Application of cylindrical IE-GSTC to physical metasurfaces

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Abstract—This work validates cylindrical IE-GSTC by applying it to physical metasurfaces, i.e. metasurfaces defined by material properties and dimensions rather than by susceptibility tensor components. Previously reported IE-GSTC which was formulated for zero thickness GSTC discontinuity is extended to handle finite thickness of physical metasurfaces. A simple analytical approach is used to extract the bianisotropic susceptibility tensor of concentric, multilayered, magneto-dielectric shell. Plane wave scattering by a physical metasurface constructed of four segments of multilayered, magneto-dielectric metasurface scatterers is used as an example problem to validate cylindrical IE-GSTC. A second example considers an opening on the cylindrical metasurface, confirming IE-GSTC can handle metasurfaces with openings. Good agreement is obtained between IE-GSTC results and full wave simulation results for both cases. The paper concludes with a novel technique to handle PEC segments within a cylindrical metasurface.

Index Terms—GSTC, Integral equation, MoM, Cylindrical, Metasurface, Boundary condition, Susceptibility, Electromagnetic discontinuity, COMSOL

I. INTRODUCTION

METASURFACES, the 2D version of volumetric metamaterials have provided unprecedented ability in the control and manipulation of electromagnetic waves. In comparison to metamaterials, metasurfaces are easier to fabricate. Metasurfaces are electrically thin surfaces constructed using subwavelength particles called metaatoms. By using engineered particles, metasurfaces can transform the phase, amplitude, polarization and propagation direction of the incident wave. Several review articles and books exist on the design, modeling and applications of planar metasurfaces [1]–[4]. Due to its deeply subwavelength thickness, physical metasurfaces can be abstracted as an infinitesimally thin layer of electric and magnetic surface polarization densities. Metasurfaces achieve its functionality by creating a spatial electromagnetic discontinuity. Mathematically, the discontinuity can be expressed by Generalized Sheet Transition Conditions (GSTCs) which relates the electric and magnetic field discontinuities to the electric and magnetic surface polarization densities [5], [6]. The surface polarization

densities are related to the average of the fields on either side of the GSTC surface through the bianisotropic susceptibility tensor model [7]. It should be noted that even though physical metasurfaces have a finite subwavelength thickness, GSTCs which assume zero metasurface thickness are a very good approximation to such physical metasurfaces, i.e. scattered field due to the physical metasurface approximates the field scattered by GSTC interface conditions.

GSTCs in combination with the bianisotropic susceptibility tensor model can be effectively used to simulate thin electromagnetic surfaces. Previous works in this approach include the application of Finite Difference Frequency Domain (FDFD) method [8] and Finite Element Method (FEM) [9] to the modeling of planar GSTCs. Even though most of the metasurfaces reported to date are planar, other canonical shapes such as cylindrical metasurfaces [10]–[13] and spherical metasurfaces [14], [15] are now being studied. Applications of conformal metasurfaces include illusion transformation [13], cloaking [12], high-gain antennas [16] and so on. GSTCs-bianisotropic susceptibility tensor model had been previously used for conformal metasurfaces. Integral Equation - Method of Moment (IE-MoM) was applied to circular cylindrical metasurfaces in [10]. This was extended to cylindrical metasurfaces of arbitrary cross-section in [11]. Conformal transformation was used to synthesize surface electromagnetic tensor quantities for non-circular cylindrical metasurface in [17]. In [18], a digital signal processing based method is outlined for analysis and synthesis of cylindrical, bianisotropic metasurfaces. Even though the paper contains physical implementation, it is not clear whether the metasurface unit cells were characterized taking curvature into effect. In [19], inverse source technique was used to synthesize surface susceptibility tensor components of metasurfaces of arbitrary shape. The work addressed metasurfaces capable of generating radiation patterns with specified half-power beamwidth, nulls, polarization etc. Recently asymptotic methods (GO, GTD) was applied to electrically large metasurfaces [20]. Even though mathematically rigorous, none of these works [10]–[12], [14], [19]–[23] have used physical metasurfaces.

One of the goals of this paper is to move from bianisotropic susceptibility tensor based conformal GSTC surfaces in [10]–[12], [14], [19]–[23] to physical metasurfaces formed by curved metasurface scattering particles. In this work, validity of the cylindrical GSTC [10] is confirmed by comparing the scattering due to a cylindrical, physical metasurface to scattering by a corresponding GSTC contour. Scattering by cylindrical GSTC contour was obtained by a slightly modified

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version of IE-GSTC formulation in [10]. The modification incorporates the finite thickness of physical metasurfaces. This modification is important to get accurate results for IE-GSTCs [10], [11], [22] when compared to physical metasurface scattering.

The organization of the paper is as follows. Section II extends the IE-MoM in [10] to handle non-zero thickness of a physical metasurface. The contents in this section closely follows [10]. Only the significant differences are explained. This is followed by susceptibility tensor extraction of cylindrical, concentric, magneto-dielectric shell using an analytical approach in section III. Section IV provides two examples. The metasurfaces in the two examples are constructed using metasurface particles which are part of the shell structure in section III. The second example has an opening which can be taken care of by IE-GSTC formulation, but with reduced accuracy. IE-GSTC along with susceptibility tensors extracted using the method in section III are used to simulate the scattering problem, the results of which is compared with full wave simulation of the physical metasurface. The last section provides detailed formulation for handling PEC segments in cylindrical metasurface with one numerical validation example.

II. IE-MoM FOR FINITE THICKNESS CYLINDRICAL METASURFACE

The IE-MoM described in [10] was for a cylindrical GSTC contour of zero thickness whose surface polarization densities are related to electric and magnetic fields on either side of the metasurface by the bianisotropic susceptibility tensor model. That formulation need to be slightly extended to include the non-zero thickness of physical metasurfaces. Instead of using a single contour for calculation of MoM coefficients and integration of surface fields, two contours one external and other internal to the metasurface are used. This is shown in Fig. 1. Throughout this paper, there is no variation of material properties or excitation along the z axis. The inner contour is Γ_2 and the outer contour is Γ_1 , separated by metasurface thickness δ . In this section, the extension of the formulation

Fig. 1. Finite thickness cylindrical metasurface

in [10] is outlined. A clear understanding of [10] is required to understand this section. The variables and notations follow

