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DFT21: Discrete Fourier Transform in the 21st century

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Abstract—Knowingly or unknowingly, *digital-data* is an integral part of our day-to-day lives. Realistically, there is probably not a single day when we do not encounter some form of *digital-data*. Typically, data originates from diverse sources in various formats out of which *time-series* is a special kind of data that captures the information about the *time-evolution* of a system under observation. However, capturing the *temporal-information* in the context of data-analysis is a highly *non-trivial* challenge. *Discrete Fourier-Transform* is one of the most widely used methods that capture the very essence of *time-series* data. While this nearly 200-year-old mathematical transform, survived the test of time, however, the nature of real-world data sources violates some of the intrinsic properties presumed to be present to be able to be processed by *DFT*. Adhoc noise and outliers fundamentally alter the true signature of the frequency domain behavior of the *signal of interest* and as a result, the *frequency-domain* representation gets corrupted as well. We demonstrate that the application of traditional digital filters as is, may not often reveal an accurate description of the pristine *time-series* characteristics of the system under study. In this work, we analyze the issues of *DFT* with *real-world* data as well as propose a method to address it by taking advantage of insights from modern *data-science* techniques and particularly our previous work SOCKS. Our results reveal that a dramatic, *never-before-seen* improvement is possible by re-imagining *DFT* in the context of *real-world* data with appropriate curation protocols. We argue that our proposed transformation *DFT21* would revolutionize the digital world in terms of accuracy, reliability, and information retrievability from *raw-data*.

Index Terms—Signal processing, data analysis, Fourier transform, DFT, FFT, data cleaning, noise reduction, signal recovery, Digital data

I. Introduction

It is the *pulse-dynamics* of our heartbeat that keep our lives moving on a *moment-to-moment* basis. The behavior of such dynamical systems can be measured as a *time-series* of an observable. *Time-series analysis* [1]–[4] is at the heart of *time-dependent* data analysis. Various methods have been developed [5], [6] over the past few centuries to address this very intricate problem. Out of all the established methods, the well cele-

brated *Fourier Transform* (or *Fourier-analysis* or *spectral-analysis*) [7], [8] stands out as the single most universally accepted transform to survive the test of time as well as broad domains of applicability. In the modern digital world, *Fourier Transform* can be singled out to be the one transform that can be called the very *pulse* of the *digital signal-processing* universe. Almost any signal with *spatio-temporal* dependency needs *FT* as a vehicle to get processed under rigorous *data-analysis* pipelines. The entire *medical-diagnosis* machinery, as well as modern spectroscopic methods, are deeply rooted in the many variants of *FT*. It is so rich in scope that, a host of associated transforms have been developed in parallel, to access many flavors of [9]–[11] it.

In the context of digital devices, since source-generated ¹ signals are almost always represented as finite samples of *digital-data*, *FT* is equivalent to *Discrete Fourier Transform (DFT)* for most modern application contexts. Since, in all practical scenarios *DFT*, for efficiency, is implemented as *Fast Fourier-Transform (FFT)* [12]–[14], we would equivalently treat *DFT* and *FFT*. The impact of *FFT* is so profound that, even a marginal improvement in its accuracy, would enable a dramatic impact in the pipelines of *digital discovery* processes.

The applications of *FFT* [15]–[18] is one of the most successful mathematical transforms to understand *time-series* characteristics of physical systems. The entire spectroscopic apparatus is fundamentally grounded on *FFT* that forms the very foundation for analytical chemistry. In the world of physical sciences, *FFT* resides [19]–[21] at the heart of many well-established spectroscopic techniques like *Raman spectroscopy* [22], [23]. The scope and importance of such spectroscopic methods even in our daily lives are so profound [24]–[29] that it is nearly unfathomable to accurately quantify the importance [30]–[35] of *FFT* on our lives. So, we emphasize that even the minutest improvement in the accuracy of the outcomes of *FFT* is enormously impactful in the progress of reliable scientific and technological outcomes.

