_n(x_1, x_2, x_3) d\nu + \left(\int_s^{\bar{s}} \sum_{n=1}^{\mathbf{n}} \eta_n(\nu) \nabla \phi_n(x_1, x_2, x_3) d\nu \right)^2 - s = 0, \\ \sum_{n=1}^{\mathbf{n}} \eta_n(s) \frac{\partial \phi_n(x_1, x_2, x_3)}{\partial \eta} + a \sum_{n=1}^{\mathbf{n}} \eta_n(s) \phi_n(x_1, x_2, x_3) = \sum_{n=1}^{\mathbf{n}} \eta_n^0(s) \phi_n(x_1, x_2, x_3), \end{cases} \quad (3.2)$$

where

$$q_0(\mathbf{x}, s) \approx \sum_{n=1}^{\mathbf{n}} \eta_n^0(s) \phi_n(x_1, x_2, x_3) \quad (3.3)$$

is the expansion of boundary value $q_0(\mathbf{x}, s)$ according to the basis functions $\{\phi_n(x_1, x_2, x_3)\}$ on boundary point $\mathbf{x} = (x_1, x_2, x_3) \in \Gamma$, and $\eta_n^0(s)$ are their expansion coefficients for $n = 1, 2, \dots, \mathbf{n}$. Using the vector form to denote that

$$\mathbf{p}(x_1, x_2, x_3, s) = \begin{pmatrix} \eta_1(s) \phi_1(x_1, x_2, x_3) \\ \vdots \\ \eta_n(s) \phi_n(x_1, x_2, x_3) \\ \vdots \\ \eta_{\mathbf{n}}(s) \phi_{\mathbf{n}}(x_1, x_2, x_3) \end{pmatrix}.$$

For clearly, the detailed first and second order derivatives of $\mathbf{p}(x_1, x_2, x_3, s)$ are given by

$$\begin{aligned} \mathbf{p}'_1 &= \frac{\partial \mathbf{p}}{\partial x_1} = \left(\frac{\eta_1(s) \partial \phi_1(x_1, x_2, x_3)}{\partial x_1}, \dots, \frac{\eta_n(s) \partial \phi_n(x_1, x_2, x_3)}{\partial x_1}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_1} \right)^T, \\ \mathbf{p}'_2 &= \frac{\partial \mathbf{p}}{\partial x_2} = \left(\frac{\eta_1(s) \partial \phi_1(x_1, x_2, x_3)}{\partial x_2}, \dots, \frac{\eta_n(s) \partial \phi_n(x_1, x_2, x_3)}{\partial x_2}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_2} \right)^T, \\ \mathbf{p}'_3 &= \frac{\partial \mathbf{p}}{\partial x_3} = \left(\frac{\eta_1(s) \partial \phi_1(x_1, x_2, x_3)}{\partial x_3}, \dots, \frac{\eta_n(s) \partial \phi_n(x_1, x_2, x_3)}{\partial x_3}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_3} \right)^T, \\ \mathbf{p}''_1 &= \frac{\partial^2 \mathbf{p}}{\partial x_1^2} = \left(\frac{\eta_1(s) \partial^2 \phi_1(x_1, x_2, x_3)}{\partial x_1^2}, \dots, \frac{\eta_n(s) \partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_1^2}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial^2 \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_1^2} \right)^T, \\ \mathbf{p}''_2 &= \frac{\partial^2 \mathbf{p}}{\partial x_2^2} = \left(\frac{\eta_1(s) \partial^2 \phi_1(x_1, x_2, x_3)}{\partial x_2^2}, \dots, \frac{\eta_n(s) \partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_2^2}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial^2 \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_2^2} \right)^T, \\ \mathbf{p}''_3 &= \frac{\partial^2 \mathbf{p}}{\partial x_3^2} = \left(\frac{\eta_1(s) \partial^2 \phi_1(x_1, x_2, x_3)}{\partial x_3^2}, \dots, \frac{\eta_n(s) \partial^2 \phi_n(x_1, x_2, x_3)}{\partial x_3^2}, \dots, \frac{\eta_{\mathbf{n}}(s) \partial^2 \phi_{\mathbf{n}}(x_1, x_2, x_3)}{\partial x_3^2} \right)^T. \end{aligned}$$

Use the notations $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3)$, $\mathbf{p}'' = (\mathbf{p}''_1, \mathbf{p}''_2, \mathbf{p}''_3)$ if necessary. For unified framework of the parabolic initial-boundary problem, roughly set the integro-differential operator $\mathbf{L}$ expressed as

$$\mathbf{L}(\mathbf{p}) = F \left(\int_s^{\bar{s}} \mathbf{p}''(x_1, x_2, x_3, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}'(x_1, x_2, x_3, \nu) d\nu, x, y, z, s \right). \quad (3.4)$$

Resultantly, from (3.2) and (3.4) to yield that

$$\begin{cases} \mathbf{L}(\mathbf{p}) = 0, \quad \mathbf{p} = \mathbf{p}(x_1, x_2, x_3, s), \quad x_1 \in (0, L), \quad x_2 \in (0, L), \quad x_3 \in (0, L), \quad s \in (s_0, \bar{s}), \\ \frac{\partial \mathbf{p}(x_1, x_2, x_3, s)}{\partial \eta} + a \mathbf{p}(x_1, x_2, x_3, s) = \mathbf{p}_0(x_1, x_2, x_3, s), \quad (x_1, x_2, x_3) \in \Gamma. \end{cases} \quad (3.5)$$

In (3.5), $\mathbf{p}_0$ is a vector corresponding to the initial value q_0 in (2.5). In details, that is

$$\mathbf{p}_0(x_1, x_2, x_3, s) = \begin{pmatrix} \eta_1^0(s)\phi_1(x_1, x_2, x_3) \\ \vdots \\ \eta_n^0(s)\phi_n(x_1, x_2, x_3) \\ \vdots \\ \eta_{\mathbf{n}}^0(s)\phi_{\mathbf{n}}(x_1, x_2, x_3) \end{pmatrix},$$

where $\eta_n^0(s), n = 1, 2, \dots, \mathbf{n}$ are coefficients of q_0 given in (3.3).

3.2. Extrapolated boundary condition

Eventually, the third kind boundary condition in (2.1) can be converted to Dirichlet boundary condition. In here, we omit the redundancies for the limitation of paper length. In short, it is instructed for three dimension case initially.

