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Bayesian Multilateration

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Abstract—Multilateration (MLAT) is the *de facto* standard to localize points of interest (POIs) in navigation and surveillance systems. Despite sensors being inherently noisy, most existing techniques i) are oblivious to noise patterns in sensor measurements; and ii) only provide point estimates of the POI's location. This often results in unreliable estimates with high variance, *i.e.*, that are highly sensitive to measurement noise. To overcome this caveat, we advocate the use of Bayesian modeling. Using Bayesian statistics, we provide a comprehensive guide to handle uncertainties in MLAT. We provide principled choices for the likelihood function and the prior distributions. Inference within the resulting model follows standard MCMC techniques. Besides coping with unreliable measurements, our framework can also deal with sensors whose location is not completely known, which is an asset in mobile systems. The proposed solution also naturally incorporates multiple measurements per reference point, a common practical situation that is usually not handled directly by other approaches. Comprehensive experiments with both synthetic and real-world data indicate that our Bayesian approach to the MLAT task provides better position estimation and uncertainty quantification when compared to the available alternatives.

Index Terms—Multilateration, Bayesian modeling, Localization, Navigation

I. INTRODUCTION

ACCURATELY estimating a target's position is fundamental in many real-world applications, such as wireless sensor networks [17], airport surveillance [9], and navigation of unmanned vehicles or robots [18]. In environments that limit the use of radar technologies and GPS, localization based on Multilateration (MLAT) is a widely used solution [10].

Multilateration is a well-known localization method that leverages distances from multiple reference points to find the position of a Point Of Interest (POI). If i) we know the location of reference points exactly and ii) measurements are noiseless, these distances allow the MLAT algorithm to localize the POI.

Over the years, several MLAT solutions have been proposed with the implicit assumption that all the involved quantities are known exactly [5], [16], [13], [2]. Although these solutions may be useful in some situations, in many real scenarios the distances to the reference points and the position of the reference points themselves may be uncertain. When this is the case, a probabilistic approach can offer a more comprehensive solution, accounting for both the uncertainty around the reference points and the distances.

Menchaca-Mendez *et al.* [11] use normal distributions to model the uncertainty in the position of each reference point. Then, they estimate the POI's position via maximum likelihood, finding the mode of a product of normals related

Fig. 1: **Noise matters.** (a) When all measurements are accurately obtained, the POI lies exactly in the intersection of the circumferences centered on the reference points with a radius equal to the observed distances; (b) When there is uncertainty in the measurements, the POI is positioned elsewhere and cannot be uniquely determined.

to each reference point. In contrast, Vasconcelos *et al.* [6], assume the locations of reference points are known, but the distances and the position of the POI are uncertain. They use Rayleigh likelihoods to factor in the distances and use a normal distribution as prior for the POI position. Despite using a Bayesian approach to marginalize the unknown quantities, their experimental evaluation considers only point estimates of the POI's location.

None of the aforementioned works explores the scenario where both the distances and the positions of the reference points are uncertain. Moreover, these approaches evaluate only point estimates of the POI's position, totally ignoring the uncertainty around predictions.

We present a fully Bayesian formulation of the multilateration problem. Our method, Bayesian Multilateration (BMLAT) accounts for the uncertainties in all variables of interest (*i.e.*, reference point positions, distances between the POI and reference points, and the POI's position). We develop a novel parameterization of the Nakagami distribution, that we use as likelihood in BMLAT. We also briefly discuss how to incorporate knowledge through priors. Furthermore, our probabilistic formulation inherently handles scenarios where multiple noisy distance measurements are available for each reference point, a common practical situation. Importantly, BMLAT is easy to implement and we can assess its full posterior using off-the-shelf Markov Chain Monte Carlo (MCMC) software [14].

Since the proposed BMLAT is flexible enough to incorporate knowledge from a given application of interest and inference follows standard Bayesian procedures, we expect it to become a valuable tool for practitioners.

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Fig. 2: **Graphical model for BMLAT.** Diamonds denote hyperparameters, circles are random variables, and shaded nodes are observations. The plate indicates repetition.

