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Coupled Alternating Neural Networks for Solving Multi-Population High-Dimensional Mean-Field Games with Stochasticity

Guofang Wang, Wang Yao, Xiao Zhang, and Zijia Niu

Abstract—Multi-population mean-field game is a critical subclass of mean-field games (MFGs). It is a theoretically feasible multi-agent model for simulating and analyzing the game between multiple heterogeneous populations of interacting massive agents. Due to the factors of game complexity, dimensionality disaster and disturbances should be taken into account simultaneously, how to solve the multi-population high-dimensional stochastic MFG problem is faced with great challenges. We present CANet, a coupled alternating neural network approach for tractably solving multi-population high-dimensional MFGs in the stochastic case. First, we provide a universal modeling framework for large-scale heterogeneous multi-agent game systems, which is strictly expressed as a multi-population MFG problem. Next, we generalize the potential variational primal-dual structure that MFGs exhibit, then phrase the multi-population MFG problem as a convex-concave saddle-point problem. Last but not least, we design a generative adversarial network (GAN) with multiple generators and multiple discriminators—the solving network, which parameterizes the value functions and the density functions of multiple populations by two sets of neural networks, respectively. Moreover, numerical experiments demonstrate the feasibility and effectiveness of our approach.

Index Terms—Multi-population model, mean-field game (MFG), neural networks, high-dimensional solution space.

I. INTRODUCTION

MEAN-FIELD games (MFGs) [1]–[6] are a class of models that simulate and analyze large populations of interacting agents. The MFGs theory can be used to solve the communication and calculation difficulties caused by large-scale multi-agent system (MAS) scenarios, that is, the core is to transform the 1-vs- N game into a 1-vs-1 game form. In other words, the communication and computational complexities are no longer related to the agent number [7]. They are widely used in economics [8]–[11], finance [12]–[15], industrial engineering [16]–[18], crowd motion [19]–[22], swarm robotics [23], [24], epidemic modeling [25], [26], data science [27]–[29], and material dynamics [30].

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From the perspective of the implementation of MFGs theory, we will find that although MFGs theoretical research is relatively advanced and mature, its numerical solution technology is far from reaching the needs of actual scenarios [31]. Many researchers have made efforts on the numerical solution of MFGs and have produced many excellent results. But most of them are directed against single-population. Specifically, the simultaneous pioneering works of MFGs theory are proposed by Lasry & Lions [1]–[3] and Huang, Malhame & Caines [4]–[6] independently. After that, some kinds of literature use rigorous analytical methods to study numerical algorithms in various situations, including continuous-state problems [35]–[47] and finite-state problems [48], [49]. In the absence of explicit solutions, there have been too many fast solvers for the solution of MFGs in two and three dimensions [50]–[55], but they all need grid-based spatial discretization. Grid-based numerical solutions will be cursed by dimensionality, namely, their computational complexity increases exponentially with the spatial dimension [32]. While the numerical solution method based on neural networks will be more suitable for the high-dimensional HJB-FPK coupled equations [33]. In terms of the numerical methods for solving high-dimensional MFGs, [31] perfectly avoids spatial grids of high-dimensional MFGs based on the machine learning framework, but their work is limited to the deterministic setting ($\sigma = 0$). Furthermore, [56] is the first document to solve the high-dimensional MFGs under a stochastic setting ($\sigma > 0$), which utilizes the natural connection between MFGs and generative adversarial neural networks (GANs) [57]. Based on the original technical difficulties (large-scale, high-dimensional), the multi-population model increases the complexity of sequential games, including heterogeneous group communication and two-level game reasoning. This makes the numerical framework for solving single-population MFGs no longer applicable. To sum up, for meeting the high requirements of more complex actual scene modeling, it is an urgent need to break through the numerical solution technology of multi-population high-dimensional stochastic MFGs.

Multi-population mean-field games, as an important subclass of MFGs, naturally arise in most practical applications [34]. Since the original contribution of [5], many researchers have studied it by analytical methods, but the technical gap in the calculation of equilibrium solution is particularly prominent. [58] and [59] dealt with ergodic MFGs of multi-population under different boundary conditions, including the existence and uniqueness results of systems. [19] and [21]

consider mean field control problems with several populations, and the authors introduce these models for crowd motion (with local and non-local interactions respectively). They have studied optimality conditions and provided numerical results. [60] studies the issue of urban settlements and residential choice by using the MFG model to describe the interactions between two populations and provides some numerical simulations. [34] and [63] systematically study the problems of mean-field games and mean-field type control problems with multi-population by analyzing the interaction of two populations. It can be seen that the above authors have studied the theory and application of multi-population MFGs, but the equilibrium solution parts generally adopt analytical methods, and their numerical examples are limited to low dimensions (ordinarily one-dimensional or two-dimensional). The numerical solution method of multi-population high-dimensional MFGs is still relatively blank.

Inspired by the above-mentioned cutting-edge works, in this paper, we are committed to solving the difficult problem of numerical solution for multi-population high-dimensional MFGs with stochasticity, which needs to take into account factors of game complexity, dimensionality disaster, and disturbances simultaneously.

Main Contribution

We present CA-Net, a coupled alternating neural network approach for tractably solving multi-population high-dimensional MFGs in the stochastic case ($\sigma > 0$). To this end, we have made the following contributions.

1) We provide a general framework for modeling large-scale heterogeneous multi-agent game systems, give a unified expression of multi-population MFGs under a two-layer game structure, and ensure the existence of equilibrium of such systems.

2) We generalize the potential variational primal-dual structure that MFGs exhibit and phrase the multi-population MFG problem as a convex-concave saddle-point problem. In this way, we circumvent spatial grids or uniformly sampling in high dimensions (the curse of dimensionality) and provide the cornerstone for the construction of the following solving network.

3) Based on the saddle-point problem, we parameterize the value functions and the density functions of multiple populations by two sets of neural networks, respectively. Then we design a generative adversarial network (GAN), a nonlinear coupled alternating neural network composed of multiple generators and multiple discriminators, that is, the solving network, referred to as CA-Net.

4) We simulate a classic task of heterogeneous MAS with numerical examples, verify and analyze the feasibility and effectiveness of multi-population high-dimensional stochastic MFG model and CA-Net algorithm. And the comparison with the baseline methods shows the advanced nature of our

Fig. 1: Illustration of a multi-population model with local interactions. Large-scale agents are divided into different populations (indicated by different colors) according to different characteristics to participate in the game. The communication between populations follows the communication topology diagram (population is the vertex, and the communication is the black solid line). A representative continuum population (bottom shaded part) with small rational agents communicates with three neighboring populations to play a non-cooperative differential game on a time horizon.

method.

To the best of our knowledge, CA-Net is the first model that solves multi-population high-dimensional MFGs with stochasticity ($\sigma > 0$).