Let $G \subset \mathbf{R}^3$ be a cube large than domain $\Omega = [0, L] \times [0, L] \times [0, L]$, $\Omega \subset G$, and 0 be the center of G . Select such a G , for any point $\mathbf{x} \in G$ which not near a corner, there exists a unique point $\mathbf{x}' \in \partial\Omega$ such that the outward distance in normal direction is given by $|\mathbf{x} - \mathbf{x}'| = a_0 l_t$ for $a_0 > 0$. Then ∂G is an extrapolated boundary of 3D domain Ω . Using the extrapolated boundary in nonlinear initial-boundary parabolic system (2.1) of $u(\mathbf{x}, t)$ to get

$$\frac{\partial u}{\partial \eta}(\mathbf{x}, t) = \lim_{\alpha \rightarrow 0} \frac{u(\mathbf{x} + \alpha \eta, t) - u(\mathbf{x}, t)}{\alpha}.$$

Since $\mathbf{x} + a_0 l_t \eta \in \partial G$, it implies $u(\mathbf{x} + a_0 l_t \eta, t) = 0$. If the number $a_0 l_t$ is sufficiently-small, then $\frac{\partial u}{\partial \eta}(\mathbf{x}, t) \approx -\frac{1}{a_0 l_t} u(\mathbf{x}, t)$. Assume that

$$\frac{\partial u}{\partial \eta}(\mathbf{x}, t) = -\frac{1}{a_0 l_t} u(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T). \quad (3.6)$$

In the case of the extrapolated boundary condition $\frac{\partial u}{\partial \eta}|_{\partial\Omega} \equiv 0$, then condition (3.6) is equivalent to the third kind boundary condition $\frac{\partial u(\mathbf{x}, t)}{\partial \eta} + a u(\mathbf{x}, t) = 0$ on $\partial\Omega$ for $t \in (0, T)$, where $a = 1/(a_0 l_t)$. That is, using (3.6), the original third kind boundary condition is convert to Dirichlet boundary condition $u(\mathbf{x}, t)|_{t=0} = -a_0 l_t u_0(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$.

3.3. Triply cubic polynomials

To propose the triply cubic approach in the construction of approximate solution, discretise the coefficient inverse problems (3.2) on the cubic domain $\Omega = [0, L] \times$

$[0, L] \times [0, L]$. Individually, set the cubical partition lattices with the coordinate in $x = x_1, y = x_2$ and $z = x_3$ direction as

$$0 = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_N = L, \quad h^i = x_i - x_{i-1}, \quad i = 1, 2, \dots, N.$$

$$0 = y_0 < y_1 < \dots < y_{j-1} < y_j < \dots < y_N = L, \quad h^j = y_j - y_{j-1}, \quad j = 1, 2, \dots, N.$$

$$0 = z_0 < z_1 < \dots < z_{k-1} < z_k < \dots < z_N = L, \quad h^k = z_k - z_{k-1}, \quad k = 1, 2, \dots, N.$$

In here, h^i, h^j, h^k are the partition interval lengths at each direction x, y and z , respectively. For domain Ω , suppose that the triply cubic polynomial at each lattice is a vector $\mathbf{p}_{ijk}(x, y, z, s)$, their indices are $j = 1, 2, \dots, N, j = 1, 2, \dots, N$ and $k = 1, 2, \dots, N$. Its detail is

$$\mathbf{p}_{ijk}(x, y, z, s) = (p_{ijk}^1(x, y, z, s), \dots, p_{ijk}^n(x, y, z, s), \dots, p_{ijk}^{\mathbf{n}}(x, y, z, s))$$

for $\mathbf{n} = N^3$. Without confusion, each cubic lattice is denoted as C_{ijk} corresponding polynomial vector $\mathbf{p}_{ijk}$. Specially, the cubic lattices with triply cubic polynomials numbers are plotted in Figure 1.

Figure 1. Cubic lattices with corresponding triply cubic polynomials (the structure is analogical a colorful cubical magic square)

We list its front (blue), above (cyan) and right (pink) numbers in Table 1, Table 2, Table 3, and omit others inside cubic lattices numbers.

Table 1. Front (blue) grids of cubic lattices in **Figure 1**.

p_{11N}	$\cdots$	p_{nN1}	$\cdots$	p_{NN1}
$\vdots$		$\vdots$		$\vdots$
p_{11n}	$\cdots$	p_{nn1}	$\cdots$	p_{Nn1}
$\vdots$		$\vdots$		$\vdots$
p_{111}	$\cdots$	p_{n11}	$\cdots$	p_{N11}

Table 2. Above (cyan) grids of cubic lattices in **Figure 1**.

p_{1NN}	$\cdots$	p_{nNN}	$\cdots$	p_{NNN}
$\vdots$		$\vdots$		$\vdots$
p_{1nN}	$\cdots$	p_{nnN}	$\cdots$	p_{NnN}
$\vdots$		$\vdots$		$\vdots$
p_{11N}	$\cdots$	p_{nN1}	$\cdots$	p_{NN1}

Table 3. Right (pink) grids of cubic lattices in **Figure 1**.

p_{NN1}	$\cdots$	p_{NnN}	$\cdots$	p_{NNN}
$\vdots$		$\vdots$		$\vdots$
p_{Nn1}	$\cdots$	p_{Nnn}	$\cdots$	p_{NNn}
$\vdots$		$\vdots$		$\vdots$
p_{N11}	$\cdots$	p_{Nn1}	$\cdots$	p_{NN1}

The idea of this paper is to construct the approximate solution with the Taylor type expansion of 3 variables. Using a Taylor type expansion for three arguments to a function $f^T(x, y, z, s)$ (upper index ‘T’ represents ‘Taylor expansion’) at point (x_0, y_0, z_0, s) . Since the original Taylor expansion contains 64 terms, and the leading term $(x - x_0)^3(y - y_0)^3(z - z_0)^3$ is 9 order polynomial. In order to reduce the complexity of computation, a Taylor type expansion is used in our calculations with 20 terms, and

its leading terms $(x - x_0)^3$, $(y - y_0)^3$, $(z - z_0)^3$, and $(x - x_0)(y - y_0)(z - z_0)$ are 3 order polynomials. The expanded polynomial in detail is in the form of

$$\begin{aligned}
& f^T(x, y, z, s) \\
= & \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^3} \frac{(x - x_0)^3}{3!} + \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y^3} \frac{(y - y_0)^3}{3!} + \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial z^3} \frac{(z - z_0)^3}{3!} \\
& + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^2 \partial y} \frac{(x - x_0)^2}{2!} (y - y_0) + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial y^2} (x - x_0) \frac{(y - y_0)^2}{2!} \\
& + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^2 \partial z} \frac{(x - x_0)^2}{2!} (z - z_0) + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y^2 \partial z} \frac{(y - y_0)^2}{2!} (z - z_0) \\
& + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial z^2} (x - x_0) \frac{(z - z_0)^2}{2!} + 3 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y \partial z^2} (y - y_0) \frac{(z - z_0)^2}{2!} \\
& + 6 \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial y \partial z} (x - x_0)(y - y_0)(z - z_0) \\
& + \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x^2} \frac{(x - x_0)^2}{2!} + \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial y^2} \frac{(y - y_0)^2}{2!} \\
& + \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial z^2} \frac{(z - z_0)^2}{2!} + 2 \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x \partial y} (x - x_0)(y - y_0) \\
& + 2 \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x \partial z} (x - x_0)(z - z_0) + 2 \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial y \partial z} (y - y_0)(z - z_0) \\
& + \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial x} (x - x_0) + \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial y} (y - y_0) \\
& + \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial z} (z - z_0) + f^T(x_0, y_0, z_0, s). \tag{3.7}
\end{aligned}$$