[10]. The 2D IEs for TM polarization given in equation (6) of [10] are modified to

$$E_{z1}^{\text{inc}}(\bar{\rho}) + \oint_{\Gamma_1} \left[E_{z1}(\bar{\rho}'_1) \frac{\partial G_0(\bar{\rho}; \bar{\rho}'_1)}{\partial n'} - j\omega\mu_0 H_{\phi1}(\bar{\rho}'_1) G_0(\bar{\rho}; \bar{\rho}'_1) \right] d\Gamma'_1 = \begin{cases} E_{z1}(\bar{\rho}) & ; \bar{\rho} \in \Omega_1 \\ 0.5E_{z1}(\bar{\rho}) & ; \bar{\rho} \in \Gamma_1 \\ 0 & ; \bar{\rho} \in \Omega_2 \end{cases} \quad (1a)$$

$$E_{z2}^{\text{inc}}(\bar{\rho}) - \oint_{\Gamma_2} \left[E_{z2}(\bar{\rho}'_2) \frac{\partial G_0(\bar{\rho}; \bar{\rho}'_2)}{\partial n'} - j\omega\mu_0 H_{\phi2}(\bar{\rho}'_2) G_0(\bar{\rho}; \bar{\rho}'_2) \right] d\Gamma'_2 = \begin{cases} 0 & ; \bar{\rho} \in \Omega_1 \\ 0.5E_{z2}(\bar{\rho}) & ; \bar{\rho} \in \Gamma_2 \\ E_{z2}(\bar{\rho}) & ; \bar{\rho} \in \Omega_2 \end{cases} \quad (1b)$$

where $\bar{\rho}'_1$ and $\bar{\rho}'_2$ denote contour integration position variables along Γ_1 and Γ_2 respectively. Such a distinction was not made in [10]. The subscripts 1, 2 on the field variables denote the fields on domain Ω_1, Ω_2 respectively. $G_0(\rho; \rho'_1) = -0.25jH_0^2(k_0|\bar{\rho} - \bar{\rho}'_1|)$ is the 2D Green's function. Derivative with respect to n denotes directional derivative along the radial direction (pointing outwards). The 2D IEs for TE polarization given in equation (7) of [10] are modified to

$$H_{z1}^{\text{inc}}(\bar{\rho}) + \oint_{\Gamma_1} \left[H_{z1}(\bar{\rho}'_1) \frac{\partial G_0(\bar{\rho}; \bar{\rho}'_1)}{\partial n'} + j\omega\epsilon_0 E_{\phi1}(\bar{\rho}'_1) G_0(\bar{\rho}; \bar{\rho}'_1) \right] d\Gamma'_1 = \begin{cases} H_{z1}(\bar{\rho}) & ; \bar{\rho} \in \Omega_1 \\ 0.5H_{z1}(\bar{\rho}) & ; \bar{\rho} \in \Gamma_1 \\ 0 & ; \bar{\rho} \in \Omega_2 \end{cases} \quad (2a)$$

$$H_{z2}^{\text{inc}}(\bar{\rho}) - \oint_{\Gamma_2} \left[H_{z2}(\bar{\rho}'_2) \frac{\partial G_0(\bar{\rho}; \bar{\rho}'_2)}{\partial n'} + j\omega\epsilon_0 E_{\phi2}(\bar{\rho}'_2) G_0(\bar{\rho}; \bar{\rho}'_2) \right] d\Gamma'_2 = \begin{cases} 0 & ; \bar{\rho} \in \Omega_1 \\ 0.5H_{z2}(\bar{\rho}) & ; \bar{\rho} \in \Gamma_2 \\ H_{z2}(\bar{\rho}) & ; \bar{\rho} \in \Omega_2 \end{cases} \quad (2b)$$

Following the same procedure as in [10], four integral equations are obtained using (1), (2) and circular metasurface synthesis equations given in equation (4) of [10]. These four equations are used to solve the four tangential fields $E_{z1}(\bar{\rho}'_1)$, $H_{\phi1}(\bar{\rho}'_1)$, $E_{\phi1}(\bar{\rho}'_1)$ and $H_{z1}(\bar{\rho}'_1)$ on the exterior contour Γ_1 . The equations are given as

$$\frac{E_{z1}(\bar{\rho}_1)}{2} - \oint_{\Gamma_1} \left[E_{z1}(\bar{\rho}'_1) \frac{\partial G_0(\bar{\rho}_1; \bar{\rho}'_1)}{\partial n'} - j\omega\mu_0 H_{\phi1}(\bar{\rho}'_1) G_0(\bar{\rho}_1; \bar{\rho}'_1) \right] d\Gamma'_1 = E_{z1}^{\text{inc}}(\bar{\rho}_1) ; \bar{\rho}_1 \in \Gamma_1 \quad (3a)$$

$$\begin{aligned}
& \oint_{\Gamma_2} \left[A_{11}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + A_{12}(\bar{\rho}_c') H_{\phi 1}(\bar{\rho}_1') + A_{13}(\bar{\rho}_c') E_{\phi 1}(\bar{\rho}_1') \right. \\
& \left. + A_{14}(\bar{\rho}_c') H_{z1}(\bar{\rho}_1') \right] \frac{\partial G_0(\bar{\rho}_2'; \bar{\rho}_2')}{\partial n'} - j\omega\mu_0 \left[A_{21}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + \right. \\
& \left. A_{22}(\bar{\rho}_c') H_{\phi 1}(\bar{\rho}_1') + A_{23}(\bar{\rho}_c') E_{\phi 1}(\bar{\rho}_1') + A_{24}(\bar{\rho}_c') H_{z1}(\bar{\rho}_1') \right] \\
& G_0(\bar{\rho}_2; \bar{\rho}_2') d\Gamma_2' + 0.5 \left[A_{11}(\bar{\rho}_c) E_{z1}(\bar{\rho}_1) + A_{12}(\bar{\rho}_c) H_{\phi 1}(\bar{\rho}_1) + \right. \\
& \left. A_{13}(\bar{\rho}_c) E_{\phi 1}(\bar{\rho}_1) + A_{14}(\bar{\rho}_c) H_{z1}(\bar{\rho}_1) \right] = E_{z2}^{\text{inc}}(\bar{\rho}_2); \bar{\rho}_2 \in \Gamma_2 \quad (3b)
\end{aligned}$$

$$\begin{aligned}
& \frac{H_{z1}(\bar{\rho}_1)}{2} - \oint_{\Gamma_1} \left[H_{z1}(\bar{\rho}_1') \frac{\partial G_0(\bar{\rho}_1; \bar{\rho}_1')}{\partial n'} + j\omega\epsilon_0 \right. \\
& \left. E_{\phi 1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' = H_{z1}^{\text{inc}}(\bar{\rho}_1); \bar{\rho}_1 \in \Gamma_1 \quad (3c)
\end{aligned}$$

$$\begin{aligned}
& \oint_{\Gamma_2} \left[A_{41}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + A_{42}(\bar{\rho}_c') H_{\phi 1}(\bar{\rho}_1') + A_{43}(\bar{\rho}_c') E_{\phi 1}(\bar{\rho}_1') \right. \\
& \left. + A_{44}(\bar{\rho}_c') H_{z1}(\bar{\rho}_1') \right] \frac{\partial G_0(\bar{\rho}_2; \bar{\rho}_2')}{\partial n'} + j\omega\epsilon_0 \left[A_{31}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + \right. \\
& \left. A_{32}(\bar{\rho}_c') H_{\phi 1}(\bar{\rho}_1') + A_{33}(\bar{\rho}_c') E_{\phi 1}(\bar{\rho}_1') + A_{34}(\bar{\rho}_c') H_{z1}(\bar{\rho}_1') \right] \\
& G_0(\bar{\rho}_2; \bar{\rho}_2') d\Gamma_2' + 0.5 \left[A_{41}(\bar{\rho}_c) E_{z1}(\bar{\rho}_1) + A_{42}(\bar{\rho}_c) H_{\phi 1}(\bar{\rho}_1) + \right. \\
& \left. A_{43}(\bar{\rho}_c) E_{\phi 1}(\bar{\rho}_1) + A_{44}(\bar{\rho}_c) H_{z1}(\bar{\rho}_1) \right] = H_{z2}^{\text{inc}}(\bar{\rho}_2); \bar{\rho}_2 \in \Gamma_2 \quad (3d)
\end{aligned}$$