II. MULTILATERATION

Suppose D is the dimension of the euclidean space we are working on¹. Let $\mathbf{t} \in \mathbb{R}^D$ be the POI position, $\mathbf{r}_k \in \mathbb{R}^D$ the position of the k -th reference point, K the number of reference points, N_k the number of observed distances for the k -th reference point, δ_{kn} the n -th observed distance for the k -th reference point, and $\delta = \{\delta_{kn}, \forall k, n \in \mathbb{Z}^+ | k \leq K \text{ and } n \leq N_k\}$ the set of observed distances.

Multilateration is a localization task based on the observations of distances to reference points. Fig. 1 illustrates a scenario with $K = 3$ reference points in $\mathbb{R}^2$. In (a) the measurements are known exactly, while in (b) there is uncertainty in the measurements. When the observations are accurate, the POI is exactly at the intersection of the circles centered on the reference points, with each radius given by the observed distances. However, this is not true when there is inaccuracy in the distance measurements. The most common way to deal with such an issue is to look for the point whose distances to the reference points are as close as possible to the observed distances, a task that can be modeled as a non-linear optimization problem.

In the following section, we use Bayesian statistics to tackle the MLAT task. We incorporate all uncertainties using probability distributions and then use the Bayes' rule to update our belief about the POI's position.

III. BAYESIAN MULTILATERATION

The graphical model of the proposed Bayesian formulation for the MLAT task is depicted in Fig. 2. Our goal is to compute the posterior probability $p(\mathbf{t}|\delta)$ given the set δ of observed distances for each reference point. The problem formulation is mathematically described by

$$p(\mathbf{t}|\delta) \propto p(\mathbf{t}) \prod_{k=1}^K \int \prod_{n=1}^{N_k} p(\delta_{kn}|\mathbf{t}, \mathbf{r}_k) p(\mathbf{r}_k) d\mathbf{r}_k, \quad (1)$$

where $p(\mathbf{t})$ and $p(\mathbf{r}_k)$ are respectively the priors for the POI and the reference point $\mathbf{r}_k$, and $p(\delta_{kn}|\mathbf{t}, \mathbf{r}_k)$ is the likelihood function. Their choices are described as follows.

¹We consider only $D = 2$ and leave scenarios w/ $D > 2$ to further work.

Fig. 3: **Reparameterized Nakagami.** Influence of the parameter θ on the shape of the likelihood. All distributions above have mode equal to δ , but become flatter as θ increases.

A. Prior Distributions

We need to define prior distributions for the POI and the reference points. In both cases, in the absence of additional knowledge, we opt for distributions that cover the entire $\mathbb{R}^D$. Specifically, we use multivariate Gaussians with diagonal covariances. Let $\mathbb{I}_D$ denote the D -dimensional identity matrix, the priors are given by

$$\begin{aligned} \mathbf{t} &\sim \mathcal{N}(\boldsymbol{\mu}, \sigma_t^2 \mathbb{I}_D), \\ \mathbf{r}_k &\sim \mathcal{N}(\boldsymbol{\eta}_k, \sigma_r^2 \mathbb{I}_D). \end{aligned} \quad (2)$$

We can easily incorporate prior knowledge by setting the hyper-parameters in Equation 2. The value $\boldsymbol{\mu} \in \mathbb{R}^D$ can be an initial guess for the position $\mathbf{t}$. It can be obtained, for instance, using some conventional MLAT solver. $\sigma_t > 0$ quantifies the uncertainty about this initial guess and a high value can be used to indicate an uninformative prior. The $\boldsymbol{\eta}_k \in \mathbb{R}^D, \forall k$, values can be the measured reference points positions, the same used in standard MLAT approaches. Finally, $\sigma_r > 0$ indicates the uncertainty regarding such positions.

B. Likelihood Function

Since δ_{kn} represents the n -th measured distance from the k -th reference point, we must choose a likelihood function such as that $p(\delta_{kn}|\mathbf{t}, \mathbf{r}_k) = 0, \forall \delta_{kn} < 0$. Thus, only likelihoods with positive support are suitable. Squared distances between random variables with different distributions can be approximated by a gamma distribution [3], [15], [7], [1]. By taking the square root of a gamma random variable, a Nakagami random variable is obtained. Thus, in our framework, we choose a Nakagami likelihood to represent the measured distances. A similar assumption was also presented in [12].