Re-denote each coefficient functions of the expanded Taylor type polynomial (3.7) as

$$\begin{aligned}
a_1(s) &= \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial x}, a_2(s) = \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial y}, a_3(s) = \frac{\partial f^T(x_0, y_0, z_0, s)}{\partial z}, \\
a_{11}(s) &= \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x^2}, a_{22}(s) = \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial y^2}, a_{33}(s) = \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial z^2}, \\
a_{12}(s) &= \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x \partial y}, a_{23}(s) = \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial y \partial z}, a_{13}(s) = \frac{\partial^2 f^T(x_0, y_0, z_0, s)}{\partial x \partial z}, \\
a_{111}(s) &= \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^3}, a_{222}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y^3}, a_{333}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial z^3}, \\
a_{112}(s) &= \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^2 \partial y}, a_{122}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial z^2}, a_{113}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x^2 \partial z}, \\
a_{223}(s) &= \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y^2 \partial z}, a_{133}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial z^2}, a_{233}(s) = \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial y \partial z^2}, \\
a_{123}(s) &= \frac{\partial^3 f^T(x_0, y_0, z_0, s)}{\partial x \partial y \partial z}. \tag{3.8}
\end{aligned}$$

Therefore, using (3.8), the Taylor type expansion (3.7) can be rewritten as

$$\begin{aligned}
& f^T(x, y, z, s) \\
= & a_{111}(s) \frac{(x-x_0)^3}{3!} + a_{222}(s) \frac{(y-y_0)^3}{3!} + a_{333}(s) \frac{(z-z_0)^3}{3!} \\
& + 3a_{112}(s) \frac{(x-x_0)^2}{2!} (y-y_0) + 3a_{122}(s) (x-x_0) \frac{(y-y_0)^2}{2!} \\
& + 3a_{113}(s) \frac{(x-x_0)^2}{2!} (z-z_0) + 3a_{223}(s) \frac{(y-y_0)^2}{2!} (z-z_0) \\
& + 3a_{133}(s) (x-x_0) \frac{(z-z_0)^2}{2!} + 3a_{233}(s) (y-y_0) \frac{(z-z_0)^2}{2!} \\
& + 6a_{123}(s) (x-x_0) (y-y_0) (z-z_0) \\
& + a_{11}(s) \frac{(x-x_0)^2}{2!} + a_{22}(s) \frac{(y-y_0)^2}{2!} + a_{33}(s) \frac{(z-z_0)^2}{2!} \\
& + 2a_{12}(s) (x-x_0) (y-y_0) + 2a_{13}(s) (x-x_0) (z-z_0) + 2a_{23}(s) (y-y_0) (z-z_0) \\
& + a_1(s) (x-x_0) + a_2(s) (y-y_0) + a_3(s) (z-z_0) + f^T(x_0, y_0, z_0, s). \tag{3.9}
\end{aligned}$$

Using the triply cubic expansion (3.9), the component p_{ijk} of the vector function $\mathbf{p}_{ijk}(x_1, x_2, x_3, s)$ can be expressed at per cubic lattice C_{ijk} in the form of:

$$\begin{aligned}
& \mathbf{p}_{ijk}(x, y, z, s) \approx p_{ijk}^n(x, y, z, s) \\
= & a_{111}(s) \frac{(x-x_{i-1})^3}{3!} + a_{222}(s) \frac{(y-y_{j-1})^3}{3!} + a_{333}(s) \frac{(z-z_{k-1})^3}{3!} \\
& + 3a_{112}(s) \frac{(x-x_{i-1})^2}{2!} (y-y_{j-1}) + 3a_{122}(s) (x-x_{i-1}) \frac{(y-y_{j-1})^2}{2!} \\
& + 3a_{113}(s) \frac{(x-x_{i-1})^2}{2!} (z-z_{k-1}) + 3a_{223}(s) \frac{(y-y_{j-1})^2}{2!} (z-z_{k-1}) \\
& + 3a_{133}(s) (x-x_{i-1}) \frac{(z-z_{k-1})^2}{2!} + 3a_{233}(s) (y-y_{j-1}) \frac{(z-z_{k-1})^2}{2!} \\
& + 6a_{123}(s) (x-x_{i-1}) (y-y_{j-1}) (z-z_{k-1}) \\
& + a_{11}(s) \frac{(x-x_{i-1})^2}{2!} + a_{22}(s) \frac{(y-y_{j-1})^2}{2!} + a_{33}(s) \frac{(z-z_{k-1})^2}{2!} \\
& + 2a_{12}(s) (x-x_{i-1}) (y-y_{j-1}) + 2a_{13}(s) (x-x_{i-1}) (z-z_{k-1}) \\
& + 2a_{23}(s) (y-y_{j-1}) (z-z_{k-1}) + a_1(s) (x-x_{i-1}) + a_2(s) (y-y_{j-1}) \\
& + a_3(s) (z-z_{k-1}) + p_{ijk}^n(x_{i-1}, y_{j-1}, z_{k-1}, s) \tag{3.10}
\end{aligned}$$

for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, N$, $k = 1, 2, \dots, N$ and $n = 1, 2, \dots, \mathbf{n}$. Via the Taylor type expansion (3.10) of $\mathbf{p}_{ijk}^n$ at point (x_0, y_0, z_0, s) with the coefficients as in (3.8), it is naturally to assume the $a_{ijk}(s), a_{ij}(s), a_i(s)$, which need be identified, take the form of

$$a_i(s) = \alpha_i s + \alpha_0, \quad i = 1, 2, 3 \tag{3.11}$$

$$a_{ij}(s) = \beta_{ij} s^2 + \beta_i s + \beta_0, \quad i = 1, 2, 3. \quad j = 1, 2, 3. \tag{3.12}$$

$$a_{ijk}(s) = \gamma_{ijk} s^3 + \gamma_{ij} s^2 + \gamma_i s + \gamma_0, \quad i = 1, 2, 3. \quad j = 1, 2, 3. \quad k = 1, 2, 3. \tag{3.13}$$

for constants $\alpha_i, \alpha_0 \in R^1$, $\beta_{ij}, \beta_i, \beta_0 \in R^1$ and $\gamma_{ijk}, \gamma_{ij}, \gamma_i, \gamma_0 \in R^1$, where R^1 is real number set. The indeterminated coefficient constants in (3.11), (3.12) and (3.13) can be calculated at each cubic lattice C_{ijk} .