where the $A_{ij}(\bar{\rho}_c')$ are elements of 4×4 matrix $\mathbf{A}(\bar{\rho}_c')$ [10]. This matrix which is a function of bianisotropic susceptibility tensor components (which vary azimuthally) relates the tangential electric and magnetic field components on the inner contour and the outer contour. $\bar{\rho}_c'$ denotes the center of the metasurface, i.e. $\bar{\rho}_c' = \bar{\rho}_1' - 0.5\delta\hat{\rho}$. Since the material properties of the metasurface varies azimuthally, the bianisotropic susceptibility tensor components and matrix $\mathbf{A}(\bar{\rho}_c')$ also varies with position. To be precise the matrix $\mathbf{A}(\bar{\rho}_c')$ is defined as

$$\mathbf{F}_2(\bar{\rho}_2') = \mathbf{A}(\bar{\rho}_c') \mathbf{F}_1(\bar{\rho}_1') \quad (4a)$$

$$\mathbf{A}(\bar{\rho}_c') = \mathbf{A}_2^{-1}(\bar{\rho}_c') \mathbf{A}_1(\bar{\rho}_c') \quad (4b)$$

where

$$\mathbf{F}_2(\bar{\rho}_2') = \begin{bmatrix} E_{z2}(\bar{\rho}_2') & H_{\phi 2}(\bar{\rho}_2') & E_{\phi 2}(\bar{\rho}_2') & H_{z2}(\bar{\rho}_2') \end{bmatrix}^T \quad (5a)$$

$$\mathbf{F}_1(\bar{\rho}_1') = \begin{bmatrix} E_{z1}(\bar{\rho}_1') & H_{\phi 1}(\bar{\rho}_1') & E_{\phi 1}(\bar{\rho}_1') & H_{z1}(\bar{\rho}_1') \end{bmatrix}^T \quad (5b)$$

The matrix $\mathbf{A}_1(\bar{\rho}_c')$ and $\mathbf{A}_2(\bar{\rho}_c')$ which are defined in equations (22), (23) in [10] are repeated in equations (6) and (7) for completeness.

The integration contours in (3a) - (3d) should be carefully noted. It is different from the corresponding equations in [10]. Similar to [10], the IEs (3a) - (3d) are converted to MoM system of linear equations using pulse basis functions and point matching. The MoM system of linear equations read

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \mathbf{Z}_{13} & \mathbf{Z}_{14} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \mathbf{Z}_{23} & \mathbf{Z}_{24} \\ \mathbf{Z}_{31} & \mathbf{Z}_{32} & \mathbf{Z}_{33} & \mathbf{Z}_{34} \\ \mathbf{Z}_{41} & \mathbf{Z}_{42} & \mathbf{Z}_{43} & \mathbf{Z}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{bmatrix} \quad (8)$$

where the unknown vectors are

$$\mathbf{x}_1 = [E_{z1,1}, \dots, E_{z1,N}]^T \quad (9a)$$

$$\mathbf{x}_2 = [H_{\phi 1,1}, \dots, H_{\phi 1,N}]^T \quad (9b)$$

$$\mathbf{x}_3 = [E_{\phi 1,1}, \dots, E_{\phi 1,N}]^T \quad (9c)$$

$$\mathbf{x}_4 = [H_{z1,1}, \dots, H_{z1,N}]^T \quad (9d)$$

N is the total number of basis functions. The $\mathbf{Z}$ matrix coefficients and the excitation vector are as follows

$$Z_{11,mn} = p_{mn}^1 \quad (10a)$$

$$Z_{12,mn} = r_{mn}^1 \quad (10b)$$

$$Z_{13,mn} = 0 \quad (10c)$$

$$Z_{14,mn} = 0 \quad (10d)$$

$$Z_{21,mn} = A_{11,n} q_{mn}^2 - A_{21,n} r_{mn}^2 \quad (11a)$$

$$Z_{22,mn} = A_{12,n} q_{mn}^2 - A_{22,n} r_{mn}^2 \quad (11b)$$

$$Z_{23,mn} = A_{13,n} q_{mn}^2 - A_{23,n} r_{mn}^2 \quad (11c)$$

$$Z_{24,mn} = A_{14,n} q_{mn}^2 - A_{24,n} r_{mn}^2 \quad (11d)$$

$$Z_{31,mn} = 0 \quad (12a)$$

$$Z_{32,mn} = 0 \quad (12b)$$

$$Z_{33,mn} = -s_{mn}^1 \quad (12c)$$

$$Z_{34,mn} = p_{mn}^1 \quad (12d)$$

$$Z_{41,mn} = A_{41,n} q_{mn}^2 + A_{31,n} s_{mn}^2 \quad (13a)$$

$$Z_{42,mn} = A_{42,n} q_{mn}^2 + A_{32,n} s_{mn}^2 \quad (13b)$$

$$Z_{43,mn} = A_{43,n} q_{mn}^2 + A_{33,n} s_{mn}^2 \quad (13c)$$

$$Z_{44,mn} = A_{44,n} q_{mn}^2 + A_{34,n} s_{mn}^2 \quad (13d)$$

where $A_{ij,n}$ is the value of matrix elements A_{ij} at the n^{th} discretization segment. It is these matrix elements which connects IE-GSTC to susceptibility tensor components. The coefficients $p_{mn}^1, q_{mn}^2, r_{mn}^1, r_{mn}^2, s_{mn}^1, s_{mn}^2$ are calculated as

$$p_{mn}^1 = \frac{1}{2} \delta_{mn} - \int_{s_{n1}} \frac{\partial G_0(\bar{\rho}_{m1}; \bar{\rho}_1')}{\partial n'} d\Gamma_1' \quad (14a)$$

$$q_{mn}^2 = \frac{1}{2} \delta_{mn} + \int_{s_{n2}} \frac{\partial G_0(\bar{\rho}_{m2}; \bar{\rho}_2')}{\partial n'} d\Gamma_2' \quad (14b)$$

$$r_{mn}^1 = j\omega\mu_0 \int_{s_{n1}} G_0(\bar{\rho}_{m1}; \bar{\rho}_1') d\Gamma_1' \quad (14c)$$

$$r_{mn}^2 = j\omega\mu_0 \int_{s_{n2}} G_0(\bar{\rho}_{m2}; \bar{\rho}_2') d\Gamma_2' \quad (14d)$$

$$s_{mn}^1 = j\omega\epsilon_0 \int_{s_{n1}} G_0(\bar{\rho}_{m1}; \bar{\rho}_1') d\Gamma_1' \quad (14e)$$

$$s_{mn}^2 = j\omega\epsilon_0 \int_{s_{n2}} G_0(\bar{\rho}_{m2}; \bar{\rho}_2') d\Gamma_2' \quad (14f)$$

s_{n1}, s_{n2} denote the n^{th} discretization segments on contours Γ_1, Γ_2 respectively. $\bar{\rho}_{m1}, \bar{\rho}_{m2}$ are the middle points of the m^{th} segments on contours Γ_1, Γ_2 respectively. δ_{mn} is the Kronecker delta function. The superscript in 1 or 2 in