On the other hand, optimal signal recovery from measured, often incomplete, or unreliable *raw data-sets* is a universally challenging problem [36]–[38]. Often, signal recovery is confused with *signal-smoothing* through traditional filtering processes with a given kernel or parameter

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¹often this means physical transducers

value e.g. *Gaussian-filter* ($\mathcal{G}$), *Wiener-filter* ($\mathcal{W}$), *Median-filter* ($\mathcal{M}$), *TotalVariation-filter* ($\mathcal{T}$), etc. While these *traditional filters* have certain utilities for smoothing and presentation purposes, however apparent visually appealing smoothness of filtered signals should not be misrepresented as accurate recovery of the underlying *pristinity* that intrinsically describes the nature of the observable under study.

In the context of *time-series* analysis, it is to be noted that, *FFT* does not by itself facilitate any underlying signal recovery process², but rather, it simply transforms *measured-signal*³ into the *frequency domain* often followed by some form of *bandpass-filtering*, while remaining completely agnostic about the true underlying signature of the *raw-data*⁴.

Being *data-driven* [39]–[41] is one of the most unbiased ways to learn about physical systems. However, if the data itself is not pre-processed to fit into basic assumptions under which it is processed, the outcomes are inaccurate. We argue that these inaccuracies are not only a margin of error issue but rather something with a dangerous side-effect when used in mission-critical systems like *medical-devices* or *aviation-systems*. Some of the basics assumptions around *FFT* are that the *raw-data* fed as input to the transform has a simple linear structure in its spectral signature⁵ and that there is some measure of equilibrium and ergodicity [42] intrinsically present in the data. However, this kind of presumption is no better than wishing for mere *good-luck* to perform rigorous, quantitative scientific analysis. We argue that optimal *signal-recovery* from *raw-data* through *appropriate curation procedure*⁶ should precede traditional *FFT* operations. In this work, we not only developed a systematic method to obtain scientifically more accurate outcomes from *FFT* operations but also propose to extend our methods to build a *data-discovery-driven* new generation of *data-curation* and filtering pipelines. By taking advantage of very high level but close to metal [43], intrinsically multi-threaded [44]–[46] and *natively-supported* by the programming language interfaces like (C++ >= -std-11) [47], [48], Rust [49]–[51], etc. We anticipate a future revolution in *data-filtering* techniques grounded on fundamental *data-discovery* principle. The discovery step would prepare the data to be ready for processing by idealized *traditional filters*. The primary insight here is, to make an initial $O(N)$ ⁷ data discovery step (e.g. noise, outlier identification) and curation step throughout the entire domain of the data followed by traditional processing techniques.

In the previous generations of filters, this step got overlooked possibly to avoid computational costs. But fortunately, we can afford that overhead now due to the lower overall cost of computation both in personal and enterprise settings. Also, intrinsic multi-threaded processing capabilities along with availability of multi-core, modern hardware and programming language interfaces provides us with a new realm of accurate filtering. We claim that, this would lead to a paradigm shift in the development of the next generation of advanced, more robust *data-filters*.

II. Method

The method primarily consists of three steps, a) data source simulation b) traditional data-filtration c) *FFT21* data curation followed by comparison of results in signal and frequency domains.

A. Data source simulation

In order to demonstrate the *signal-recovery* process of arbitrarily contaminated data we choose a simple harmonic function as in Eq. 1 with $\omega = 3$ to generate a set number of ($N \approx 200$)⁸ simulated sample points ($\mathcal{S}(f(t), N)$) in the *time-domain* range $0 \leq t \leq 2\pi$ that we treat as *raw input-data* (..||). As a simple illustration to demonstrate the *proof of principle* case study of *data-curation*, we generated pristine, uncontaminated data using the following harmonic function as in Eq. 1.

$$f(t) = \sin(\omega t) \quad (1)$$

For the sake of simplicity, we choose $\omega = \omega_0 = 3$ ⁹ and call it the *resonance frequency* in this context. The basic reasoning for the simple choice of $f(t)$ ¹⁰ in Eq. 1 is that, a simple analytical harmonic function is easy to sample, make visual observations for features (or differences w.r.t reference signal), the *frequency-spectrum* is precisely known (e.g. ω_0). Any contamination of this function (or signal) followed by filtering (or de-contamination) should recover $f(t)$ ¹¹ precisely. A signal contaminant is added to the *pristine* function $f(t)$ in the form of a Weiner process (or for all practical purposes Gaussian random number generator)¹² as demonstrated in Eq. 2.

$$\hat{f}(t) = f(t) + \mathcal{W}_t \quad (2)$$

$\hat{f}(t)$ is then sampled ($\mathcal{S}(\hat{f}(t), N)$)¹³ with the same number of sample points ($N \approx 200$) as that of $f(t)$.