Remark 3.1. *Clearly, the idea can be extended to different polynomials expansion.*

Remark 3.2. *The comparison with existing methods is not considered into this paper yet. To compare the convergence order, error estimation and occupied CPU memory usage with indirect approach are left to future works.*

3.4. Objective function minimization

Citing [3] to do the minimization for a proper objective function given in the form of quadratic criteria as

$$J(\mathbf{p}) = \int_0^\infty \int_\Omega \left| \mathbf{L}(\mathbf{p}(\mathbf{x}, t)) \right|^2 C_\lambda^2(\mathbf{x}) \, d\mathbf{x} dt. \quad (3.14)$$

That is

$$J(\mathbf{p}) = \int_0^\infty \int_0^L \int_0^L \int_0^L \left| \mathbf{L}(\mathbf{p}(x, y, z, t)) \right|^2 C_\lambda^2(x, y, z) \, dx dy dz dt,$$

where Carleman's weighted function $C_\lambda(x, y, z) = \exp(-\lambda x - \lambda y - \lambda z)$, which appeared in Carleman estimates (cf. [17]) for the operators $\frac{d^2}{dx^2}, \frac{d^2}{dy^2}, \frac{d^2}{dz^2}$, and λ is parameter. Corresponding to partition C_{ijk} on spatial domain Ω , the criteria functional (3.14) is equivalent to

$$J(\mathbf{p}) = \lim_{\bar{s} \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N J_{\lambda, i, j, k}(\mathbf{p}), \quad (3.15)$$

where resulting functional in (3.15) at each cubic lattice with respect to the (i, j, k) -th coefficients $\alpha_i(s)$, $\alpha_{ij}(s)$, $\alpha_{ijk}(s)$ of the triply cubic polynomials is:

$$J_{\lambda, i, j, k}(\mathbf{p}) = \int_{s_0}^{\bar{s}} \int_{z_{k-1}}^{z_k} \int_{y_{j-1}}^{y_j} \int_{x_{i-1}}^{x_i} \left| \mathbf{L}(\mathbf{p}_{ijk}(x, y, z, t)) \right|^2 C_{\lambda, i, j, k}^2(x, y, z) \, dx dy dz dt, \quad (3.16)$$

where $C_{\lambda, i, j, k}(x, y, z) = \exp[-\lambda(x - x_{i-1}) - \lambda(y - y_{j-1}) - \lambda(z - z_{k-1})]$ are Carleman's weighted function, and $s_0 = 0$ or $s_0 > 0$. The calculation of $J_{\lambda, i, j, k}$ can be implemented by using Gauss-Legendre quadrature (cf. [3, 18]) as

$$\hat{J}_{\lambda, \mathbf{i}, \mathbf{j}, \mathbf{k}} = \sum_{\mathbf{t}=1}^{\mathbf{T}} w_{\mathbf{t}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{k}=1}^{\mathbf{K}} w_{\mathbf{k}} \left| \mathbf{L}(\mathbf{p}_{ijk}(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, \mathbf{t})) \right|^2 C_{\lambda, \mathbf{i}, \mathbf{j}, \mathbf{k}}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}), \quad (3.17)$$

where $w_{\mathbf{i}}, w_{\mathbf{j}}, w_{\mathbf{k}}, w_{\mathbf{t}}$ are Gauss weights, and $x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, \mathbf{t}$ are Gauss points at x, y, z and t direction, respectively. $\mathbf{T}, \mathbf{I}, \mathbf{J}, \mathbf{K}$ are the maximum number of indices $\mathbf{t}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ for Gauss points at each direction t, x, y, z . Take the usual norms as

$$\|a_i\| = \max_{s \in [s_0, \bar{s}]} |a_i(s)|, \quad \|a_{ij}\| = \max_{s \in [s_0, \bar{s}]} |a_{ij}(s)|, \quad \|a_{ijk}\| = \max_{s \in [s_0, \bar{s}]} |a_{ijk}(s)|.$$

Use the notation $\mathbf{n}$ without confusion with n . The sequential minimizer of $J_{\lambda,i,j,k}(p_{ijk}^n)$ can be searched on a bounded set

$$G_{ijk}(\mathbf{n}) = \left\{ (a_i(s), a_{ij}(s), a_{ijk}(s)) \mid \|a_i\| \leq \mathbf{n}, \|a_{ij}\| \leq \mathbf{n}, \|a_{ijk}\| \leq \mathbf{n} \right\}$$

for a positive number $\mathbf{n} > 0$. The corresponding set of p_{ijk}^n is denoted by

$$G(\mathbf{n}) = \left\{ p_{ijk}^n(s) \mid \|p_{ijk}^n\| \leq \mathbf{n} \right\}.$$

Using the result of [11] and the convexity of $J_{\lambda,i,j,k}(\mathbf{p})$, the uniqueness of minimizer for the functional (3.16) and the convergence of posed approach can be proven.

Definition 3.3. Suppose that there exists a vector function $\mathbf{p}^*(x, y, z, s)$, such that

$$\begin{cases} \hat{\mathbf{L}}(\mathbf{p}^*) = 0, \mathbf{p}^* = \mathbf{p}^*(x, y, z, s), & x \in (0, L), y \in (0, L), z \in (0, L), s \in (s_0, \bar{s}), \\ \frac{\partial \mathbf{p}^*(x, y, z, s)}{\partial \eta} + a\mathbf{p}^*(x, y, z, s) = \mathbf{p}_0^*(s), & (x, y, z) \in \Gamma. \end{cases}$$

where

$$\hat{\mathbf{L}}(\mathbf{p}^*) \equiv F^* \left(\int_s^{\bar{s}} \mathbf{p}^{*''}(x, y, z, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}^{*'}(x, y, z, \nu) d\nu, x, y, z, s \right).$$

Then $\mathbf{p}^*(x, y, z, s)$ is called the exact solution to (3.5).

Assume that $\mathbf{p}^* \in C^3(\Omega)$ and satisfy

$$\max_{s \in [s_0, \bar{s}]} \|\mathbf{p}^*(x, y, z, s)\|_{C^3(\Omega)} \leq c_0 \mathbf{n}, \quad (3.18)$$

where c_0 is constant. It means that $\mathbf{p}^* \in G(c_0 \mathbf{n})$. The function F is approximately known as

$$\begin{aligned} & F \left(\int_s^{\bar{s}} \mathbf{p}''(x, y, z, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}'(x, y, z, \nu) d\nu, x, y, z, s \right) \\ &= F^* \left(\int_s^{\bar{s}} \mathbf{p}^{*''}(x, y, z, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}^{*'}(x, y, z, \nu) d\nu, x, y, z, s \right) + \hat{\varepsilon}(x, y, z, s), \end{aligned}$$

where $\hat{\varepsilon}(x, y, z, s)$ is error function, and

$$\max_{s \in [s_0, \bar{s}]} \|\hat{\varepsilon}(x, y, z, s)\|_{C^3(\Omega)} \leq \varepsilon, \quad \max_{s \in [s_0, \bar{s}]} |\mathbf{p}_0(s) - \mathbf{p}_0^*(s)| \leq \varepsilon \quad (3.19)$$

for a constant ε . At each node $(x_{i-1}, y_{j-1}, z_{k-1})$, denote that F and F^* take the form of

$$\begin{aligned} F_{ijk} &= F \left(\int_s^{\bar{s}} \mathbf{p}_{ijk}''(x, y, z, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}_{ijk}'(x, y, z, \nu) d\nu, x, y, z, s \right) \Big|_{\substack{x = x_{i-1}, \\ y = y_{j-1}, \\ z = z_{k-1}}} \\ F_{ijk}^* &= F \left(\int_s^{\bar{s}} \mathbf{p}_{ijk}^{*''}(x, y, z, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}_{ijk}^{*'}(x, y, z, \nu) d\nu, x, y, z, s \right) \Big|_{\substack{x = x_{i-1}, \\ y = y_{j-1}, \\ z = z_{k-1}}} \end{aligned}$$