$$\mathbf{A}_1(\bar{\rho}'_c) = \begin{bmatrix} \frac{j\omega\epsilon_0\chi_{ee}^{\phi z}(\bar{\rho}'_c)}{2} & \frac{j\omega c_0^{-1}\chi_{em}^{\phi\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\epsilon_0\chi_{ee}^{\phi\phi}(\bar{\rho}'_c)}{2} & \frac{2+j\omega c_0^{-1}\chi_{em}^{\phi z}(\bar{\rho}'_c)}{2} \\ \frac{j\omega\epsilon_0\chi_{ee}^{zz}(\bar{\rho}'_c)}{2} & \frac{-2+j\omega c_0^{-1}\chi_{em}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\epsilon_0\chi_{ee}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega c_0^{-1}\chi_{em}^{zz}(\bar{\rho}'_c)}{2} \\ \frac{-2+j\omega c_0^{-1}\chi_{me}^{\phi z}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{\phi\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega c_0^{-1}\chi_{me}^{\phi\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{\phi z}(\bar{\rho}'_c)}{2} \\ \frac{j\omega c_0^{-1}\chi_{me}^{zz}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{2+j\omega c_0^{-1}\chi_{me}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{zz}(\bar{\rho}'_c)}{2} \end{bmatrix} \quad (6)$$

$$\mathbf{A}_2(\bar{\rho}'_c) = \begin{bmatrix} \frac{j\omega\epsilon_0\chi_{ee}^{\phi z}(\bar{\rho}'_c)}{-2} & \frac{j\omega c_0^{-1}\chi_{em}^{\phi\phi}(\bar{\rho}'_c)}{-2} & \frac{j\omega\epsilon_0\chi_{ee}^{\phi\phi}(\bar{\rho}'_c)}{-2} & \frac{2-j\omega c_0^{-1}\chi_{em}^{\phi z}(\bar{\rho}'_c)}{2} \\ \frac{j\omega\epsilon_0\chi_{ee}^{zz}(\bar{\rho}'_c)}{-2} & \frac{-2-j\omega c_0^{-1}\chi_{em}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\epsilon_0\chi_{ee}^{z\phi}(\bar{\rho}'_c)}{-2} & \frac{j\omega c_0^{-1}\chi_{em}^{zz}(\bar{\rho}'_c)}{-2} \\ \frac{-2-j\omega c_0^{-1}\chi_{me}^{\phi z}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{\phi\phi}(\bar{\rho}'_c)}{-2} & \frac{j\omega c_0^{-1}\chi_{me}^{\phi\phi}(\bar{\rho}'_c)}{-2} & \frac{j\omega\mu_0\chi_{mm}^{\phi z}(\bar{\rho}'_c)}{-2} \\ \frac{j\omega c_0^{-1}\chi_{me}^{zz}(\bar{\rho}'_c)}{-2} & \frac{j\omega\mu_0\chi_{mm}^{z\phi}(\bar{\rho}'_c)}{-2} & \frac{2-j\omega c_0^{-1}\chi_{me}^{z\phi}(\bar{\rho}'_c)}{2} & \frac{j\omega\mu_0\chi_{mm}^{zz}(\bar{\rho}'_c)}{-2} \end{bmatrix} \quad (7)$$

$p_{mn}^1, q_{mn}^2, r_{mn}^1, r_{mn}^2, s_{mn}^1, s_{mn}^2$ represents whether the integration is performed in the outer or inner contour. The excitation vector components are given by

$$b_{1,m} = E_{z1}^{\text{inc}}(\bar{\rho}_{1,m}), \quad b_{2,m} = E_{z2}^{\text{inc}}(\bar{\rho}_{2,m}) \quad (15a)$$

$$b_{3,m} = H_{z1}^{\text{inc}}(\bar{\rho}_{1,m}), \quad b_{4,m} = H_{z2}^{\text{inc}}(\bar{\rho}_{2,m}) \quad (15b)$$

These four excitation vectors provide four possibilities: $b_{1,m}$ is for TM excitation external to the metasurface, $b_{2,m}$ is for TM excitation internal to the metasurface, $b_{3,m}$ is for TE excitation external to the metasurface and $b_{4,m}$ is for TE excitation internal to the metasurface.

III. SUSCEPTIBILITY EXTRACTION OF CYLINDRICAL, MULTILAYERED MAGNETO-DIELECTRIC SHELL

In this section, the bianisotropic susceptibility tensor components of a cylindrical, multilayered, magneto-dielectric shell is extracted. The geometry is shown in Fig. 2. Even though a three layered geometry is shown, the method can be used for any number of layers. The medium inside and outside the shell is vacuum. TM_z polarization is assumed. Following equation

Fig. 2. Three layered cylindrical, magneto-dielectric shell

(4) in [10], the difference between the field components across the cylindrical metasurface $\Delta E_z, \Delta H_\phi, \Delta H_z, \Delta E_\phi$ is related to the average of fields across the metasurface

$E_{z,\text{av}}, H_{\phi,\text{av}}, E_{\phi,\text{av}}, H_{z,\text{av}}$, through the following metasurface synthesis expressions.

$$\begin{bmatrix} -\Delta H_z \\ \Delta H_\phi \end{bmatrix} = j\omega\epsilon_0 \begin{bmatrix} \chi_{ee}^{\phi\phi} & \chi_{ee}^{\phi z} \\ \chi_{ee}^{z\phi} & \chi_{ee}^{zz} \end{bmatrix} \begin{bmatrix} E_{\phi,\text{av}} \\ E_{z,\text{av}} \end{bmatrix} + j\omega\sqrt{\mu_0\epsilon_0} \begin{bmatrix} \chi_{em}^{\phi\phi} & \chi_{em}^{\phi z} \\ \chi_{em}^{z\phi} & \chi_{em}^{zz} \end{bmatrix} \begin{bmatrix} H_{\phi,\text{av}} \\ H_{z,\text{av}} \end{bmatrix} \quad (16a)$$

$$\begin{bmatrix} \Delta E_z \\ -\Delta E_\phi \end{bmatrix} = j\omega\sqrt{\mu_0\epsilon_0} \begin{bmatrix} \chi_{me}^{\phi\phi} & \chi_{me}^{\phi z} \\ \chi_{me}^{z\phi} & \chi_{me}^{zz} \end{bmatrix} \begin{bmatrix} E_{\phi,\text{av}} \\ E_{z,\text{av}} \end{bmatrix} + j\omega\mu_0 \begin{bmatrix} \chi_{mm}^{\phi\phi} & \chi_{mm}^{\phi z} \\ \chi_{mm}^{z\phi} & \chi_{mm}^{zz} \end{bmatrix} \begin{bmatrix} H_{\phi,\text{av}} \\ H_{z,\text{av}} \end{bmatrix} \quad (16b)$$

For the case of TM_z polarization, they reduce to

$$\Delta E_z = j\omega\sqrt{\mu_0\epsilon_0}\chi_{me}^{\phi z}E_{z,\text{av}} + j\omega\mu_0\chi_{mm}^{\phi\phi}H_{\phi,\text{av}} \quad (17a)$$

$$\Delta H_\phi = j\omega\epsilon_0\chi_{ee}^{zz}E_{z,\text{av}} + j\omega\sqrt{\mu_0\epsilon_0}\chi_{em}^{z\phi}H_{\phi,\text{av}} \quad (17b)$$