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. Control Formulation of Multi-Population Model

We consider a multi-agent system (MAS) with N agents indexed by $i \in I = \{1, \dots, N\}$, who take part in the game are not all homogeneous, and can be subdivided in P agent populations indexed by $p(i) \in Q = \{1, \dots, P\}$, as illustrated in Fig. 1. And each continuum population p with small rational agents plays a non-cooperative differential game on a time horizon $[0, T]$. To model the set of P distinct and interconnected populations, we assume a graph $G = (Q, E)$ is given, where $Q = \{1, \dots, P\}$ is the set of vertices, one per each agent population, and $E \subseteq Q \times Q$ is the set of all weighted edges. More specifically, $\arg(p, q)$ belongs to E if the corresponding weight $v_{p,q} > 0$. A positive scalar weight $v_{p,q}$ means the relative influence that population p assigns to population q . For example, $v_{p,q}$ could be calculated as the number of agents of population q over the total number of agents of all the neighbor populations. Although most results are easily extended to more general graphs, possibly directed, time-varying, for the sake of simplicity, we henceforward assume that $G = (Q, E)$ is a connected undirected graph. We denote the set of neighbors of p by $V(p) = \{q \in Q | (p, q) \in E\}$, and the cardinality of the set $V(p)$ by $|V(p)|$.

$$J_p(u^p, \rho^p, \rho^{-p}) = \mathbb{E} \left[\int_0^T L_p(x^p(t), u^p(t)) + F_p(x^p(t), \rho^p(x^p(t), t), \rho^{-p}(x^{-p}(t), t)) dt + G_p(x^p(T), \rho^p(x^p(T), T), \rho^{-p}(x^{-p}(T), T)) \right] \quad (2)$$

In the same agent population, we argue that the agents share a common characteristic (e.g. type, function, angle range, or upper bound of velocity). Each agent $i \in I$ belongs to a population $p(i) \in Q$ and is characterized by a state $x^p(t) \in \mathbb{R}^n$ at time $t \in [0, T]$. More precisely, we consider that the state of the generic player i of population p is a stochastic process $\{x_{p \in Q}^p\}$ in $\mathbb{R}^{Pn}$ with the following forward McKean-Vlasov dynamics

$$dx^p = h^p(x^p, \rho^p, \rho^{-p}, u^p)dt + \sigma^p dw^p \quad (1)$$

where $u^p : [0, T] \rightarrow \mathcal{U}_p \subseteq \mathbb{R}^m$ is the control input (strategy), $\rho^p : \mathbb{R}^n \times [0, T] \rightarrow \mathcal{P}_2(\mathbb{R}^n)$ is the probability density function, $\rho^{-p} = (\rho^q)_{q \in V(p)} : \mathbb{R}^n \times [0, T] \rightarrow \mathcal{P}_2(\mathbb{R}^n)^{|V(p)|}$ describes the distribution vector of neighbor populations, $h^p : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)^{|V(p)|+1} \times \mathcal{U}_p \rightarrow \mathbb{R}^n$ is the nonlinear evolution function, $\sigma^p \in \mathbb{R}^{n \times m}$ is a fixed coefficient matrix of population dependent volatility term, w^p denotes an m -dimensional Wiener process (standard Brownian motion) comes from the environment, where each component w_k^p is independent of w_l^p for all $k \neq l$, $p = 1, \dots, P$.

Then, for the generic player i of population p , $p = 1, \dots, P$, we consider the following cost function (2), as shown at the bottom of the previous page.

In Eq. (2), $L_p : \mathbb{R}^n \times \mathcal{U}_p \rightarrow \mathbb{R}$ is a running cost incurred by an agent based solely on their actions, $F_p : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)^{|V(p)|+1} \rightarrow \mathbb{R}$ is a running cost incurred by an agent based on their interaction with rest of the same population and the neighbor populations, and $G_p : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)^{|V(p)|+1} \rightarrow \mathbb{R}$ is a terminal cost incurred by an agent based on their final state and the final distribution of the whole related populations. The terms F and G are called mean field terms because they encode the interaction of a single agent with the rest of the populations.

B. The Formulation of Multi-Population Mean-Field Game

Mean-field games (MFGs) are a class of problems that encode large populations of interacting agents into systems of coupled partial differential equations. As an important subclass of MFGs, multi-population mean-field game not only has the advantages of MFG, e.g., it overcomes the difficulties of large-scale and information sharing, but allows for the representation of some more realistic situations, such as [60]–[62].

For every population $p = 1, \dots, P$, each agent i of population $p(i)$ forecasts a distribution of the population $\{\rho^p(\cdot, t)\}_{t=0}^T$, and aims at minimizing their cost which eventually reaches the Nash equilibrium where no agent can reduce

individual cost by changing its control strategy unilaterally. That is, for every $x^p \in \mathbb{R}^n$,

$$J_p(\hat{u}^p, \rho^p, \rho^{-p}) \leq J_p(u^p, \rho^p, \rho^{-p}), \quad \forall u^p : [0, T] \rightarrow \mathcal{U}_p \quad (3)$$

where $\hat{u}^p$ is the equilibrium strategy of an agent at state x^p . Here, we assume that agents are small, and their unilateral actions do not alter the density ρ^p . Literature [34] and [63] of multi-population MFG provide sufficient and necessary conditions for the solution to the coupled system of forward-backward stochastic partial differential equations (SPDEs), respectively.

Remark 1: Under appropriate assumptions, the solution to the MFG provides an approximate Nash equilibrium for the corresponding game with a large but finite number of agents [34].

From Eq. (3), we have that each agent i of population $p(i)$ solves an optimal control problem that has a value function

$$\Phi_p(x^p, t) = \inf_{u^p} J_p(u^p, \rho^p, \rho^{-p}), \quad \text{s.t. Eq. (1)} \quad (4)$$

From the optimal control theory (for details, see reference [64], sections 2.2, 2.3, and 3.4.2 or reference [63], section 4.1), we have that Φ_p solves the HJB equation (5), as shown at the bottom of the page.

where the Hamiltonian, $H_p : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)^{|V(p)|+1} \times \mathbb{R}^n \rightarrow \mathbb{R}$, is defined as

$$H_p(x^p, \rho^p, \rho^{-p}, z^p) = \sup_{u^p} \{-L_p(x^p, u^p) - z^{p\top} h^p(x^p, \rho^p, \rho^{-p}, u^p)\} \quad (6)$$

Hence we have

$$z^p = \nabla_{x^p} \Phi_p(x^p, t) \quad (7)$$

$$h^p(x^p, \rho^p, \rho^{-p}, \hat{u}^p) = -\nabla_{z^p} H_p(x^p, \rho^p, \rho^{-p}, \nabla \Phi_p(x^p, t)) \quad (8)$$

According to Ito's lemma (for details, see reference [64], sections 2.2 and 3.2.2), ρ^p satisfies the forward equation of x^p , i.e. the FPK equation (9), as shown at the bottom of the page, where $\rho_0^p \in \mathcal{P}_2(\mathbb{R}^n)$ is the given population density at time $t = 0$.

The above discussion provides an alternative to optimizing the strategy u^p individually for each agent of the multi-population model. One can obtain the optimal strategy for all agents of multi-population simultaneously by solving the coupled SPDEs system given by the HJB (5) and the FPK (9). Following this macroscopic approach, our first step is to accurately establish a multi-population agent model and reduce the immense computational challenges and communication energy consumption caused by heterogeneous multi-agent systems.

$$\text{HJB:} \quad -\partial_t \Phi_p(x^p, t) - \frac{\sigma^{p2}}{2} \Delta \Phi_p(x^p, t) + H_p(x^p, \rho^p, \rho^{-p}, \nabla \Phi_p(x^p, t)) = F_p(x^p, \rho^p(x^p, t), \rho^{-p}(x^{-p}, t)) \quad (5)$$

$$\Phi_p(x^p, T) = G_p(x^p, \rho^p(x^p, T), \rho^{-p}(x^{-p}, T))$$

$$\begin{aligned} \text{FPK:} \quad \partial_t \rho^p(x^p, t) - \frac{\sigma^{p2}}{2} \Delta \rho^p(x^p, t) - \text{div}(\rho^p(x^p, t) \nabla_{z^p} H_p(x^p, \rho^p, \rho^{-p}, \nabla \Phi_p(x^p, t))) &= 0 \\ \rho^p(x^p, 0) &= \rho_0^p(x^p) \end{aligned} \quad (9)$$

Remark 2: Instead of including all agents' states in the value function [65]–[67], the MFGs' deterministic flows of probability density describe the population's current information. Utilizing the density distribution can benefit from two aspects: 1) the dimension of the density distribution is independent of the number of agents so the dimension disaster is overcome theoretically and 2) the density distribution can be obtained by locally solving the FPK equation so the information sharing between all agents is not necessary.

Remark 3: From an engineering angle, the proposed multi-population MFG model can provide an accurate and feasible scheme for the large-scale heterogeneous MAS. Because it breaks the dimensionality curse of the system theoretically and ensures the Nash equilibrium solution under the limit conditions.