Theorem 3.4. Assume that above assumptions are satisfied. Let $(\alpha_0, \alpha_{00}, \alpha_{000})$, $(\alpha_i, \alpha_{ij}, \alpha_{ijk})$ and $(\alpha'_i, \alpha'_{ij}, \alpha'_{ijk})$ be the corresponding coefficients of p_0^n , p_{ijk}^n and $p_{ijk}^{n'}$, respectively. Let h_i, h_j, h_k and h_{i0}, h_{j0}, h_{k0} be the discrete lengths of the subintervals at x, y and z direction, respectively. Assume that $(\alpha_0, \alpha_{00}, \alpha_{000}) \in G_{ijk}(\mathbf{n})$. Then there exist 3 sufficiently-small positive numbers $h_{i0}(\mathbf{n})$, $h_{j0}(\mathbf{n})$ and $h_{k0}(\mathbf{n})$ such that for all $h_i \in (0, h_{i0})$, $h_j \in (0, h_{j0})$ and $h_k \in (0, h_{k0})$

$$\begin{aligned} &(\alpha_i, \alpha_{ij}, \alpha_{ijk}), (\alpha'_i, \alpha'_{ij}, \alpha'_{ijk}) \in G_{ijk}(\mathbf{n}), \\ &p_{ijk}^n(x_{i-1}, y_{j-1}, z_{k-1}, s), p_{ijk}^{n'}(x_{i-1}, y_{j-1}, z_{k-1}, s) \in G(\mathbf{n}), \\ &F_{ijk} \in G(c_0\mathbf{n}), \quad i = 1, 2, \dots, N, \quad j = 1, 2, \dots, N, \quad k = 1, 2, \dots, N. \end{aligned}$$

All the functionals $J_{\lambda, i, j, k}(\mathbf{p})$ are strictly convex on $G(\mathbf{n})$. The unique minimizer pair $(\alpha_i^*, \alpha_{ij}^*, \alpha_{ijk}^*)$ of $J_{\lambda, i, j, k}(\mathbf{p}^*)$ is an interior point of $G_{ijk}(\mathbf{n})$. It can be regarded as the solution of operation equation $\Psi_{\lambda, i, j, k}(\alpha_i, \alpha_{ij}, \alpha_{ijk}) = (\alpha_i^*, \alpha_{ij}^*, \alpha_{ijk}^*)$ with the map: $\Psi_{\lambda, i, j, k} : G_{ijk}(\mathbf{n}) \rightarrow G_{ijk}(\mathbf{n})$. Its corresponding solution $p_{ijk}^{*n}(x, y, z, s)$ is satisfying operator equation $\Phi_{\lambda, i, j, k} p_{ijk}^n(x, y, z, s) = p_{ijk}^{*n}(x, y, z, s)$ with map: $\Phi_{\lambda, i, j, k} : G(\mathbf{n}) \rightarrow G(\mathbf{n})$. Further, operator equation $\Xi_{ijk} F_{ijk} = F_{ijk}^*$ is satisfied for map $\Xi_{ijk} : G(c_0\mathbf{n}) \rightarrow G(c_0\mathbf{n})$.

Citing [1], [2] and [11] to get the complete proof. It is omitted in here.

Theorem 3.4 evident that the convergence of the triply cubic polynomials approach to the exact solution is guaranteed for initial guess $(\alpha_0, \alpha_{00}, \alpha_{000})$ in the set $G_{ijk}(\mathbf{n})$. Further, it briefly verified that the globally convergence of proposed triply cubic polynomials approach in three dimension. To render the details in Section 4.

Theorem 3.5. If the equivalently approximate field $\tilde{f}(\mathbf{x}, \nu)$ can be founded, the unknown coefficient $k(\mathbf{x})$ is calculated by the expression (2.3). The inversion formula for $\tilde{k}(\mathbf{x})$ is deduced as

$$\tilde{k}(\mathbf{x}) = \nabla^2 \tilde{f} + (\nabla \tilde{f})^2 - s, \quad \forall s \in [s_0, \bar{s}]. \quad (3.20)$$

Obviously, there is no need to compute the solutions for getting the $\tilde{k}(\mathbf{x})$ (cf. [11]).

4. Convergence and error estimate

4.1. Convergence order

Theorem 4.1. If all assumptions are held, then the triply cubic polynomials approach is globally converged with the order of $o(h^3)$.

Proof. Firstly, in one coordinate χ case, let $f_\tau(\chi)$ (the index ‘ τ ’ denote the ‘Taylor expansion’) be a function on the interval $\chi \in [a, b]$. The remainder of the Taylor type expansion is as follow:

$$R_\tau = \int_a^b f_\tau(\chi) d\chi - \sum_{n=1}^N A_n f_\tau(\chi_n) = \frac{f_\tau^{(N+1)}(\bar{\chi})}{(N+1)!} \int_a^b w^2(\chi) d\chi, \quad (4.1)$$

where $w(\chi) = (\chi - \chi_1)(\chi - \chi_2) \cdots (\chi - \chi_n)$, $a < \bar{\chi} < b$. The interval $[a, b]$ is taken $[s, \bar{s}]$ as integral intervals and χ is $d\nu$ as variable. In our case $\mathbf{N} = 3$ for cubic polynomials $\mathbf{p}_{ijk}$, the convergence order is $o(h^3)$ if setting $h_i = h_j = h_k = h$ for triple integrals.

Secondly, for a function $g(\chi)$, the error estimate of Gauss-Legendre (GL) quadrature for the interval $[-1, 1]$ is as follows:

$$E_G = \int_{-1}^1 g(\chi) d\chi - \sum_{k=1}^{\kappa} \frac{2g(\xi_k^{(\kappa)})}{(1 - \xi_k^{(\kappa)})[p'_\kappa(\xi_k^{(\kappa)})]^2} + \frac{2^{2\kappa+1}(\kappa!)^4}{(2\kappa+1)[(2\kappa)!]^3} g^{(2\kappa)}(\bar{\chi}), \quad (4.2)$$

where $\xi_k^{(\kappa)}$ are the roots of the GL polynomials $p_\kappa(\chi)$, and $-1 < \bar{\chi} < 1$. For regularity, it need to transform the interval $[-1, 1]$ to $[x_{i-1}, x_i]$, $[y_{j-1}, y_j]$, $[z_{k-1}, z_k]$ and $[s_0, \bar{s}]$ at (3.17). Clearly, the convergence order is $o(h^{2\kappa})$. In the case $\kappa = 6$ (usual take Gauss points $\mathbf{I} = \mathbf{J} = \mathbf{K} = \mathbf{T} = 6$ in numerical experiments) for four variables x, y, z, ν at (3.17), the convergence order is $o(h^{12})$ of $\hat{J}$ if setting $h_i = h_j = h_k = h$ as triple integral intervals and $h_t = h$ as t integral interval.