The Nakagami distribution is described in terms of the shape parameter $m \geq \frac{1}{2}$ and the spread parameter $\Omega > 0$:

$$\text{Naka}(x|m, \Omega) = \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} e^{-\frac{m}{\Omega} x^2}. \quad (3)$$

We choose those parameters so that the distribution mode is at the calculated distance $\hat{\delta}_k = \|\mathbf{t} - \mathbf{r}_k\|$ to the corresponding reference point. The spread constitutes a free hyperparameter of the model. Thus, for the k -th reference point:

$$\hat{\delta}_k = \frac{\sqrt{2}}{2} \sqrt{\frac{(2m_k - 1)\Omega_k}{m_k}} \implies m_k = \frac{\Omega_k}{2(\Omega_k - \hat{\delta}_k^2)}. \quad (4)$$

Fig. 4: **Results for the artificial dataset.** (a) In terms of RMSE, BMLAT outperforms the other methods for any number of references and observation. (b) Additionally, BMLAT is the method that places higher density on the true POI value.

In the above formulation, we need to ensure that $\Omega_k > \hat{\delta}_k^2$. Thus, we introduce the hyperparameter θ_k and define $\Omega_k = \theta_k + \hat{\delta}_k^2$. This ensures that the previous constraint will be met for every $\theta_k > 0$. Furthermore, by substituting the value of θ_k in Eq. (4), we conclude that $m_k = \frac{\theta_k + \hat{\delta}_k^2}{2\theta_k}$. From this definition, the likelihood function depends only on the calculated distance $\hat{\delta}_k$ and the spread parameter θ_k . Thus, for simplicity, we define a reparameterization of the Nakagami distribution denoted by $\mathfrak{N}\text{aka}$. For any $\delta \geq 0$ and any positive θ , it is written as:

$$\mathfrak{N}\text{aka}(\delta, \theta) \triangleq \text{Naka}\left(m = \frac{\theta + \delta^2}{2\theta}, \Omega = \theta + \delta^2\right). \quad (5)$$

To illustrate the influence of θ , we evaluate the Nakagami likelihood function according to the above parameterization considering the calculated distances $\delta \in \{20, 30, 40\}$ and $\frac{3\delta}{20} \leq \theta \leq \frac{10\delta}{20}$. The results can be seen in Fig. 3. Regardless of the θ value, the mode remains at the calculated distance δ . Moreover, the larger the θ , the wider the curve of the likelihood. Furthermore, one can show experimentally that there is a linear relation between θ and the variance of the likelihood, given by $\text{var}[\mathfrak{N}\text{aka}(\delta, \theta)] \approx \frac{\theta}{2}$.

The final reparametrized likelihood function is given by

$$\delta_{kn} \sim \mathfrak{N}\text{aka}(\|t - \mathbf{r}_k\|, \theta_k). \quad (6)$$

C. Bayesian inference

The choices that we have made for the priors and the likelihood function do not result in a conjugate model for the Eq. (1). Thus, we follow a MCMC strategy to sample from the posterior $p(\mathbf{t}|\delta)$ of the POI given the measured distances. In this work, we use the NUTS algorithm [4] as implemented in the Python library pyStan [14]. The use of other approximate inference methods is left for future investigations.

IV. EXPERIMENTS

We evaluate our proposal using both real and artificial datasets. We use MLAT [10], LMLAT [5], PMLAT [11], and MMLAT [6] as baselines. For a fair comparison, we use bootstrap to derive uncertainty estimates for these methods. More specifically, we run each classical method 2000 times, each one with a random measure of one among the N_k distances to each reference point. We emphasize that the proposed BMLAT can directly cope with all the available measurements and compute a posterior distribution for the POI. In all experiments, we use the result of the linear method (LMLAT) to define the parameters of the POI's prior distribution, where $\boldsymbol{\mu}$ is the sample mean and σ_t^2 is the scale of the covariance matrix.

A. Synthetic data

Data. The data was generated according to the generative model described in Section III, with $\boldsymbol{\mu} = \mathbf{0}$, $\sigma_t = 10$, and $\theta_k = 8 \forall k$. We fix $\boldsymbol{\eta}_k$ at a distance 40 from the POI and distribute it around the POI, so that the distance between two adjacent points is constant. We vary the number of reference points ($K \in \{3, 4, 5\}$); the number of distances to each reference point ($N_k \in \{10, 50, 100\}, \forall k$); and the standard deviation of the reference points distribution ($\sigma_r \in \{4, 5.5, 7, 8.5, 10\}$).