To compute the optimal control policy of the large-scale heterogeneous MAS, the mean-field equations (5) and (9) need to be solved in real-time. However, the HJB and FPK are two high-dimensional nonlinear SPDEs that are also solved in different directions. Therefore, the analytical solution is nearly impossible to obtain. In this article, a novel CA-Net algorithm induced from the idea of the generative adversarial network (GAN) is developed to find the solution of the coupled HJB-FPK equations in high dimensions.

III. GAN-BASED APPROACH FOR HIGH-DIMENSIONAL SPDEs OF MULTI-POPULATION PROBLEM

We phrase the multi-population MFG problem as a saddle-point problem and parameterize the value functions and the density functions of multi-population, then use a coupled alternating neural network (CA-Net) to find the equilibrium solution.

A. Variational Primal-Dual Formulation of Multi-Population Mean-Field Game

According to the natural connection between MFGs and GANs, we deduce the multi-population MFG formulation to a primal-dual convex-concave one, then can obtain a tailored coupled alternating network to find the equilibrium solution.

An multi-population MFG system consists of the two equations (5) and (9) is called potential, if there exist functionals $\mathcal{F}_p, \mathcal{G}_p : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}$, $p = 1, \dots, P$ such that

$$\delta_{\rho^p} \mathcal{F}_p = F_p(x^p, \rho^p, \rho^{-p}) \quad \text{and} \quad \delta_{\rho^p} \mathcal{G}_p = G_p(x^p, \rho^p, \rho^{-p})$$

where

$$\begin{aligned} \int_{\mathbb{R}^n} F_p(x^p, \rho^p) \mu(x^p) dx^p &= \lim_{h \rightarrow 0} \frac{\mathcal{F}_p(\rho^p + h\mu) - \mathcal{F}_p(\rho^p)}{h}, \forall \mu \\ \int_{\mathbb{R}^n} G_p(x^p, \rho^p) \mu(x^p) dx^p &= \lim_{h \rightarrow 0} \frac{\mathcal{G}_p(\rho^p + h\mu) - \mathcal{G}_p(\rho^p)}{h}, \forall \mu \end{aligned}$$

That is, there exist functionals $\mathcal{F}, \mathcal{G}$ such that their variational derivatives with respect to ρ are the interaction and terminal costs F and G from (5) and (9). A key feature of potential MFG is that it can be equivalently expressed as a convex-concave saddle point optimization problem. So the multi-population MFG problem in (5) and (9) can be reformulate

as a variational problem [3], [51]. For the generic player i of population p , we get:

$$\begin{aligned} & \inf_{\rho^p, u^p} \int_0^T \int_{\mathbb{R}^n} \rho^p(x^p, t) L_p(x^p(t), u^p(t)) dx^p dt \\ & + \int_0^T \mathcal{F}_p(x^p(t), \rho^p(x^p(t), t), \rho^{-p}(x^{-p}(t), t)) dt \\ & + \mathcal{G}_p(x^p(T), \rho^p(x^p(T), T), \rho^{-p}(x^{-p}(T), T)) \\ \text{s.t. } & \partial_t \rho^p - \frac{\sigma^{p2}}{2} \Delta \rho^p + \nabla \cdot (\rho^p h^p) = 0, \quad \rho^p(x^p, 0) = \rho_0^p(x^p) \\ & \partial_t \rho^q - \frac{\sigma^{q2}}{2} \Delta \rho^q + \nabla \cdot (\rho^q h^q) = 0, \quad \rho^q(x^q, 0) = \rho_0^q(x^q) \\ & \rho^{-p} = (\rho^q)_{q \in V(p)} \end{aligned} \quad (10)$$

Now, the running cost L_p could be regarded as the Lagrangian function corresponding to the Legendre transform of the Hamiltonian H_p (6), and $\mathcal{F}_p, \mathcal{G}_p$ are mean field interaction terms corresponding to F_p, G_p in (2).

In order to obtain the Lagrangian dual problem of (10) [62], [68], we start with introducing the value function $\Phi_p(x^p, t)$ as a Lagrange multiplier, then insert the constraint (FPK equations) into the objective (cost function) to get the following extended cost function:

$$\begin{aligned} & \sup_{\Phi_p} \inf_{\rho^p, u^p} \int_0^T \int_{\mathbb{R}^n} \rho^p(x^p, t) L_p(x^p(t), u^p(t)) dx^p dt \\ & + \int_0^T \mathcal{F}_p(x^p(t), \rho^p(x^p(t), t), \rho^{-p}(x^{-p}(t), t)) dt \\ & + \mathcal{G}_p(x^p(T), \rho^p(x^p(T), T), \rho^{-p}(x^{-p}(T), T)) \\ & - \int_0^T \int_{\Omega} \Phi_p(x^p, t) \left(\partial_t \rho^p - \frac{\sigma^{p2}}{2} \Delta \rho^p \right. \\ & + \nabla \cdot (\rho^p(x^p, t) h^p(x^p, \rho^p, \rho^{-p}, u^p)) \left. \right) dx^p dt \\ & - \sum_{q \in V(p)} \int_0^T \int_{\Omega} \Phi_q(x^q, t) \left(\partial_t \rho^q - \frac{\sigma^{q2}}{2} \Delta \rho^q \right. \\ & + \nabla \cdot (\rho^q(x^q, t) h^q(x^q, \rho^q, \rho^{-q}, u^q)) \left. \right) dx^q dt \end{aligned} \quad (11)$$

Following the approach of [51], [69], [70], we write the constraint $\partial_t \rho^p - \frac{\sigma^{p2}}{2} \Delta \rho^p + \nabla \cdot (\rho^p h^p) = 0$, $p = 1, \dots, P$ in weak form, i.e.,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} \rho^p \partial_t \Phi_p + \frac{\sigma^{p2}}{2} \Delta \Phi_p \rho^p + \nabla \Phi_p \cdot (\rho^p h^p) dx^p dt \\ & + \int_{\mathbb{R}^n} \Phi_p(x^p, 0) \rho_0^p(x^p) dx^p - \int_{\mathbb{R}^n} \Phi_p(x^p, T) \rho^p(x^p, T) dx^p = 0 \end{aligned} \quad (12)$$

Finally, utilizing (12) and integrating by parts, we can rewrite our problem as

$$\begin{aligned} & \inf_{\rho^p} \sup_{\Phi_p} \int_0^T \int_{\mathbb{R}^n} \left(\partial_t \Phi_p + \frac{\sigma^{p2}}{2} \Delta \Phi_p - H_p(x^p, \nabla \Phi_p) \right) \rho^p(x^p, t) dx^p dt \\ & + \int_0^T \mathcal{F}_p(x^p(t), \rho^p(x^p(t), t), \rho^{-p}(x^{-p}(t), t)) dt \\ & + \mathcal{G}_p(x^p(T), \rho^p(x^p(T), T), \rho^{-p}(x^{-p}(T), T)) \\ & + \int_{\mathbb{R}^n} \Phi_p(x^p, 0) \rho_0^p(x^p) dx^p - \int_{\mathbb{R}^n} \Phi_p(x^p, T) \rho^p(x^p, T) dx^p \\ & + \sum_{q \in V(p)} \left\{ \int_0^T \int_{\mathbb{R}^n} \left(\partial_t \Phi_q + \frac{\sigma^{q2}}{2} \Delta \Phi_q + \nabla \Phi_q \cdot h^q \right) \right. \\ & \times \rho^q(x^q, t) dx^q dt + \int_{\mathbb{R}^n} \Phi_q(x^q, 0) \rho_0^q(x^q) dx^q \\ & \left. - \int_{\mathbb{R}^n} \Phi_q(x^q, T) \rho^q(x^q, T) dx^q \right\} \end{aligned} \quad (13)$$

This formula can also be deduced in the context of HJB equations in probability measure spaces [52], or by integrating the HJB (5) and the FPK (9) equations for ρ^p and Φ_p , respectively [71].