Combining the results of (4.1) and (4.2), roughly, we obtain the convergence order of our proposed globally convergent algorithm at each cubic lattice as

$$\mathbf{p}(x, y, z, s) \approx \mathbf{p}_{ijk}(x, y, z, s) + o(h^3), \quad J_{\lambda,i,j,k}(\mathbf{p}) \approx \hat{J}_{\lambda,i,j,k}(\mathbf{p}) + o(h^{12}).$$

Therefore, the total convergence order is deduced by

$$\begin{aligned} & \left| J_{\lambda,i,j,k}(\mathbf{p}) - \hat{J}_{\lambda,i,j,k}(\mathbf{p}_{ijk}) \right| \\ & \leq \left| \int_{s_0}^{\bar{s}} \int_{z_{j-1}}^{z_j} \int_{y_{j-1}}^{y_j} \left[\int_{x_{i-1}}^{x_i} |\mathbf{L}(\mathbf{p}(x, y, z, t))|^2 C_{\lambda,i,j,k}^2(x, y, z) dx \right. \right. \\ & \quad \left. \left. - \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y, z, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y, z) \right] dy dz dt \right| \\ & \quad + \left| \int_{s_0}^{\bar{s}} \int_{z_{j-1}}^{z_j} \left[\int_{y_{j-1}}^{y_j} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y, z, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y, z) dy \right. \right. \\ & \quad \left. \left. - \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y_{\mathbf{j}}, z, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z) \right] dz dt \right| \\ & \quad + \left| \int_{s_0}^{\bar{s}} \left[\int_{z_{j-1}}^{z_j} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y_{\mathbf{j}}, z, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z) dz \right. \right. \\ & \quad \left. \left. - \sum_{\mathbf{k}=1}^{\mathbf{K}} w_{\mathbf{k}} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}) \right] dt \right| \\ & \quad + \left| \int_{s_0}^{\bar{s}} \sum_{\mathbf{k}=1}^{\mathbf{K}} w_{\mathbf{k}} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, t))|^2 C_{\lambda,i,j}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}) dt \right| \end{aligned}$$

$$\begin{aligned}
& - \sum_{\mathbf{t}=1}^{\mathbf{T}} w_{\mathbf{t}} \sum_{\mathbf{k}=1}^{\mathbf{K}} w_{\mathbf{k}} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} |\mathbf{L}(\mathbf{p}(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, \mathbf{t}))|^2 C_{\lambda, i, j, k}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}) \\
& + |\hat{J}_{\lambda, i, j, k}(\mathbf{p}) - \hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk})| \\
& \approx O(h^{2\mathbf{I}+1}) + O(h^{2\mathbf{J}+1}) + O(h^{2\mathbf{K}+1}) + O(h^{2\mathbf{T}+1}) + o(h^{12}) \approx o(h^{12}).
\end{aligned} \tag{4.3}$$

Consequently, the triply cubic polynomials approach induced globally convergence order is $o(h^{12})$. It completes the proof of Theorem 4.1.

4.2. Error estimate

Consider the error estimate of proposed triply cubic approach. The exact solution function $\mathbf{p}^*(x, y, z, s)$ in Definition 3.3 can be expanded using Taylor type series at each cube C_{ijk} with the exact form and remainder as

$$\begin{aligned}
\mathbf{p}^*(x, y, z, s) &= a_{111}^*(s) \frac{(x - x_{i-1})^3}{3!} + a_{222}^*(s) \frac{(y - y_{j-1})^3}{3!} + a_{333}^*(s) \frac{(z - z_{k-1})^3}{3!} \\
&+ 3a_{112}^*(s) \frac{(x - x_{i-1})^2}{2!} (y - y_{j-1}) + 3a_{122}^*(s) (x - x_{i-1}) \frac{(y - y_{j-1})^2}{2!} \\
&+ 3a_{113}^*(s) \frac{(x - x_{i-1})^2}{2!} (z - z_{k-1}) + 3a_{223}^*(s) \frac{(y - y_{j-1})^2}{2!} (z - z_{k-1}) \\
&+ 3a_{133}^*(s) (x - x_{i-1}) \frac{(z - z_{k-1})^2}{2!} + 3a_{233}^*(s) (y - y_{j-1}) \frac{(z - z_{k-1})^2}{2!} \\
&+ 6a_{123}^*(s) (x - x_{i-1}) (y - y_{j-1}) (z - z_{k-1}) \\
&+ a_{11}^*(s) \frac{(x - x_{i-1})^2}{2!} + a_{22}^*(s) \frac{(y - y_{j-1})^2}{2!} + a_{33}^*(s) \frac{(z - z_{k-1})^2}{2!} \\
&+ 2a_{12}^*(s) (x - x_{i-1}) (y - y_{j-1}) + 2a_{13}^*(s) (x - x_{i-1}) (z - z_{k-1}) \\
&+ 2a_{23}^*(s) (y - y_{j-1}) (z - z_{k-1}) \\
&+ a_1^*(s) (x - x_{i-1}) + a_2^*(s) (y - y_{j-1}) + a_3^*(s) (z - z_{k-1}) \\
&+ p_{ijk}(x_{i-1}, y_{j-1}, z_{k-1}, s) + R_s \frac{(\bar{x} - x_{i-1})^4 + (\bar{y} - y_{j-1})^4 + (\bar{z} - z_{k-1})^4}{4!}.
\end{aligned} \tag{4.4}$$

In (4.4), $R_s = p^{*(4)}(\bar{x}, \bar{y}, \bar{z}, s)$ is remainder of the Taylor type expansion for exact solution $\mathbf{p}^*(x, y, z, s)$, and $x_{i-1} < \bar{x} < x_i$, $y_{j-1} < \bar{y} < y_j$, $z_{k-1} < \bar{z} < z_k$. By (4.1), (4.2) and (4.3) to get the error estimate as

$$\begin{aligned}
& \left| J_{\lambda, i, j, k}(\mathbf{p}^*) - \hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk}) \right| \\
& \leq \left| J_{\lambda, i, j, k}(\mathbf{p}) - \hat{J}_{\lambda, i, j, k}(\mathbf{p}^*) \right| + \left| \hat{J}_{\lambda, i, j, k}(\mathbf{p}^*) - \hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk}) \right| \\
& \leq \left| \sum_{\mathbf{t}=1}^{\mathbf{T}} w_{\mathbf{t}} \sum_{\mathbf{k}=1}^{\mathbf{K}} w_{\mathbf{k}} \sum_{\mathbf{j}=1}^{\mathbf{J}} w_{\mathbf{j}} \sum_{\mathbf{i}=1}^{\mathbf{I}} w_{\mathbf{i}} \left(|\mathbf{L}(\mathbf{p}^*(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, \mathbf{t}))|^2 - |\mathbf{L}(\mathbf{p}_{ijk}(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}, \mathbf{t}))|^2 \right) \right. \\
& \quad \left. C_{\lambda, i, j, k}^2(x_{\mathbf{i}}, y_{\mathbf{j}}, z_{\mathbf{k}}) \right| + \frac{2^{2\kappa+1}(\kappa!)^4}{(2\kappa+1)[(2\kappa)!]^3} f^{(2\kappa)}(\theta_x, \theta_y, \theta_z, \tau),
\end{aligned} \tag{4.5}$$