Results. Fig. 4 shows the mean and the 95% confidence interval for the RMSE and the posterior density of the POI, along with 30 runs for each method and each scenario. We can see that both metrics have monotonic behaviour about the increase of σ_r . The results also slightly improve as K or N_k increase (mainly in the first two columns). In addition, BMLAT consistently obtains better results in all settings.

Fig. 5: **Does BMLAT concentrate on the right spot?** The plots show posterior contours for BMLAT (black curve) for synthetic and real-world data. Colored circles represent distance observations, with different colors for each reference point. The star denotes the true POI location. (a) Shows the effect of increasing the variance around each reference’s position. (b) Shows results for five real-world problems. In all cases, the BMLAT posterior concentrates more tightly on the true POI.

dataset	Distance / RMSE					Posterior density				
	BMLAT	LMLAT	MLAT	PMLAT	MMLAT	BMLAT	LMLAT	MLAT	PMLAT	MMLAT
0	2.02e+00	1.06e+01	8.58e+00	5.26e+00	4.03e+00	6.53e-02	2.37e-03	3.58e-03	1.01e-02	1.62e-02
1	1.51e+00	3.56e+00	3.14e+00	2.62e+00	3.97e+00	1.11e-01	1.96e-02	2.44e-02	3.37e-02	1.13e-02
2	1.84e+00	6.86e+00	5.40e+00	3.77e+00	2.74e+00	7.58e-02	4.81e-03	8.42e-03	1.81e-02	3.67e-02
3	1.18e+00	2.76e+00	2.32e+00	2.02e+00	2.91e+00	1.82e-01	3.52e-02	4.85e-02	7.18e-02	1.56e-02
4	1.26e+00	2.44e+00	2.72e+00	2.06e+00	2.15e+00	1.58e-01	3.90e-02	3.25e-02	5.79e-02	4.52e-02
mean	1.56e+00	5.25e+00	4.43e+00	3.15e+00	3.16e+00	1.18e-01	2.02e-02	2.35e-02	3.83e-02	2.50e-02

TABLE I: **Results for the real dataset.** Average distance of the posterior samples to the POI for the real datasets. BMLAT obtained the best performance among all methods, with lower RMSE and placing higher density on the true POI location.

Fig. 5a shows the POI posterior distribution for each method considering the scenarios with $K = 3$ reference points and $N_k = 50, \forall k$, observed distances. All methods have posterior samples approximately centred on the POI, but the BMLAT defines a region with a consistently smaller radius.

B. Real world data

Data. We use a dataset of indoor measurements using Wi-Fi received signal strength indicator (RSSI)[8], which we convert into distances following the methodology by Zhu and Feng [18]. We focus on 5 POIs, each with approximate 400 distances measurements from 4 reference points.

Results. We evaluate our method with $\sigma_r = 2$ for all datasets and θ_k equals to the variance of the observed distances for the k -th reference point. Table I shows that the BMLAT has significantly better RMSE and posterior density than the other methods. In Fig. 5b, we can see that the posterior distribution of the BMLAT delimits a considerably smaller region.

V. CONCLUDING REMARKS

We propose using Bayesian statistics to model the uncertainties that arise when applying multilateration to real-

world settings. Moreover, we follow standard Bayesian inference strategies to marginalize all the unknown quantities and produce a posterior distribution for the POI position. The priors and the likelihood within our framework were carefully chosen to better represent the characteristics of the localization problem. We have also presented practical guidelines for choosing prior hyperparameters. Importantly, our approach naturally incorporates multiple distance observations for each reference point, a common scenario. Remarkably, BMLAT outperforms classical methods on artificial and real-world data.

In the future, we will evaluate the BMLAT in scenarios where $D > 2$. Moreover, to enable BMLAT in embedded devices, it is worth investigating variational approximations for the posterior in Eq. (1). These approximations may provide a valuable trade-off between accuracy and computational effort. Finally, the model itself can be enhanced by including priors to its hyperparameters, which, for instance, would allow different noise variances for each reference point.

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