In [71], it has been shown that all MFG systems coincide with an infinite-dimensional two-player general-sum game, and the potential MFGs are the ones that correspond to zero-sum games. In its elaboration, player 1 represents the whole population and their strategy is the population's probability distribution. While player 2 represents the generic agent and their strategy is the value function. The goal of player 2 is to provide a strategy that yields the best response of a generic agent against the population. This elaboration is consistent with the intuition behind generative adversarial networks (GANs). By drawing from a natural connection between MFGs and GANs, [56] presents APAC-Net, an alternating population and agent control neural network, which is the first model for solving high-dimensional MFGs of a single population in the stochastic case. We extended this methodology to high-dimensional MFGs of multiple characteristic populations in the stochastic setting ($\sigma^p > 0$) and solve the HJB-FPK coupled system (5)-(9) for a certain population p .

Remark 4: Under mild assumptions on $\mathcal{F}_p$ and $\mathcal{G}_p$, each spatial integral is really an expectation from ρ^p , $p = 1, \dots, P$. The formulation (13) is the cornerstone of our approach.

B. CA-Net

Recall that from Riemann integral to Lebesgue integral, the former focuses on grid-points of the definition domain, while the latter focuses on values of the function. The proposed CA-Net does not discretize the domain and solve for the function values on grid-points, but avoids these problems by parameterizing the function and solving for the function itself. This is also the fundamental reason why CA-Net can overcome the dimensional disaster of MFG equations. Another important point is that although the idea is inspired by GANs, and the training process of single-population MFG solving networks is analog to that of GANs (a single generator and discriminator). For multi-population MFG solving network, it is a nonlinear coupled alternating neural network formed by multiple generators and multiple discriminators. We show the structure and training process of our CA-Net in the diagram Fig. 2.

We initialize P pairs of neural networks $N_\omega^p(x^p, t)$ and $N_\theta^p(y^p, t)$, $p = 1, \dots, P$. We then let

$$\begin{aligned}\Phi_\omega^p(x^p, t) &= (1-t)N_\omega^p(x^p, t) + tG_p(x^p) \\ \mathcal{G}_\theta^p(y^p, t) &= (1-t)y^p + tN_\theta^p(y^p, t)\end{aligned}\quad (14)$$

where $y^p \sim \rho_0^p$ are samples drawn from the initial distribution, and $\mathcal{G}_\theta^p$ produces samples from ρ_0^p at $t = 0$. While, we set $\rho^p(\cdot, t) = \mathcal{G}_\theta^p(\cdot, t) \# \rho_0^p$, it is the push-forward of ρ_0^p . In this setting, we train $\mathcal{G}_\theta^p(\cdot, t)$ to produce samples from $\rho^p(\cdot, t)$. We note that Φ_ω^p and $\mathcal{G}_\theta^p$ automatically satisfy the terminal and initial condition, respectively.

Our approach for training neural networks includes parallel alternately training P pairs of $\mathcal{G}_\theta^p$ (the density distribution of population p) and Φ_ω^p (the value function for the generic agent of population p). Intuitively, in order to obtain the equilibrium of multi-population MFG, we are training a coupled alternating neural networks (CA-Net) about multi-group distributions and agent controls. Specifically, we train Φ_ω^p by

Fig. 2: Visualization of the structure and training process of CA-Net. CA-Net is composed of multiple generators (Bottom) and multiple discriminators (Top). Its training process is divided into two coupled alternating training parts. Initialize the weights of Φ_ω^p and $\mathcal{G}_\theta^p$, $p = 1, \dots, P$ randomly, the first step is to train the value function network Φ_ω^p , batch a random sample points $y_b^p, y_b^q, t_b, b = 1, \dots, B$, get x_b^p, x_b^q forward through the density function network $\mathcal{G}_\theta^p$. The weight ω of Φ_ω^p is updated by the gradient descent algorithm to minimize the loss function $\ell_0 + \ell_t + \ell_{\text{neighbors}} + \ell_{\text{HJB}}$. The second step is to train the density function network $\mathcal{G}_\theta^p$, batch a random sample points $y_b^p, y_b^q, t_b, b = 1, \dots, B$, update the weight θ of $\mathcal{G}_\theta^p$ through the value function network Φ_ω^p and the gradient descent algorithm to minimize the loss function ζ_t . Then loop the two parts to achieve the optimal result.

first sampling a batch $\{y_b^p\}_{b=1}^B$ from the given initial density ρ_0^p , and $\{t_b\}_{b=1}^B$ uniformly from $[0, 1]$, i.e. we are sampling from $\rho_0^p \times \text{Unif}([0, 1])$. Next, we compute the push-forward states $x_b^p = \mathcal{G}_\theta^p(y_b^p, t_b)$ for $b = 1, \dots, B$. It should be emphasized that we construct the loss functions of CA-Net according to the variational primal-dual formulation (13) of multi-population MFG. So, here is the main loss item for training the discriminator,

$$\begin{aligned}\text{loss}_{\Phi^p} &= \frac{1}{B} \sum_{b=1}^B \Phi_\omega^p(x_b^p, 0) + \frac{1}{B} \sum_{b=1}^B \{ \partial_t \Phi_\omega^p(x_b^p, t_b) \\ &\quad + \frac{\sigma^{p2}}{2} \Delta \Phi_\omega^p(x_b^p, t_b) - H_p(\nabla_{x^p} \Phi_\omega^p(x_b^p, t_b)) \}\end{aligned}\quad (15)$$

Of course, we need to consider adding a dual penalty term from neighbors.

$$\begin{aligned} \text{penalty}_{\text{neighbors}} = & \eta \sum_{q \in V(p)} \left\{ \frac{1}{B} \sum_{b=1}^B [\partial_t \Phi_{\omega}^q(x_b^q, t_b) + \frac{\sigma^{q2}}{2} \Delta \Phi_{\omega}^q(x_b^q, t_b) \right. \\ & \left. + \nabla_{x^q} \Phi_{\omega}^q(x_b^q, t_b) \cdot h^q(x_b^q, t_b)] \right\} \end{aligned} \quad (16)$$

And we can optionally add a regularization term to penalize deviations from the HJB equations [31], [72].

$$\begin{aligned} \text{penalty}_{\text{HJB}} = & \lambda \frac{1}{B} \sum_{b=1}^B \left\| \partial_t \Phi_{\omega}^p(x_b^p, t_b) + \frac{\sigma^{p2}}{2} \Delta \Phi_{\omega}^p(x_b^p, t_b) \right. \\ & \left. - H_p(\nabla_{x^p} \Phi_{\omega}^p(x_b^p, t_b)) + F_p(x_b^p, t_b) \right\| \end{aligned} \quad (17)$$

This extra regularization term has also been found effective in, e.g., Wasserstein GANs [73]. Finally, we backpropagate the loss to the weights of Φ_{ω}^p . To train the generator, we again sample $\{y_b^p\}_{b=1}^B$ and $\{t_b\}_{b=1}^B$ as before, and compute

$$\begin{aligned} \text{loss}_{G^p} = & \frac{1}{B} \sum_{b=1}^B \left\{ \partial_t \Phi_{\omega}^p(G_{\theta}^p(y_b^p), t_b) + \frac{\sigma^{p2}}{2} \Delta \Phi_{\omega}^p(G_{\theta}^p(y_b^p), t_b) \right. \\ & \left. - H_p(\nabla_{x^p} \Phi_{\omega}^p(G_{\theta}^p(y_b^p), t_b)) + F_p(G_{\theta}^p(y_b^p), t_b) \right\} \end{aligned} \quad (18)$$

Finally, we backpropagate this loss to the weights of G_{θ}^p . Please refer to Algorithm 1 for detailed operation flow.