$\mathcal{S}(f(t), N)$ and $\mathcal{S}(\hat{f}(t), N)$ are plotted in Fig. 1 and Fig. 2 respectively. Next step is to recover the underlying pristine signal from $\mathcal{S}(\hat{f}(t), N)$ using *traditional filters* to see how much restoration is possible.

²often pedagogically, confusingly, or in some cases erroneously thought to be done so by simply removing *high-frequency* components

³pristine or contaminated

⁴e.g. whether or not the *raw-data* satisfies the eligibility criteria of *FFT* processing

⁵e.g. the signal can be written as a linear sum of harmonic functions with various frequencies

⁶appropriateness is contextual

⁷ N is the number of sample points in the data

⁸arbitrarily chosen

⁹arbitrary choice

¹⁰Let's call it pristine-signal

¹¹e.g. as if inverted back to *pristine*

¹²the actual type of the distribution is irrelevant

¹³e.g. sampling $\mathcal{S}$ of $f(t)$ with N sample-points

Fig. 1: Plot of the harmonic function $f(t) = \sin(\omega t)$ for $\omega = 3$

Fig. 2: Plot of the contaminated harmonic function from Eq. 2 for $\omega = 3$

B. Traditional filtering

Traditional data filters have been widely used in the context of signal smoothing and recovery. Some common filters ($\mathcal{F}_k$) are *Wiener-filter* ($\mathcal{W}$), *Gaussian-filter* ($\mathcal{G}$), *Median-filter* ($\mathcal{M}$), and *TotalVariation-filter* ($\mathcal{T}$). It is reasonable to see how the application of filters affects the recovery process for various filter parameters e.g. Kernel size ($\mathcal{K}$).

Fig. 3, is a demonstration of the effect of various *de-noising* filters on contaminated signal (simulated). Fig. 4, is a demonstration of the effect of various *de-noising* filters with relatively bigger *filter kernel size* applied on contaminated signal (simulated). Ideally, for a given plot label, the filtered data should collapse with the pristine signal after the application of a given filter $\mathcal{F}_k$. However, it is clear from Fig. 3 (a)-(d), that, although some filters perform better (e.g. *median-filter*) than others, however, a near collapse is almost impossible to achieve despite comparing with various filters. The situation does not improve if we have a bigger kernel size as shown in

Fig. 3: Comparison of the effect of various standard *de-noising* filters (a) *Gaussian*, (b) *Wiener*, (c) *median* (d) *TotalVariation* filter respectively. The *kernel-size* $\mathcal{K} = 5$ for (a)-(c) is and $\mathcal{K} = 0.3$ for *TotalVariation* filter.

Fig. 4. As a matter of fact, it performs much worse in comparison to small kernel size filter.

In Fig. 5 we demonstrate similar comparison like before in the frequency domain for the *small-kernel* size filter application. Plot labels (a)-(d) represent different filters and within a given plot, the *window-labels* $W_{k \in \{1,2,3\}}^p \in \{a, b, c, d\}$ represent three representative comparison windows out of which W_2^p is the one that highlights the ω_0 . Rest of the *window-labels* represent frequencies introduced by the as in Eq. 2.

Simultaneously for all the frequency amplitude values, despite testing most of the commonly available filters in the literature the signal recovery is a difficult task. Even if we change the parameters of the filters from smaller to bigger parametric values, the situation does not improve if not get worse. This calls for a dramatic paradigm

Fig. 4: Comparison of the effect of various standard *denoising* filters. (a) *Gaussian*, (b) *Wiener*, (c) *median* (d) *TotalVariation* filter respectively. The *kernel-size* $\mathcal{K} = 25$ for (a)-(c) is and $\mathcal{K} = 5$ for *TotalVariation* filter.

shift and complete rethinking of the problem of getting very accurate *FFT* outcomes. Our core insight is that the computationally driven *data-discovery* centric *data-curation* is the missing link.