Since there are four unknown susceptibility components in the above equation, two different excitation problems are solved to obtain the four susceptibility components. The two excitations used are shown in Fig. 3. The first problem is that of a infinite, electric line source at origin and the second excitation problem is that of a cylindrical surface electric current density of radius a_s . a_s can be of arbitrary value which satisfies the condition, $a_s > a_4$. Both of these problems are simple, boundary value problems, which can be either solved analytically [24] or simulated using COMSOL. In the case of analytical solution,

Fig. 3. Bianisotropic susceptibility tensor extraction using two sources: (1) Infinite electric line source at origin, $\vec{J} = \delta(x)\delta(y)\hat{z}\text{Am}^{-2}$ shown as a blue dot (2) Cylindrical electric surface current density of radius a_s , $\vec{J}_s = \delta(\rho - a_s)\hat{z}\text{Am}^{-1}$ shown as a dashed, blue circle.

the vector magnetic potential, electric field and magnetic field in each region can be expressed in terms of Bessel functions.

$$A_{z,i} = C_i J_0(k_i \rho) + D_i Y_0(k_i \rho) \quad (18a)$$

$$E_{z,i} = -j\omega \left[C_i J_0(k_i \rho) + D_i Y_0(k_i \rho) \right] \quad (18b)$$

$$H_{\phi,i} = \frac{k_i}{\mu_i} \left[C_i J_1(k_i \rho) + D_i Y_1(k_i \rho) \right] \quad (18c)$$

where $J_0(\cdot)$, $Y_0(\cdot)$ are Bessel function of first and second kind respectively. k_i, μ_i are the wavenumber and permeability of region i . For the case of cylindrical surface current density excitation of three layered, magneto-dielectric cylinder, the total number of regions is 6 (the surface current density discontinuity splits the external medium into two separate regions). Applying continuity of electric and magnetic fields across the region boundaries and magnetic field discontinuity caused by surface current density, the unknowns C_i, D_i can be obtained.

Once the two excitation problems are solved, the susceptibility tensor components are obtained as follows

$$j \frac{\omega}{c_0} \begin{bmatrix} E_{z,av}^{(1)} & \eta_0 H_{\phi,av}^{(1)} & 0 & 0 \\ 0 & 0 & \frac{E_{z,av}^{(1)}}{\eta_0} & H_{\phi,av}^{(1)} \\ E_{z,av}^{(2)} & \eta_0 H_{\phi,av}^{(2)} & 0 & 0 \\ 0 & 0 & \frac{E_{z,av}^{(2)}}{\eta_0} & H_{\phi,av}^{(2)} \end{bmatrix} \begin{bmatrix} \chi_{me}^{\phi z} \\ \chi_{mm}^{\phi\phi} \\ \chi_{ee}^{zz} \\ \chi_{em}^{z\phi} \end{bmatrix} = \begin{bmatrix} \Delta E_z^{(1)} \\ \Delta H_{\phi}^{(1)} \\ \Delta E_z^{(2)} \\ \Delta H_{\phi}^{(2)} \end{bmatrix} \quad (19)$$

where the superscript (1) denotes line source excitation fields and (2) denotes the cylindrical surface current density excitation.

IV. NUMERICAL VALIDATION

This section contains two examples of cylindrical metasurfaces which use IE-GSTC and the susceptibility extraction method described in the previous sections. COMSOL software is used for full wave simulations.

A. Four segment cylindrical metasurface excited by plane wave

The cylindrical metasurface shown in Fig. 4 consists of four segments (quadrants). Each segment is a multilayered, concentric, magneto-dielectric shell. The susceptibility tensor components of each segment can be extracted using the method in the previous section. Referring to Fig. 4, different annular regions are identified as follows: region 1: $0 \leq \rho < a_1$, region 2: $a_1 \leq \rho < a_2$, region 3: $a_2 \leq \rho < a_3$, region 4: $a_3 \leq \rho < a_4$ and region 5: $a_4 \leq \rho < \infty$. The metasurface is surrounded by vacuum, i.e. region 1 and region 5 are vacuum. The material properties of each region (represented by subscript number) in the four quadrants are given in table I. The susceptibility tensors of the quadrants are also given. The dimensions are: $a_1 = 5\lambda$, $a_2 = 5\lambda + \frac{\lambda}{24}$, $a_3 = a_2 + \frac{\lambda}{24}$, $a_4 = a_3 + \frac{\lambda}{20}$.

The structure is excited by a plane wave $\vec{E}^{\text{inc}}(x) = e^{-jk_0 x} \hat{z}$. The total field along a line from $(0,0)$ to $(10\lambda, 0)$ is plotted in Fig. 5 and Fig. 6. Good agreement between IE-GSTC and full wave results obtained using COMSOL software

Fig. 4. Four segment cylindrical metasurface.

Quadrant	Material properties	Susceptibility tensor
Quadrant 1	$\epsilon_{r1} = 1, \mu_{r1} = 1$ $\epsilon_{r2} = 9, \mu_{r2} = 2$ $\epsilon_{r3} = 4, \mu_{r3} = 1$ $\epsilon_{r4} = 9, \mu_{r4} = 2$ $\epsilon_{r5} = 1, \mu_{r5} = 1$	$\chi_{me}^{\phi z} = j2.55236 \times 10^{-3}$ $\chi_{mm}^{\phi\phi} = 6.08697 \times 10^{-2}$ $\chi_{ee}^{zz} = 1.97682 \times 10^{-1}$ $\chi_{em}^{z\phi} = j1.42107 \times 10^{-2}$
Quadrant 2	$\epsilon_{r1} = 1, \mu_{r1} = 1$ $\epsilon_{r2} = 4, \mu_{r2} = 1$ $\epsilon_{r3} = 3, \mu_{r3} = 1$ $\epsilon_{r4} = 4, \mu_{r4} = 1$ $\epsilon_{r5} = 1, \mu_{r5} = 1$	$\chi_{me}^{\phi z} = -j1.79047 \times 10^{-5}$ $\chi_{mm}^{\phi\phi} = 5.27727 \times 10^{-3}$ $\chi_{ee}^{zz} = 1.86202 \times 10^{-2}$ $\chi_{em}^{z\phi} = j2.78948 \times 10^{-4}$
Quadrant 3	$\epsilon_{r1} = 1, \mu_{r1} = 1$ $\epsilon_{r2} = 4, \mu_{r2} = 1$ $\epsilon_{r3} = 3, \mu_{r3} = 1$ $\epsilon_{r4} = 6, \mu_{r4} = 2$ $\epsilon_{r5} = 1, \mu_{r5} = 1$	$\chi_{me}^{\phi z} = j1.45598 \times 10^{-3}$ $\chi_{mm}^{\phi\phi} = 9.25367 \times 10^{-3}$ $\chi_{ee}^{zz} = 2.80732 \times 10^{-2}$ $\chi_{em}^{z\phi} = -j9.74236 \times 10^{-4}$
Quadrant 4	$\epsilon_{r1} = 1, \mu_{r1} = 1$ $\epsilon_{r2} = 1, \mu_{r2} = 1$ $\epsilon_{r3} = 6, \mu_{r3} = 2$ $\epsilon_{r4} = 1, \mu_{r4} = 1$ $\epsilon_{r5} = 1, \mu_{r5} = 1$	$\chi_{me}^{\phi z} = j2.83566 \times 10^{-4}$ $\chi_{mm}^{\phi\phi} = 6.01626 \times 10^{-3}$ $\chi_{ee}^{zz} = 1.44824 \times 10^{-2}$ $\chi_{em}^{z\phi} = -j3.78786 \times 10^{-5}$