IV. NUMERICAL EXPERIMENTS

In this section, We demonstrate the potential of the proposed CA-Net on large-scale heterogeneous MAS control problems. First, we show the feasibility of using multi-population MFG to model large-scale heterogeneous MAS and the convergence of using CA-Net to solve the corresponding problems in high-dimensional space in Sec. IV-A. Second, we also illustrate the behaviors of the multi-population MFG solutions for different values of stochastic characteristic σ in Sec. IV-B. Third, we further reveal the relationship between algorithm parameters and costs in Sec. IV-C. Last, to demonstrate the competitiveness of our approach, we compare the CA-Net algorithm's performance against baseline methods in Sec. IV-D.

A. Multi-Group Quadcopters Trajectory Planning Example

This experiment is inspired by [56], [74], [75], we examine a realistic scenario where the dynamics are that of a quadcopter (a.k.a. quadrotor craft), which is an aerial vehicle with four rotary wings (similar to many of the consumer drones seen today), and utilizes its four rotors to propel itself across space from an initial state to target state. The designed test environment is a 3-D real flight space and has a total of a thousand drones, which are divided into five populations with different velocity ranges to perform tasks—flying from their initial distribution to the common terminal distribution. Here, graph $G = (Q, E)$ is considered as a global communication between groups. The dynamics of the generic quadrotor craft i of population p , $p = 1, \dots, P$, are given as,

$$\begin{cases} \ddot{x}_p = \frac{u_p}{m_p} (\sin(\phi_p) \sin(\psi_p) + \cos(\phi_p) \cos(\psi_p) \sin(\theta_p)) \\ \ddot{y}_p = \frac{u_p}{m_p} (-\cos(\psi_p) \sin(\phi_p) + \cos(\phi_p) \sin(\theta_p) \sin(\psi_p)) \\ \ddot{z}_p = \frac{u_p}{m_p} \cos(\theta_p) \cos(\phi_p) - g \\ \ddot{\psi}_p = \tilde{\tau}_{\psi_p} \\ \ddot{\theta}_p = \tilde{\tau}_{\theta_p} \\ \ddot{\phi}_p = \tilde{\tau}_{\phi_p} \end{cases} \quad (19)$$

where x_p, y_p , and z_p are the usual Euclidean spatial coordinates of the quadcopter, and ϕ_p, θ_p and ψ_p are the angular

coordinates of roll, pitch, and yaw, respectively. The constant m_p is the quadcopter's mass, and g is the gravitation acceleration constant on Earth. The variables $u_p, \tilde{\tau}_{\psi_p}, \tilde{\tau}_{\theta_p}, \tilde{\tau}_{\phi_p}$ are the controls representing thrust, and angular acceleration, respectively [76]. In order to fit a control framework, the above second-order system is turned into a first-order system:

$$\begin{cases} \dot{x}_p = v_{x_p} \\ \dot{y}_p = v_{y_p} \\ \dot{z}_p = v_{z_p} \\ \dot{\psi}_p = v_{\psi_p} \\ \dot{\theta}_p = v_{\theta_p} \\ \dot{\phi}_p = v_{\phi_p} \\ \dot{v}_{x_p} = \frac{u_p}{m_p} h_7(\psi_p, \theta_p, \phi_p) \\ \dot{v}_{y_p} = \frac{u_p}{m_p} h_8(\psi_p, \theta_p, \phi_p) \\ \dot{v}_{z_p} = \frac{u_p}{m_p} h_9(\theta_p, \phi_p) - g \\ \dot{v}_{\psi_p} = \tilde{\tau}_{\psi_p} \\ \dot{v}_{\theta_p} = \tilde{\tau}_{\theta_p} \\ \dot{v}_{\phi_p} = \tilde{\tau}_{\phi_p} \end{cases} \quad (20)$$

where

$$\begin{cases} h_7(\psi_p, \theta_p, \phi_p) = \sin(\phi_p) \sin(\psi_p) + \cos(\phi_p) \cos(\psi_p) \sin(\theta_p) \\ h_8(\psi_p, \theta_p, \phi_p) = -\cos(\psi_p) \sin(\phi_p) + \cos(\phi_p) \sin(\theta_p) \sin(\psi_p) \\ h_9(\theta_p, \phi_p) = \cos(\theta_p) \cos(\phi_p) \end{cases} \quad (21)$$

which we will compactly denote as $\dot{\mathbf{x}}^p = \mathbf{h}^p(\mathbf{x}^p, \mathbf{u}^p)$, where $\mathbf{h}^p$ is a 12-dimensional vector function in the right-hand side of (20), $\mathbf{x}^p = [x_p, y_p, z_p, \psi_p, \theta_p, \phi_p, v_{x_p}, v_{y_p}, v_{z_p}, v_{\psi_p}, v_{\theta_p}, v_{\phi_p}]^T \in \mathbb{R}^{12}$ is the state with velocities $\mathbf{v}^p = [v_{x_p}, v_{y_p}, v_{z_p}, v_{\psi_p}, v_{\theta_p}, v_{\phi_p}]^T \in \mathbb{R}^6$, and $\mathbf{u}^p = [u_p, \tilde{\tau}_{\psi_p}, \tilde{\tau}_{\theta_p}, \tilde{\tau}_{\phi_p}]^T \in \mathbb{R}^4$ is the control. As can be seen, (20) is a 12-dimensional system that is highly coupled, nonlinear and high-dimensional. In the stochastic case, we add a noise term to the dynamics: $d\mathbf{x}^p = \mathbf{h}^p(\mathbf{x}^p, \mathbf{u}^p)dt + \sigma^p d\mathbf{W}_t^p$, where $\mathbf{W}$ denotes a Wiener process (standard Brownian motion). The interpretation here is that we are modeling the situation when the quadcopter suffers from noisy measurements.

For the Lagrangian cost function, we consider

$$\begin{aligned} L_p(\mathbf{u}^p) = & \frac{1}{2} \|\mathbf{u}^p(t)\|_2^2 \\ = & \frac{1}{2} \left(u_p^2(t) + \tilde{\tau}_{\psi_p}^2(t) + \tilde{\tau}_{\theta_p}^2(t) + \tilde{\tau}_{\phi_p}^2(t) \right) \end{aligned} \quad (22)$$

We consider the Hamiltonian in (6) where $\mathbf{z}^p = [z_1^p, z_2^p, \dots, z_{12}^p]^T \in \mathbb{R}^{12}$. Noting the optimality conditions of (6) for the quadcopter problem are obtained by

$$\begin{aligned} & -\nabla_{\mathbf{u}^p} L_p(\mathbf{u}^p) - \mathbf{z}^{pT} \nabla_{\mathbf{u}^p} \mathbf{h}^p = 0 \\ \Rightarrow & - \begin{bmatrix} u_p \\ \tilde{\tau}_{\psi_p} \\ \tilde{\tau}_{\theta_p} \\ \tilde{\tau}_{\phi_p} \end{bmatrix} - \begin{bmatrix} z_1^p \\ z_2^p \\ z_3^p \\ z_4^p \\ z_5^p \\ z_6^p \\ z_7^p \\ z_8^p \\ z_9^p \\ z_{10}^p \\ z_{11}^p \\ z_{12}^p \end{bmatrix}^T \begin{bmatrix} h_7/m_p & 0 & 0 & 0 \\ h_8/m_p & 0 & 0 & 0 \\ h_9/m_p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0 \\ \Rightarrow & - \begin{bmatrix} u_p \\ \tilde{\tau}_{\psi_p} \\ \tilde{\tau}_{\theta_p} \\ \tilde{\tau}_{\phi_p} \end{bmatrix} - \begin{bmatrix} \frac{1}{m_p} (z_7^p h_7 + z_8^p h_8 + z_9^p h_9) \\ z_{10}^p \\ z_{11}^p \\ z_{12}^p \end{bmatrix} = 0 \end{aligned} \quad (23)$$

we can derive an expression for the controls as

$$\begin{aligned} u_p = & -\frac{1}{m_p} (z_7^p h_7 + z_8^p h_8 + z_9^p h_9), \\ \tilde{\tau}_{\psi_p} = & -z_{10}^p, \tilde{\tau}_{\theta_p} = -z_{11}^p, \tilde{\tau}_{\phi_p} = -z_{12}^p \end{aligned} \quad (24)$$