where $f(\theta_x, \theta_y, \theta_z, \tau) = |\mathbf{L}(\mathbf{p}_{ijk}(\theta_x, \theta_y, \theta_z, \tau))|^2 C_{\lambda, i, j, k}^2(\theta_x, \theta_y, \theta_z)$ and $x_{i-1} < \theta_x < x_i$, $y_{j-1} < \theta_y < y_j$, $z_{k-1} < \theta_z < z_k$ and $\mathbf{t} - 1 < \tau < \mathbf{t}$. For the functional $\mathbf{L}(\mathbf{p})$ of (4.5), from Taylor type expansion (3.7), (3.18), (3.19) and exact expansion (4.4), one can estimate that

$$\begin{aligned}
& |\mathbf{L}(\mathbf{p}^*(x_i, y_j, z_k, t))|^2 - |\mathbf{L}(\mathbf{p}_{ijk}(x_i, y_j, z_k, t))|^2 \\
&= \left| F\left(\int_s^{\bar{s}} \mathbf{p}^{*''}(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}^{*'}(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right|^2 \\
&\quad - \left| F\left(\int_s^{\bar{s}} \mathbf{p}_{ijk}''(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}_{ijk}'(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right|^2 \\
&= \left| F^*\left(\int_s^{\bar{s}} \mathbf{p}^{*''}(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}^{*'}(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right. \\
&\quad \left. + F_{ijk}\left(\int_s^{\bar{s}} \mathbf{p}_{ijk}''(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}_{ijk}'(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right| \\
&\quad \times \left| F^*\left(\int_s^{\bar{s}} \mathbf{p}^{*''}(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}^{*'}(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right. \\
&\quad \left. - F_{ijk}\left(\int_s^{\bar{s}} \mathbf{p}_{ijk}''(x_i, y_j, z_k, \nu) d\nu, \int_s^{\bar{s}} \mathbf{p}_{ijk}'(x_i, y_j, z_k, \nu) d\nu, x_i, y_j, z_k, s\right) \right| \\
&\leq 4c_0 n'' \hat{\varepsilon}.
\end{aligned} \tag{4.6}$$

Error $\hat{\varepsilon}(x_i, y_j, z_k, \mathbf{t})$ in (4.6) is a sufficiently-small number depended on the node (x_i, y_j, z_k) of C_{ijk} and time $\mathbf{t}$. Notice that $\mathbf{p} = \mathbf{p}_{ijk}$ at cubic lattice C_{ijk} , and $\mathbf{p}_{ijk} \rightarrow \mathbf{p}^*$ as $h \rightarrow 0$ in the order of $o(h^3)$. Thus, from (4.5) and (4.6) to obtain the resultant error estimate as

$$\left| J_{\lambda, i, j, k}(\mathbf{p}^*) - \hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk}) \right| \leq \frac{2^{13}(6!)^4}{13(12!)^3} \mathbf{f} + 4c_0 n'' \mathbf{w}_t \mathbf{w}_k \mathbf{w}_j \mathbf{w}_i \epsilon, \tag{4.7}$$

where $\mathbf{f} = f^{(12)}(\theta_x, \theta_y, \theta_z, \tau)$, $\mathbf{w}_t = \sum_{\mathbf{t}=1}^6 w_t$, $\mathbf{w}_k = \sum_{\mathbf{k}=1}^6 w_k$, $\mathbf{w}_j = \sum_{\mathbf{j}=1}^6 w_j$, $\mathbf{w}_i = \sum_{\mathbf{i}=1}^6 w_i$. That is $\hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk}) \rightarrow J_{\lambda, i, j, k}(\mathbf{p}^*)$ in the order of $o(h^{12})$ with the estimate (4.7).

Consider the error estimate of $\mathbf{p}_{ijk} \rightarrow \mathbf{p}^*$ in the form of norm at cubic lattice C_{ijk} . Assume that there are exact coefficient pair $a_i^*(s), a_{ij}^*(s), a_{ijk}^*(s)$ for exact solution $\mathbf{p}^*(x, y, z, s)$. From (3.11) to suppose the exact coefficients are yielded that

$$a_i^*(s) = \alpha_i^* s + \alpha_0^*, \quad i = 1, 2, 3. \tag{4.8}$$

$$a_{ij}^*(s) = \beta_{ij}^* s^2 + \beta_i^* s + \beta_0^*, \quad i = 1, 2, 3, j = 1, 2, 3. \tag{4.9}$$

$$a_{ijk}^*(s) = \gamma_{ijk}^* s^3 + \gamma_{ij}^* s^2 + \gamma_i^* s + \gamma_0^*, \quad i = 1, 2, 3, j = 1, 2, 3, k = 1, 2, 3. \tag{4.10}$$

By (3.11) and (4.8), there is $\epsilon_1 > 0$ to get estimate as

$$\|a_i(s) - a_i^*(s)\| = \max_{s \in [s_0, \bar{s}]} |a_i(s) - a_i^*(s)| \leq (\alpha_i - \alpha_i^*) \bar{s} + (\alpha_0 - \alpha_0^*) \leq \epsilon_1. \tag{4.11}$$

As the argument in (4.11), there are $\epsilon_2, \epsilon_3 > 0$ for (4.9) and (4.10) such that

$$\|a_{ij}(s) - a_{ij}^*(s)\| \leq (\beta_{ij} - \beta_{ij}^*)\bar{s}^2 + (\beta_i - \beta_i^*)\bar{s} + (\beta_0 - \beta_0^*) \leq \epsilon_2, \quad (4.12)$$

$$\|a_{ijk}(s) - a_{ijk}^*(s)\| \leq (\gamma_{ijk} - \gamma_{ijk}^*)\bar{s}^3 + (\gamma_{ij} - \gamma_{ij}^*)\bar{s}^2 + (\gamma_i - \gamma_i^*)\bar{s} + (\gamma_0 - \gamma_0^*) \leq \epsilon_3. \quad (4.13)$$