TABLE I

Material properties and susceptibility tensor of metasurface segments shown in Fig. 4. Permittivity and permeability have subscripts which represent corresponding regions.

can be seen. For the IE-GSTC result a small gap can be around the metasurface radius of $x = 5\lambda$. This is because IE results is valid only for regions interior and exterior to the metasurface. Minor discrepancies between the two results can be seen around the metasurface. This is probably due to MoM being inefficient in the near field of the scatterer as well as GSTC being an approximate interface condition. In Fig. 5 and Fig. 6, the results from the zero-thickness IE-GSTC [10] are plotted for the same problem to show that the formulation in this paper is required to get more accurate results.

B. Cylindrical metasurface with opening excited by a line source

This example as shown in Fig. 7 is a four segment metasurface but with an opening. The opening subtends an angle

Fig. 5. IE-GSTC vs. full wave simulation for example A. $|E_z(x)|$ vs. x from $(0, 0)$ to $(10\lambda, 0)$

Fig. 6. IE-GSTC vs. full wave simulation for example A. $\angle E_z(x)$ vs. x from $(0, 0)$ to $(10\lambda, 0)$

of $\pi/2$ at the center and is equally divided between first and fourth quadrants. The material properties of the quadrants are same as that of previous example. The excitation is an infinite, electric, line source of $\vec{J}(\vec{r}) = \delta(x)\delta(y)\hat{z} \text{ Am}^{-2}$. In IE-GSTC an opening is handled by setting $\mathbf{A}(\vec{\rho}_c) = \text{diag}(1, 1, 1, 1)$ [10]. This ensures that tangential fields on either side of the opening are equal. This is only an approximation because the opening has a width of fraction of a wavelength. In COMSOL full wave simulation, line source is modelled using the external current density option on a circle of radius much smaller than wavelength ($\sim \lambda/100$). IE-GSTC results and COMSOL

Fig. 7. Four segment cylindrical metasurface with opening

results on a circle of radius 8λ are plotted in Fig. 8 and Fig. 9. There is good agreement between the results. The discrepancies for $\phi < 1$ radian and $\phi > 5.5$ radian could be caused by edge diffraction which the GSTC does not take into account. Another reason could be the IE-GSTC approximation for opening, which ignores the finite width of the metasurface.

Fig. 8. IE-GSTC vs. full wave simulation for example B. $|E_z(\phi)|$ vs. ϕ on a circle of radius 8λ .

V. IE-GSTC MODELING OF CYLINDRICAL METASURFACE - PEC COMPOSITE STRUCTURES

The problem under consideration is shown in Fig. 11. Similar to the previous sections, there is no variation of material properties or excitation along the z axis. The inner contour is Γ_2 and the outer contour is Γ_1 , separated by metasurface thickness δ . The circular boundary which separates the external domain Ω_1 and internal domain Ω_2 is partly GSTC (mathematical abstraction for physical metasurface) and

Fig. 9. IE-GSTC vs. full wave simulation for example B. $\angle E_z(\phi)$ vs. ϕ on a circle of radius 8λ .

partly PEC. GSTC is denoted by G and PEC by P. Therefore Γ_{1G} , Γ_{1P} represent the outer contour of GSTC and PEC sections. Similarly Γ_{2G} , Γ_{2P} represent the inner contour of GSTC and PEC sections. The PEC sections do not necessarily need to be contiguous. Considering TM_z polarization and referring to Fig. 11, the governing integral equations read

$$\frac{E_{z1}(\bar{\rho}_1)}{2} - \oint_{\Gamma_1} \left[E_{z1}(\bar{\rho}_1') \frac{\partial G_0(\bar{\rho}_1; \bar{\rho}_1')}{\partial n'} - j\omega\mu_0 H_{\phi1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' = E_{z1}^{inc}(\bar{\rho}_1) ; \bar{\rho}_1 \in \Gamma_1 \quad (20a)$$

$$\frac{E_{z2}(\bar{\rho}_2)}{2} + \oint_{\Gamma_2} \left[E_{z2}(\bar{\rho}_2') \frac{\partial G_0(\bar{\rho}_2; \bar{\rho}_2')}{\partial n'} - j\omega\mu_0 H_{\phi2}(\bar{\rho}_2') G_0(\bar{\rho}_2; \bar{\rho}_2') \right] d\Gamma_2' = E_{z2}^{inc}(\bar{\rho}_2) ; \bar{\rho}_2 \in \Gamma_2 \quad (20b)$$

On the GSTC part of the boundary, the external and internal tangential electric and magnetic fields are related through matrix $\mathbf{A}(\bar{\rho}_c)$

$$\begin{bmatrix} E_{z2}(\bar{\rho}_2) \\ H_{\phi2}(\bar{\rho}_2) \end{bmatrix} = \begin{bmatrix} A_{11}(\bar{\rho}_c) & A_{12}(\bar{\rho}_c) \\ A_{21}(\bar{\rho}_c) & A_{22}(\bar{\rho}_c) \end{bmatrix} \begin{bmatrix} E_{z1}(\bar{\rho}_1) \\ H_{\phi1}(\bar{\rho}_1) \end{bmatrix} \quad (21)$$

$\bar{\rho}_c$ denotes the center of the metasurface, i.e. $\bar{\rho}_c = \bar{\rho}_1 - 0.5\delta\hat{\rho}$. For PEC segments, there is no relation between the tangential components on inner and outer domain. By using the fact that total tangential electric field on PEC surface is zero, the integral equation (20a) is converted to two IEs, one for

Fig. 10. Composite metasurface-PEC system

$\bar{\rho}_1 \in \Gamma_{1G}$ and another for $\bar{\rho}_1 \in \Gamma_{1P}$.

$$\frac{E_{z1}(\bar{\rho}_1)}{2} - \int_{\Gamma_{1G}} \left[E_{z1}(\bar{\rho}_1') \frac{\partial G_0(\bar{\rho}_1; \bar{\rho}_1')}{\partial n'} - j\omega\mu_0 H_{\phi1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' - \int_{\Gamma_{1P}} \left[-j\omega\mu_0 H_{\phi1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' = E_{z1}^{inc}(\bar{\rho}_1) ; \bar{\rho}_1 \in \Gamma_{1G} \quad (22a)$$

$$- \int_{\Gamma_{1G}} \left[E_{z1}(\bar{\rho}_1') \frac{\partial G_0(\bar{\rho}_1; \bar{\rho}_1')}{\partial n'} - j\omega\mu_0 H_{\phi1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' - \int_{\Gamma_{1P}} \left[-j\omega\mu_0 H_{\phi1}(\bar{\rho}_1') G_0(\bar{\rho}_1; \bar{\rho}_1') \right] d\Gamma_1' = E_{z1}^{inc}(\bar{\rho}_1) ; \bar{\rho}_1 \in \Gamma_{1P} \quad (22b)$$