Algorithm 1: CA-Net

1 **Require:** σ^p diffusion parameter, G_p terminal cost, H_p Hamiltonian, F_p interaction term.
2 **Require:** Initialize neural networks N_ω^p and N_θ^p , batch size B
3 **Require:** Set Φ_ω^p and G_θ^p as in (14)
4 **while** not converged **do**
5 **train** Φ_ω^p :
6 Sample batch $\{(y_b^p, t_b)\}_{b=1}^B, \{(y_b^q, t_b)\}_{b=1}^B, q \in V(p)$ where $y_b^p \sim \rho_0^p, y_b^q \sim \rho_0^q$ and $t_b \sim \text{Unif}(0, T)$
7 $x_b^p \leftarrow G_\theta^p(y_b^p, t_b)$ for $b = 1, \dots, B$.
8 $\ell_0 \leftarrow \frac{1}{B} \sum_{b=1}^B \Phi_\omega^p(x_b^p, 0)$
9 $\ell_t \leftarrow \frac{1}{B} \sum_{b=1}^B \{\partial_t \Phi_\omega^p(x_b^p, t_b) + \frac{\sigma^{p2}}{2} \Delta \Phi_\omega^p(x_b^p, t_b) - H_p(\nabla_{x^p} \Phi_\omega^p(x_b^p, t_b))\}$
10 $\ell_{\text{neighbors}} \leftarrow \eta \sum_{q \in V(p)} \frac{1}{B} \sum_{b=1}^B \{\partial_t \Phi_\omega^q(x_b^q, t_b) + \frac{\sigma^{q2}}{2} \Delta \Phi_\omega^q(x_b^q, t_b) + \nabla_{x^q} \Phi_\omega^q(x_b^q, t_b) \cdot h^q(x_b^q, t_b)\}$
11 $\ell_{\text{HJB}} \leftarrow \lambda \frac{1}{B} \sum_{b=1}^B \|\partial_t \Phi_\omega^p(x_b^p, t_b) + \frac{\sigma^{p2}}{2} \Delta \Phi_\omega^p(x_b^p, t_b) - H_p(\nabla_{x^p} \Phi_\omega^p(x_b^p, t_b)) + F_p(x_b^p, t_b)\|$
12 Backpropagate the loss $\ell_{\text{total}} = \ell_0 + \ell_t + \ell_{\text{neighbors}} + \ell_{\text{HJB}}$ to ω weights.
13 **train** G_θ^p :
14 Sample batch $\{(y_b^p, t_b)\}_{b=1}^B$ where $y_b^p \sim \rho_0^p$ and $t_b \sim \text{Unif}(0, T)$
15 $\zeta_t \leftarrow \frac{1}{B} \sum_{b=1}^B \{\partial_t \Phi_\omega^p(G_\theta^p(y_b^p, t_b), t_b) + \frac{\sigma^{p2}}{2} \Delta \Phi_\omega^p(G_\theta^p(y_b^p, t_b), t_b) - H_p(\nabla_{x^p} \Phi_\omega^p(G_\theta^p(y_b^p, t_b), t_b)) + F_p(G_\theta^p(y_b^p, t_b), t_b)\}$
16 Backpropagate the loss $\zeta_{\text{total}} = \zeta_t$ to θ weights.
17 **end**

We therefore can compute the Hamiltonian

$$\begin{aligned}
H_p(t, \mathbf{x}^p, \mathbf{z}^p) = & -L_p(\mathbf{u}^p) - [v_{x_p} v_{y_p} v_{z_p}] \begin{bmatrix} z_1^p \\ z_2^p \\ z_3^p \end{bmatrix} \\
& - [v_{\psi_p} v_{\theta_p} v_{\phi_p}] \begin{bmatrix} z_4^p \\ z_5^p \\ z_6^p \end{bmatrix} + \frac{1}{m_p^2} (z_7^p h_7 + z_8^p h_8 + z_9^p h_9)^2 \\
& + z_9^p g + (z_{10}^{p2} + z_{11}^{p2} + z_{12}^{p2})
\end{aligned} \quad (25)$$

Finally, using (7) and (24), we compute the controls $\mathbf{u}^p$ using the CA-Net with

$$\begin{aligned}
u_p = & -\frac{1}{m_p} \left(h_7 \frac{\partial \Phi_p}{\partial v_{x_p}} + h_8 \frac{\partial \Phi_p}{\partial v_{y_p}} + h_9 \frac{\partial \Phi_p}{\partial v_{z_p}} \right), \\
\tilde{\tau}_{\psi_p} = & -\frac{\partial \Phi_p}{\partial v_{\psi_p}}, \tilde{\tau}_{\theta_p} = -\frac{\partial \Phi_p}{\partial v_{\theta_p}}, \tilde{\tau}_{\phi_p} = -\frac{\partial \Phi_p}{\partial v_{\phi_p}}
\end{aligned} \quad (26)$$

We assume that the initial density of the five populations are Gaussian distributions ρ_0^p , $p = 1, 2, 3, 4, 5$. And their centers in the spatial coordinates (x, y, z) are located in $(-1, -5, -8)$, $(4 \cos(\frac{2\pi}{5}) - 5, 4 \sin(\frac{2\pi}{5}) - 5, -6)$, $(4 \cos(\frac{4\pi}{5}) - 5, 4 \sin(\frac{4\pi}{5}) - 5, -6)$, $(4 \cos(\frac{6\pi}{5}) - 5, 4 \sin(\frac{6\pi}{5}) - 5, -4)$ and $(4 \cos(\frac{8\pi}{5}) - 5, 4 \sin(\frac{8\pi}{5}) - 5, -2)$ with same standard deviation 0.5, respectively. Noting that we set all other initial coordinates to zero (i.e. initial angular position, initial velocity, and initial angular velocity are all set to zero). We set the terminal cost to be a simple norm difference between the agents current position and the terminal distribution center position $(8, 8, 8)$, and we also want the agents to have zero velocity at terminal point, i.e.

$$\begin{aligned}
G_p(\rho^p(\cdot, T), \rho^{-p}(\cdot, T)) = & \int_{\mathbb{R}^n} \|(x_p, y_p, z_p, \psi_p, \dots, v_{\phi_p}) \\
& - (8, 8, 8, 0, \dots, 0)\|_2 \rho^p(\mathbf{x}^p, T) d\mathbf{x}^p + \sum_{q \in V(p)} \int_{\mathbb{R}^n} \|(x_q, y_q, z_q, \\
& \psi_q, \dots, v_{\phi_q}) - (8, 8, 8, 0, \dots, 0)\|_2 \rho^q(\mathbf{x}^q, T) d\mathbf{x}^q
\end{aligned} \quad (27)$$

In our experiments, we set the final time to be $T = 4$.

We choose the interaction term to penalize congestion (a.k.a. interactive running cost) in our experiments, where the congestion is in the spatial positions, to encourage agents to spread out (and thus make mid-air collisions less likely). The first and second terms of the right formula of (28) are intra-group and inter-group congestion terms, respectively:

$$\begin{aligned}
\mathcal{F}_p(\rho^p(x_p, y_p, z_p), \rho^{-p}(x_{-p}, y_{-p}, z_{-p})) = & \gamma_{\text{cong}} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \\
& e^{(-\frac{1}{2} \|(x_p, y_p, z_p) - (x_{-p}, y_{-p}, z_{-p})\|_2^2)} d\rho^p(x_p, y_p, z_p) d\rho^{-p}(x_{-p}, y_{-p}, z_{-p}) \\
& + \beta_{\text{cong}} \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{q \in V(p)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{(-\frac{1}{2} \|(x_p, y_p, z_p) - (x_q, y_q, z_q)\|_2^2)} \\
& d\rho^p(x_p, y_p, z_p) d\rho^q(x_q, y_q, z_q)
\end{aligned} \quad (28)$$

where in our experiments we put intra-group congestion coefficient $\gamma_{\text{cong}} = 20$, and inter-group congestion coefficient $\beta_{\text{cong}} = 1e + 4$.