Then, the estimate for the triply cubic polynomials is as follows,

$$\begin{aligned} & \|\mathbf{p}^*(x, y, z, s) - \mathbf{p}_{ijk}(x, y, z, s)\| \\ &= \max_{s \in [s_0, \bar{s}]} \max_{0 \leq x, y, z \leq L} |\mathbf{p}^*(x, y, z, s) - \mathbf{p}_{ijk}(x, y, z, s)| \\ &\leq [a_{111}(s) - a_{111}^*(s)] \frac{(x - x_{i-1})^3}{3!} + [a_{222}(s) - a_{222}^*(s)] \frac{(y - y_{j-1})^3}{3!} \\ &\quad + [a_{333}(s) - a_{333}^*(s)] \frac{(z - z_{k-1})^3}{3!} \\ &\quad + 3[a_{112}(s) - a_{112}^*(s)] \frac{(x - x_{i-1})^2}{2!} (y - y_{j-1}) + 3[a_{122}(s) - a_{122}^*(s)] (x - x_{i-1}) \frac{(y - y_{j-1})^2}{2!} \\ &\quad + 3[a_{113}(s) - a_{113}^*(s)] \frac{(x - x_{i-1})^2}{2!} (z - z_{k-1}) + 3[a_{223}(s) - a_{223}^*(s)] \frac{(y - y_{j-1})^2}{2!} (z - z_{k-1}) \\ &\quad + 3[a_{133}(s) - a_{133}^*(s)] (x - x_{i-1}) \frac{(z - z_{k-1})^2}{2!} + 3[a_{233}(s) - a_{233}^*(s)] (y - y_{j-1}) \frac{(z - z_{k-1})^2}{2!} \\ &\quad + 6[a_{123}(s) - a_{123}^*(s)] (x - x_{i-1}) (y - y_{j-1}) (z - z_{k-1}) \\ &\quad + [a_{11}(s) - a_{11}^*(s)] \frac{(x - x_{i-1})^2}{2!} + [a_{22}(s) - a_{22}^*(s)] \frac{(y - y_{j-1})^2}{2!} \\ &\quad + [a_{33}(s) - a_{33}^*(s)] \frac{(z - z_{k-1})^2}{2!} + 2[a_{12}^*(s) - a_{12}^*(s)] (x - x_{i-1}) (y - y_{j-1}) \\ &\quad + 2[a_{13}(s) - a_{13}^*(s)] (x - x_{i-1}) (z - z_{k-1}) + 2[a_{23}(s) - a_{23}^*(s)] (y - y_{j-1}) (z - z_{k-1}) \\ &\quad + [a_1(s) - a_1^*(s)] (x - x_{i-1}) + [a_2(s) - a_2^*(s)] (y - y_{j-1}) + [a_3(s) - a_3^*(s)] (z - z_{k-1}) \\ &\quad + R_s \frac{(\bar{x} - x_{i-1})^4 + (\bar{y} - y_{j-1})^4 + (\bar{z} - z_{k-1})^4}{4!} \end{aligned}$$

Use the estimate (4.11), (4.12) and (4.13), set $\bar{x} - x_{i-1} = \bar{y} - y_{j-1} = \bar{z} - z_{k-1} = h$, $(x - x_{i-1}) = (y - y_{j-1}) = (z - z_{k-1}) = h$, and $R_s \leq R_0$. Therefore

$$\begin{aligned} & \|\mathbf{p}^*(x, y, z, s) - \mathbf{p}_{ijk}(x, y, z, s)\| \\ &\leq 3\epsilon_3 \frac{h^3}{3!} + 24\epsilon_3 h^3 + 15\epsilon_2 \frac{h^2}{2!} + 6\epsilon_2 h^2 + 3\epsilon_1 h + R_0 \frac{3h^4}{4!}. \end{aligned} \quad (4.14)$$

Clearly, $\mathbf{p}_{ijk}(x, y, z, s) \rightarrow \mathbf{p}^*(x, y, z, s)$ in the form of norm as $h \rightarrow 0$ at the order of $o(h^3)$ on each cubic lattice C_{ijk} with the estimate (4.14).

Finally, from estimate errors (4.7) and (4.14) to find that for the total objective function (3.15), the convergence is in the order of $o(h^3)$ for constructed triply cubic polynomials $\mathbf{p}_{ijk}(x, y, z, s)$ approach, and in the order of $o(h^{12})$ for performance index $J(\mathbf{p})$ at domain Ω .

5. Computational procedure

Render the computational procedure for the completeness of proposed triply cubic polynomials paradigm in the following steps:

Step 0 Given a partition number N for domain Ω .

Set $x_1 = x, x_2 = y, x_3 = z$. Divide the cube domain $[0, L] \times [0, L] \times [0, L]$ into $N \times N \times N$ cubic lattices C_{ijk} with coordinates $x_i = \frac{i-1}{N}, y_j = \frac{j-1}{N}, z_k = \frac{k-1}{N}$ for $i = 1, 2, \dots, N, j = 1, 2, \dots, N, k = 1, 2, \dots, N$ at direction x, y, z .

- Define η_n for $n = 1, 2, \dots, \mathbf{n}$.
- Define sequences of triply cubic polynomials p_{ijk} for each cubic lattice C_{ijk} .
- Define $\mathbf{p}'_{ijk}, \mathbf{p}''_{ijk}$ at each cubic lattice C_{ijk} .
- Define $a_i(s), a_{ij}(s), a_{ijk}(s)$ at each cubic lattice C_{ijk} .
- Combine $\mathbf{p}_{ijk}(i, j, k, s)$ at each cubic lattice C_{ijk} .
- Take the base functions $\phi_n(x_1, x_2, x_3)$ for $n = 1, 2, \dots, \mathbf{n}$ as given in (3.1).

Step 1 Execute iteration $\mathbf{p}_{ijk}(i, j, k, s)$ at each cubic lattice C_{ijk} .

Step 2 Compute $\nabla \mathbf{p}_{ijk}(i, j, k, s), \Delta \mathbf{p}_{ijk}(i, j, k, s)$ at each cubic lattice C_{ijk} . Set $\mathbf{L}(\mathbf{p}_{ijk})$ as $L^p(t, i, j, k)$.

Step 3 Use $L^p(t, i, j, k)$ to calculate $\hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk})$ as Gauss quadrature approximate given at (3.17). Then, solve $a_i(s), a_{ij}(s), a_{ijk}(s)$ from $\hat{J}_{\lambda, i, j, k}(\mathbf{p}_{ijk})$ for each cubic lattice C_{ijk} .

Step 4 Calculate $\nabla \eta(i, j, k, s), \Delta \eta(i, j, k, s)$ respectively, and combine $\phi_k(x_1, x_2, x_3)$ to get $q(i, j, k, s)$ to recover the unknown coefficient $\tilde{k}(x, y, z)$ at each lattice C_{ijk} .

Step 5 Summarizing of C_{ijk} for i, j, k , get the total surface plot and contour plot of $\tilde{k}(x, y, z)$ at total domain Ω .

Remark 5.1. For hyperbolic and elliptic cases, the triply cubic polynomial approach can be extended directly. (cf.[6, 7, 8, 9]).

As well approach, experimental issue are acquired to evident the theoretic algorithm in our later works. Naturally, we omit it for the restrict of paper length. The task of this article is to address the new and novel point for constructing a triple cubic polynomial in finding the profile of unknown coefficient.

The detailed calculation issues (source data collections, redundant data elimination, domain partition, mesh expanding and recovery treatment) are exclusive in this paper.

6. Conclusions

As theoretical results, triple cubic polynomials approach for globally convergent algorithm is presented in this paper for solving the coefficient inverse problems in three

dimensional case. It can be efficiently used to recover the unknown coefficients occurred in nonlinearity (cf. [19]). The theoretical issue subject to the parabolic initial-boundary equation is accomplished. Convergence proof and error estimate are provided for the proposed approach.

Upon with theoretic survey, the triple cubic polynomials approach supply a method to solve the wide applications in practical sense (cf. [20, 21]). This original investigation can be utilized to a variety of coefficient inverse problems as well as related areas.

The globally convergent algorithm consist of triple cubic polynomials approach exposit the possibility to achieve numerical approximation in high-dimensional case. As perspective of the future work, it would be a promise direction to tailor the triple cubic polynomials approach as a powerful and substantial methodology in theoretical research, laboratory experiments and industrial fields.

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