By using equation (21) and zero total, tangential electric field on PEC surface (20b) is transformed to two equations, one for $\bar{\rho}_2 \in \Gamma_{2G}$ and another for $\bar{\rho}_2 \in \Gamma_{2P}$.

$$\begin{aligned} & \frac{1}{2} \left[A_{11}(\bar{\rho}_c) E_{z1}(\bar{\rho}_1) + A_{12}(\bar{\rho}_c) H_{\phi1}(\bar{\rho}_1) \right] + \\ & \int_{\Gamma_{2G}} \left[A_{11}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + A_{12}(\bar{\rho}_c') H_{\phi1}(\bar{\rho}_1') \right] \frac{\partial G_0(\bar{\rho}_2; \bar{\rho}_2')}{\partial n'} d\Gamma_2' \\ & + \int_{\Gamma_{2G}} -j\omega\mu_0 \left[A_{21}(\bar{\rho}_c') E_{z1}(\bar{\rho}_1') + A_{22}(\bar{\rho}_c') H_{\phi1}(\bar{\rho}_1') \right] \\ & G_0(\bar{\rho}_2; \bar{\rho}_2') d\Gamma_2' + \int_{\Gamma_{2P}} -j\omega\mu_0 H_{\phi2}(\bar{\rho}_2') G_0(\bar{\rho}_2; \bar{\rho}_2') d\Gamma_2' \\ & = E_{z2}^{inc}(\bar{\rho}_2) ; \bar{\rho}_2 \in \Gamma_{2G} \quad (23a) \end{aligned}$$

$$\begin{aligned}
& \int_{\Gamma_{2G}} \left[A_{11}(\bar{\rho}'_c) E_{z1}(\bar{\rho}'_1) + A_{12}(\bar{\rho}'_c) H_{\phi1}(\bar{\rho}'_1) \right] \frac{\partial G_0(\bar{\rho}_2; \bar{\rho}'_2)}{\partial n'} d\Gamma'_2 \\
& + \int_{\Gamma_{2G}} -j\omega\mu_0 \left[A_{21}(\bar{\rho}'_c) E_{z1}(\bar{\rho}'_1) + A_{22}(\bar{\rho}'_c) H_{\phi1}(\bar{\rho}'_1) \right] \\
& G_0(\bar{\rho}_2; \bar{\rho}'_2) d\Gamma'_2 + \int_{\Gamma_{2P}} -j\omega\mu_0 H_{\phi2}(\bar{\rho}'_2) G_0(\bar{\rho}_2; \bar{\rho}'_2) d\Gamma'_2 \\
& = E_{z2}^{\text{inc}}(\bar{\rho}_2) ; \bar{\rho}_2 \in \Gamma_{2P} \quad (23b)
\end{aligned}$$

The four equations (22a), (22b), (23b) and (23b) are used to solve the four unknowns. The unknowns are tangential electric and magnetic fields on Γ_{1G} , tangential magnetic field on Γ_{1G} and tangential magnetic field on Γ_{2P} . These surface fields are expressed as a weighted, linear combination of pulse basis function $p_n(\cdot)$.

$$E_{z1}(\bar{\rho}'_1) = \sum_{n=1}^{N_G} E_{z1G,n} p_n(\bar{\rho}'_1) ; \bar{\rho}'_1 \in \Gamma_{1G} \quad (24a)$$

$$H_{\phi1}(\bar{\rho}'_1) = \sum_{n=1}^{N_G} H_{\phi1G,n} p_n(\bar{\rho}'_1) ; \bar{\rho}'_1 \in \Gamma_{1G} \quad (24b)$$

$$H_{\phi1}(\bar{\rho}'_1) = \sum_{n=1}^{N_P} H_{\phi1P,n} p_n(\bar{\rho}'_1) ; \bar{\rho}'_1 \in \Gamma_{1P} \quad (24c)$$

$$H_{\phi2}(\bar{\rho}'_2) = \sum_{n=1}^{N_P} H_{\phi2P,n} p_n(\bar{\rho}'_2) ; \bar{\rho}'_2 \in \Gamma_{2P} \quad (24d)$$

where N_G , N_P are the number of basis functions for GSTC and PEC part respectively on either the inner or outer surface. Therefore the total number of unknowns are: $2(N_G + N_P)$. By using the above expansions and testing functions: $\delta(\bar{\rho}_1 - \bar{\rho}_{1G,m})$, $m = 1, 2, \dots, N_G$ for (22a), $\delta(\bar{\rho}_1 - \bar{\rho}_{1P,m})$, $m = 1, 2, \dots, N_P$ for (22b), $\delta(\bar{\rho}_2 - \bar{\rho}_{2G,m})$, $m = 1, 2, \dots, N_G$ for (23a) and $\delta(\bar{\rho}_2 - \bar{\rho}_{2P,m})$, $m = 1, 2, \dots, N_P$ for (23b) a MoM system of equations can be derived. The MoM system of linear equations read

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \mathbf{Z}_{13} & \mathbf{Z}_{14} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \mathbf{Z}_{23} & \mathbf{Z}_{24} \\ \mathbf{Z}_{31} & \mathbf{Z}_{32} & \mathbf{Z}_{33} & \mathbf{Z}_{34} \\ \mathbf{Z}_{41} & \mathbf{Z}_{42} & \mathbf{Z}_{43} & \mathbf{Z}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{bmatrix} \quad (25)$$

where the unknown vectors are

$$\mathbf{x}_1 = [E_{z1G,1}, \dots, E_{z1G,N_G}]^T \quad (26a)$$

$$\mathbf{x}_2 = [H_{\phi1G,1}, \dots, H_{\phi1G,N_G}]^T \quad (26b)$$

$$\mathbf{x}_3 = [H_{\phi1P,1}, \dots, H_{\phi1P,N_P}]^T \quad (26c)$$

$$\mathbf{x}_4 = [H_{\phi2P,1}, \dots, H_{\phi2P,N_P}]^T \quad (26d)$$

The $\mathbf{Z}$ matrix coefficients and the excitation vector are as follows

$$Z_{11,mn} = p_{mn}^{1G1G} ; (N_G \times N_G) \quad (27a)$$

$$Z_{12,mn} = r_{mn}^{1G1G} ; (N_G \times N_G) \quad (27b)$$

$$Z_{13,mn} = r_{mn}^{1G1P} ; (N_G \times N_P) \quad (27c)$$

$$Z_{14,mn} = 0 ; (N_G \times N_P) \quad (27d)$$

$$Z_{21,mn} = -t_{mn}^{1P1G} ; (N_P \times N_G) \quad (28a)$$

$$Z_{22,mn} = r_{mn}^{1P1G} ; (N_P \times N_G) \quad (28b)$$

$$Z_{23,mn} = r_{mn}^{1P1P} ; (N_P \times N_P) \quad (28c)$$

$$Z_{24,mn} = 0 ; (N_P \times N_P) \quad (28d)$$

$$Z_{31,mn} = A_{11,n} q_{mn}^{2G2G} - A_{21,n} r_{mn}^{2G2G} ; (N_G \times N_G) \quad (29a)$$