In all experiments, our neural networks have three hidden layers, with 100 hidden units per layer. We use a residual neural network (ResNet) for both networks, with a skip connection weight of 0.5. For Φ_ω^p , we use the Tanh activation function, and for G_θ^p , we use the ReLU activation function. For training, we use ADAM with $\beta = (0.5, 0.9)$, learning rate $4e - 4$ for Φ_ω^p , learning rate $1e - 4$ for G_θ^p , weight decay of $1e - 4$ for both networks, batch size 250, $\lambda = 1$ (the HJB penalty parameter), and $\eta = 2.5e - 3$ (the neighbors penalty parameter) in Algorithm 1. In the following figures, time is represented by color. Specifically, blue denotes starting time, red denotes final time, and the intermediate colors denote intermediate times. As in standard machine learning methods, all the plots in Sec. IV are generated using validation data, i.e., data not used in training, to gauge generalizability of CA-Net.

Fig. 3 shows the trajectories of $N = 1000$ quadcopters under multi-population MFG method. Through the observation and analysis of the obtained trajectories corresponding to eight

Fig. 3: Illustration of multi-group quadcopters trajectory planning problem. The initial density (Bottom Left) and target density (Top Right) are Gaussians at the bottom and top, respectively, of these subplots domain. The five groups of agents start from the initial distribution, aiming to move towards the target distribution while avoiding intra-group and inter-group congestion. The colors in each subplot represent time, thus presenting the trajectories of the agents over time. Under different parameter configurations, the trajectories of the agents are different. A stands for noise $\sigma = 1e-2$, B for intra-group congestion coefficient $\gamma_{\text{cong}} = 20$ and C for inter-group congestion coefficient $\beta_{\text{cong}} = 1e+4$. Comparing sub-pictures (a), (b), (c), (d), (e), (f), (g) and (h), it can be seen that with σ , the agents add an envelope of uncertainty; with γ_{cong} , the agents of intra-group spread out more as expected; with β_{cong} , the clusters are more diffuse.

Fig. 4: Log HJB residual errors comparison with eight cases in Fig. 3. The horizontal axis is the number of iterations and the vertical axis is ℓ_{HJB} in Algorithm 1.

groups of parameters, we can see that with noise σ , the agents add an envelope of uncertainty, so we do see the agents are not as streamlined as in the noiseless cases, and not getting as close to the terminal point of $(8, 8, 8)$ at terminal time. It is the comparison of subgraphs (e), (f), (g) and (h) with (a), (b),

(c) and (d), respectively. Furthermore, in the presence of intra-group congestion coefficient γ_{cong} , the agents of intra-group spread out more as expected. It is the comparison of subgraphs between (d) and (a), (c) and (b), (h) and (e), (g) and (f). And in the presence of inter-group congestion coefficient β_{cong} , the clusters are more diffuse, and their terminal distributions do not fully gather to the expected terminal point of $(8, 8, 8)$. It is the comparison of subgraphs between (b) and (a), (c) and (d), (f) and (e), (g) and (h). We also plot the HJB residual errors in Fig. 4, i.e. ℓ_{HJB} in Algorithm 1, which shows the convergence of CA-Net. Without an efficient strategy control, the HJB residuals under different cases are relatively high. HJB residuals drop fast after we apply a series of controls. After around $4e+4$ iterations, the error curves tend to be stable and close to zero, when we obtain an optimal control for quadcopters. Besides, it is obvious that the CA-Net algorithm converges faster without noise, and the red curve (no A, no B, with C)—without noise, without intra-group congestion, but with inter-group congestion—converges the fastest.

B. Effect of Stochasticity Parameter σ

We investigate the effect of the stochasticity parameter σ on the behavior of the HJB residual errors and the multi-population MFG solutions.

Fig. 5: Comparison of log HJB residual errors with stochasticity parameter $\sigma = 0.02, 0.04$ and 0.06 .

Fig. 7: Comparison of average speed with different stochasticity parameters.

Fig. 6: Comparison of controls with different stochasticity parameters. The horizontal axis is time and the vertical axis is u_p (Top Left), $\tilde{\tau}_{\psi_p}$ (Top Right), $\tilde{\tau}_{\theta_p}$ (Bottom Left), $\tilde{\tau}_{\phi_p}$ (Bottom Right), respectively.

In Fig. 5, we show the convergence behavior of the HJB residual errors of quadcopters with different stochasticity parameters. The HJB residual error decreases more and more slowly with the increase of noise σ . This means that the smaller σ , the faster the CA-Net algorithm converges.

In Fig. 6, we show the controls for multi-group quadcopters trajectory planning using $\sigma = 0.02, 0.04$, and 0.06 . As σ increases, the control curves of the agent fluctuates more.

In Fig. 7, we show the optimal velocity of quadcopters obtained by Algorithm 1. The initial average speed of drones is zero and it increases first and then decreases gradually when they approach the target distribution to minimize their loss functions. Moreover, before approaching the peak, when σ is larger, the quadcopter tends to choose a faster speed. After the strategic speed change, at the terminal time $T = 4$, the smaller σ is, the closer the quadcopter terminal speed is to zero.

C. Comparison with Costs

In order to further reveal the role of some important parameters in the design of CA-Net algorithm. We show the transport $\mathcal{L}_p$, intra-running $\mathcal{F}_p^{\text{intra}}$, inter-running $\mathcal{F}_p^{\text{inter}}$, terminal $\mathcal{G}_p$ and objective J_p costs in Table I under the change of parameters.

From Table I, first, compare the influence of the σ parameter and fix other parameters. From the first line to the third line, it can be seen that the larger stochasticity σ , the greater the transport cost $\mathcal{L}_p$, the smaller the intragroup collision cost $\mathcal{F}_p^{\text{intra}}$, and the greater the total cost J_p . Comparing lines 6 and 10, 7 and 11, 8 and 12, 9 and 13, we can see that if σ exists, the total cost J_p is relatively large. Second, comparing the influence of parameter η , it can be seen from the fourth and fifth lines that if the neighbor dual penalty item exists, the smaller the transport cost $\mathcal{L}_p$, the greater the intragroup $\mathcal{F}_p^{\text{intra}}$ and intergroup $\mathcal{F}_p^{\text{inter}}$ cost, the constant terminal cost $\mathcal{G}_p$, and the smaller the total cost J_p . Third, compare the influence of the parameter λ . It can be seen from lines 13 and 14 that if the HJB penalty item exists, the smaller the transport cost $\mathcal{L}_p$, the smaller the total cost J_p . Fourth, compare the influence of the parameter γ_{cong} . It can be seen from lines 6 and 8 or lines 10 and 12 that if the intra-group congestion coefficient γ_{cong} exists, the intragroup collision cost $\mathcal{F}_p^{\text{intra}}$ will be relatively small. Fifth, compare the impact of parameter β_{cong} . It can be seen from lines 6 and 7 or lines 10 and 11 that if the inter-group congestion coefficient β_{cong} exists, the intergroup collision cost $\mathcal{F}_p^{\text{inter}}$ is relatively small. Sixth, compare the influence of the network parameter Width, i.e. the network width of each layer. By observing lines 13 and 15, we can see that smaller networks can also achieve convergence, and the total cost J_p is small. As shown in Fig. 8, for networks with the same depth, the training convergence speed is faster if each layer is appropriately widened. The larger networks have good robustness and better generalization in complex cases. Finally, comparing the parameter Time Steps, that is, lines 13 and 16, the Time Steps is reduced from 600 to 300, and the total cost fluctuates by nearly 1%, which means that our network is convergent and stable.

TABLE I: Comparison with Costs

Number	Width	Time Steps	σ	η	λ	γ_{cong}	β_{cong}	$\mathcal{L}_p$	$\mathcal{F}_p^{\text{intra}}$	$\mathcal{F}_p^{\text{inter}}$	$\mathcal{G}_p$	J_p
1	100	600	0.02	2.50e-03	1	20	1.00e+05	1.68e+01	2.33e-02	5.03e-02	1.45e+01	3.14e+01
2	100	600	0.04	2.50e-03	1	20	1.00e+05	1.79e+01	1.85e-02	5.54e-02	1.49e+01	3.29e+01
3	100	600	0.06	2.50e-03	1	20	1.00e+05	1.99e+01	1.26e-02	3.15e-02	1.47e+01	3.47e+01
4	100	600	0.01	0	1	20	1.00e+05	1.64e+01	2.33e-02	4.30e-02	1.46e+01	3.11e+01
5	100	600	0.01	1.00e-02	1	20	1.00e+05	1.52e+01	2.86e-02	5.22e-02	1.46e+01	2.98e+01
6	100	600	0	2.50e-03	1	0	0	1.56e+01	4.73e-02	1.41e-01	1.41e+01	2.98e+01
7	100	600	0	2.50e-03	1	0	1.00e+05	1.57e+01	4.71e-02	6.85e-02	1.45e+01	3.03e+01
8	100	600	0	2.50e-03	1	20	0	1.65e+01	3.11e-02	5.64e-02	1.41e+01	3.07e+01
9	100	600	0	2.50e-03	1	20	1.00e+05	1.55e+01	2.86e-02	6.08e-02	1.47e+01	3.03e+01
10	100	600	0.01	2.50e-03	1	0	0	1.59e+01	3.76e-02	9.28e-02	1.40e+01	3.01e+01
11	100	600	0.01	2.50e-03	1	0	1.00e+05	1.61e+01	3.30e-02	5.83e-02	1.49e+01	3.11e+01
12	100	600	0.01	2.50e-03	1	20	0	1.66e+01	2.49e-02	5.40e-02	1.42e+01	3.08e+01
13	100	600	0.01	2.50e-03	1	20	1.00e+05	1.65e+01	2.43e-02	6.27e-02	1.47e+01	3.13e+01
14	100	600	0.01	2.50e-03	0	20	1.00e+05	1.68e+01	2.76e-02	5.89e-02	1.46e+01	3.15e+01
15	50	600	0.01	2.50e-03	1	20	1.00e+05	1.53e+01	2.27e-02	6.45e-02	1.51e+01	3.04e+01
16	100	300	0.01	2.50e-03	1	20	1.00e+05	1.63e+01	2.57e-02	5.97e-02	1.45e+01	3.09e+01

All values were approximated using the validation points; σ , stochasticity; η , neighbor penalty coefficient; λ , HJB penalty coefficient; γ_{cong} , intra-group congestion coefficient; β_{cong} , inter-group congestion coefficient; $\mathcal{L}_p$, transport costs; $\mathcal{F}_p^{\text{intra}}$, intra-running costs; $\mathcal{F}_p^{\text{inter}}$, inter-running costs; $\mathcal{G}_p$, terminal costs; J_p , objective costs.

Fig. 8: Comparison of log HJB residual errors under different network widths.

D. Comparison with Baselines

As the last part of the numerical experiment, we verify the superiority of our method by comparing it to some typical baseline methods, including cluster game [77], HJB-NN [65], Lax-Friedrichs [62], ML [31], APAC-Net [56]. For more details of these baseline methods, please refer to the related documents [31], [56], [62], [65], [77], etc. Below we give the performance comparison of the above algorithms in Table II.

For cluster game [77], it is a game model for small-scale multi-group agent control problems, and that communicates through a given point-to-point communication topology. The equilibrium calculation adopts an analytical solution based on grid calculation. The solution space is generally low-dimensional, and special interference is usually not considered.

HJB-NN [65] is a neural network approach for solving high-dimensional optimal control problems. This method can be used for large-scale group modeling. And the information interaction between agents is point-to-point communication, which means that the communication cost is relatively high. Literature [62] describes a multi-population social network MFG problem, which is solved analytically by an adjoint method. Its corresponding algorithm uses a numerical scheme called the Lax-Friedrichs scheme [64] belonging to the finite difference method. Therefore, the numerical solution is cursed by dimension and is only suitable for low-dimensional solution space. Literature [31] provides a machine learning framework for solving high-dimensional MFG and mean-field control problems. However, it is limited to solving the single group deterministic MFGs. APAC-Net [56] is an alternating population and agent control neural network for solving high-dimensional stochastic MFGs. It is indeed suitable for large-scale homogeneous agent systems. Our proposed CA-Net is a nonlinear coupled alternating neural network tailored for multi-population high-dimensional stochastic MFGs, which can be induced by optimal control problems of large-scale multi-agent heterogeneous systems. To sum up, we can see the superiority of the CA-Net algorithm in all-around performance from Table II.

V. CONCLUSION

We formally propose a numerical solution scheme for the multi-population high-dimensional stochastic MFG system. The execution algorithm here is CA-Net, a nonlinear coupled alternating neural network. To this end, the proposed modeling framework strictly represents the large-scale heterogeneous multi-agent game system as a multi-population MFG model.

TABLE II: Comparison with baseline methods

Method	Communication	Computation	Scale (N)	Dimension (n)	Population (P)	Stochasticity (σ)
Cluster game [77]	$\mathcal{O}(N)$	$\mathcal{O}(n!)$	Small	Low	Multiple	No
HJB-NN [65]	$\mathcal{O}(N)$	$\mathcal{O}(n)$	Large	High	Single	No
Lax-Friedrichs [62]	$\mathcal{O}(1)$	$\mathcal{O}(n!)$	Large	Low	Multiple	Yes
ML [31]	$\mathcal{O}(1)$	$\mathcal{O}(n)$	Large	High	Single	No
APAC-Net [56]	$\mathcal{O}(1)$	$\mathcal{O}(n)$	Large	High	Single	Yes
CA-Net	$\mathcal{O}(1)$	$\mathcal{O}(n)$	Large	High	Multiple	Yes

$\mathcal{O}()$, infinitesimal of the same order.

Our algorithm avoids the use of spatial grids by parameterizing the value functions and the density functions of multiple populations by two sets of neural networks, respectively. We design a multiple generators and multiple discriminators GAN, where density functions act as generative networks and value functions act as discriminative networks. Unlike other GANs, CA-Net incorporates the structure of multi-population MFGs, which absolves the network from learning an entire multi-population MFG solution from the ground up—provides numerical solutions for the multi-population MFG and improves the solution efficiency. Consequently, our method is geared toward multi-population high-dimensional instances of MFGs that are beyond reach with existing solution methods. CA-Net, therefore, sets the stage for solving realistic multi-population high-dimensional MFGs arising in, e.g., crowd motion, swarm robotics, epidemic modeling. To verify the feasibility of the multi-population MFG model, we provide a practical numerical simulation case, multi-group quadcopters trajectory planning. The convergence results about HJB residual error, control, average speed show the effectiveness of the CA-Net algorithm. To reveal the role of multi-dimensional parameters in the design of the CA-Net, we show the costs of transport, intra-running, inter-running, terminal, and objective under the change of these parameters. Finally, the comparison with baseline methods—cluster game, HJB-NN, Lax-Friedrichs, ML, APAC-Net—makes clear the advanced nature of our solution method. In the future, we will consider the application and adaptation of the CA-Net method in more complex scenarios of heterogeneous multi-agent systems, such as complex communication conditions (time-varying, partial rejection) and obstacles (static, dynamic).

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