$$Z_{32,mn} = A_{12,n} q_{mn}^{2G2G} - A_{22,n} r_{mn}^{2G2G} ; (N_G \times N_G) \quad (29b)$$

$$Z_{33,mn} = 0 ; (N_G \times N_P) \quad (29c)$$

$$Z_{34,mn} = -r_{mn}^{2G2P} ; (N_G \times N_P) \quad (29d)$$

$$Z_{41,mn} = A_{11,n} t_{mn}^{2P2G} - A_{21,n} r_{mn}^{2P2G} ; (N_P \times N_G) \quad (30a)$$

$$Z_{42,mn} = A_{12,n} t_{mn}^{2P2G} - A_{22,n} r_{mn}^{2P2G} ; (N_P \times N_G) \quad (30b)$$

$$Z_{43,mn} = 0 ; (N_P \times N_P) \quad (30c)$$

$$Z_{44,mn} = -r_{mn}^{2P2P} ; (N_P \times N_P) \quad (30d)$$

where $A_{ij,n}$ is the value of matrix elements A_{ij} at the n^{th} discretization segment. The constants used in the above Z matrix elements are calculated as follows

$$p_{mn}^{1G1G} = \frac{1}{2} \delta_{mn} - \int_{s_{1G,n}} \frac{\partial G_0(\bar{\rho}_{1G,m}; \bar{\rho}'_1)}{\partial n'} d\Gamma'_1 \quad (31a)$$

$$r_{mn}^{1G1G} = j\omega\mu_0 \int_{s_{1G,n}} G_0(\bar{\rho}_{1G,m}; \bar{\rho}'_1) d\Gamma'_1 \quad (31b)$$

$$r_{mn}^{1G1P} = j\omega\mu_0 \int_{s_{1P,n}} G_0(\bar{\rho}_{1G,m}; \bar{\rho}'_1) d\Gamma'_1 \quad (31c)$$

$$t_{mn}^{1P1G} = \int_{s_{1G,n}} \frac{\partial G_0(\bar{\rho}_{1P,m}; \bar{\rho}'_1)}{\partial n'} d\Gamma'_1 \quad (31d)$$

$$r_{mn}^{1P1G} = j\omega\mu_0 \int_{s_{1G,n}} G_0(\bar{\rho}_{1P,m}; \bar{\rho}'_1) d\Gamma'_1 \quad (31e)$$

$$r_{mn}^{1P1P} = j\omega\mu_0 \int_{s_{1P,n}} G_0(\bar{\rho}_{1P,m}; \bar{\rho}'_1) d\Gamma'_1 \quad (31f)$$

$$q_{mn}^{2G2G} = \frac{1}{2} \delta_{mn} + \int_{s_{2G,n}} \frac{\partial G_0(\bar{\rho}_{2G,m}; \bar{\rho}'_2)}{\partial n'} d\Gamma'_2 \quad (31g)$$

$$r_{mn}^{2G2G} = j\omega\mu_0 \int_{s_{2G,n}} G_0(\bar{\rho}_{2G,m}; \bar{\rho}'_2) d\Gamma'_2 \quad (31h)$$

$$r_{mn}^{2G2P} = j\omega\mu_0 \int_{s_{2P,n}} G_0(\bar{\rho}_{2G,m}; \bar{\rho}'_2) d\Gamma'_2 \quad (31i)$$

$$t_{mn}^{2P2G} = \int_{s_{2G,n}} \frac{\partial G_0(\bar{\rho}_{2P,m}; \bar{\rho}'_2)}{\partial n'} d\Gamma'_2 \quad (31j)$$

$$r_{mn}^{2P2G} = j\omega\mu_0 \int_{s_{2G,n}} G_0(\bar{\rho}_{2P,m}; \bar{\rho}'_2) d\Gamma'_2 \quad (31k)$$

$$r_{mn}^{2P2P} = j\omega\mu_0 \int_{s_{2P,n}} G_0(\bar{\rho}_{2P,m}; \bar{\rho}'_2) d\Gamma'_2 \quad (31l)$$

s_{n1} , s_{n2} denote the n^{th} discretization segments on contours Γ_1 , Γ_2 respectively. $\bar{\rho}_{m1}$, $\bar{\rho}_{m2}$ are the middle points of the m^{th} segments on contours Γ_1 , Γ_2 respectively. δ_{mn} is the Kronecker delta function. The first part of superscript in the above coefficients represents whether the testing is performed on inner or outer contour. It also shows whether the testing is done on GSTC part or PEC part. The second part represents whether the integration is performed in the outer or inner

contour's GSTC part or PEC part. For example in t_{mn}^{1P1G} , the testing is done on exterior contour's PEC part (i.e. 1P) and integration is performed on the exterior contour's GSTC part (i.e. 1G). The excitation vector components are given by

$$b_{1,m} = E_{z1}^{\text{inc}}(\bar{\rho}_{m1G}), \quad b_{2,m} = E_{z1}^{\text{inc}}(\bar{\rho}_{m1P}) \quad (32a)$$

$$b_{3,m} = E_{z2}^{\text{inc}}(\bar{\rho}_{m2G}), \quad b_{4,m} = E_{z2}^{\text{inc}}(\bar{\rho}_{m2P}) \quad (32b)$$

These four excitation vectors provide the possibility for TM_z excitation internal and external to the metasurface-PEC system. For validating the formulation, consider the metasurface-PEC system example shown in Fig. 11. The four metasurface segments are identical to what is shown in table I. The first quadrant in table I corresponds to metasurface segment in the range $0 \leq \phi \leq \pi/4$ in Fig. 11 and so on. The dimensions are same as used in previous examples. The excitation is a plane wave $\bar{E}^{\text{inc}}(x) = e^{-jk_0 x} \hat{z}$. In Fig. 12 and Fig. 13, IE-GSTC-PEC results are compared with COMSOL full wave simulations on a circle of radius 10λ .

Fig. 11. Four segment metasurface with PEC backing

VI. CONCLUSION

In this work, IE-GSTC for circular cylindrical metasurface is validated for the case of physical metasurfaces. The essence of this work is that of circular metasurface scattering particles can be abstracted as few numbers, i.e. susceptibility tensor components. As far as to the author's knowledge, this work is one of the few works in which GSTC and bianisotropic susceptibility tensor model is validated for a physical, conformal metasurface. Even though the examples and susceptibility extraction was for TM polarization, it is easily extendable to TE polarization. The combination can be used to analyze and design circular metasurfaces for the case of circularly polarized incident wave. This work is important in the sense that it pushes the previous works on conformal GSTC [10]–[12], [14] to physical implementation and optimization. The IE-GSTC formulation is extended to handle PEC segments which could be useful for practical scattering reduction and radiation problems [25].

Fig. 12. IE-GSTC-PEC vs. full wave simulation for four segment metasurface with PEC backing. $|E_z(\phi)|$ vs. ϕ on a circle of radius 10λ .

Fig. 13. IE-GSTC-PEC vs. full wave simulation for four segment metasurface with PEC backing. $\angle E_z(\phi)$ vs. ϕ on a circle of radius 10λ .

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