C. Data curation

In most practical scenarios of *time-series* data analysis, we are typically interested in a given *frequency-band* of interest called *bandwidth* ($\mathcal{B}_{\omega_l}^{\omega_h}$). We argue that, the complementary *band-width* $\mathcal{B}'$ (e.g. orthogonal to this range) composed of low-pass ($\mathcal{B}_{\omega_l'}^{\omega_{l'}}$) and high-pass ($\mathcal{B}_{\omega_l''}^{\omega_{l''}}$) can be curated using some optimal *curation-protocol*¹⁴ (say $\mathcal{C}^{\mathcal{P}^*}$) such that, the true¹⁵ *bandwidth* ($\mathcal{B}_{\omega_l}^{\omega_h*}$) is optimally

¹⁴The details of which is not relevant to this work

¹⁵or as close as possible to the truth

Fig. 5: Visual demonstration of the differences between the spectrum of original *raw-signal*, *noise-contaminated* signal, and *filtered-signal*. Data point labels, L_1, L_1', L_1'' ($L \in \{a, b, c, d\}$) indicate the zero frequency Fourier amplitudes of *pristine-signal*, *contaminated* signal and *filtered-signal* with respect to a given filter type respectively. Highlighting windows W_1^{a-d}, W_2^{a-d} & W_3^{a-d} represent three representative data points.

recovered from the *raw-signal* (..||). The entire effective *band-width* in the context of *time-series* data can be thought of as Eq. 3 below.

$$\mathcal{B}_{total} = \mathcal{B}_{\omega_l'}^{\omega_{l'}} + \mathcal{B}_{\omega_l}^{\omega_h} + \mathcal{B}_{\omega_l''}^{\omega_{l''}} \quad (3)$$

For the sake of simplicity we would focus on *high-pass* curation since most *ad hoc* outliers and noise processes are centered around this *band-width*. It is to be noted that, the frequency scale (ω) is inversely related¹⁶ to the time-scale (or length scale) parameter $\#$ in ref [52]. So, depending on the curation frequency band, appropriate values of the $\#$'s can be chosen. For the sake

¹⁶typically $\omega = \frac{2\pi}{\#}$

of simplicity of our discussion, we would curate only the *high-frequency* components as that is where most random noise lies.

The fundamental basis of our proposed methods is first, to scan the entire data set to *identify local-fluctuations* of values with respect to a specified length ($\mathcal{L}$) of *data-records* and a pre-defined *threshold-parameter* (λ) in a recursive fashion for various scales of $\mathcal{L}$. $\mathcal{L}$ defines the *length-scale* (or *time-scale* or equivalently *frequency band*) of interest.

The details of the basic idea behind the *identification* of the points of fluctuations and basic curation steps are outlined in Figs. 3 & 4 in ref. [52] although a future customization of the core-algorithms and parameters would be needed to *carve-out* a module specific to the *DFT* problem described here but the details of the module is beyond the scope of this work. In the present context, we primarily demonstrate that, it is *as a matter of principle*, possible to optimally recover the underlying harmonic signal perturbed by simple noise processes, that is significantly accurate in its *Fourier-Transformed* space as well as in original *signal-space*.

III. Results

Fig. 6 demonstrates that the *FFT* of the curated signal using *FFT21* is almost non-discernible from the pristine counterpart. This is dramatic compared to those with *traditional filters* tested even with different scale parameters as demonstrated in Fig.3(small-scale) & Fig. 4(bigger-scale). From Fig. 7 we observe that the restored signal makes a visually non-discernible overlap with the pristine frequency amplitudes at almost all sample points except for a few points highlighted by the *yellow rectangle*. All the error bars are approximately within the symbol sizes.

Table I demonstrates the percent deviation of various

TABLE I: Comparison of frequency amplitude (%) deviations (under perturbed and filtered conditions) from pristine signal amplitudes. (cont=contaminated, $\mathcal{G}$ =Gaussian, $\mathcal{M}$ =Median, $\mathcal{W}$ =Wiener, $\mathcal{T}$ =Total Variation)

col-1	col-2	col-3	col-4	col-5	col-6	col-7	col-8	col-9	col-10
sl.	ω/ω'	1	1'	2	2'	$\omega_0 = 3$	ω_0'	4	4'
	Filter								
1	cont.	1658.8	∞	421.9	∞	-1.5	$\Delta\omega_0^0$	400.7	∞
2	$\mathcal{G}$	1626.1	∞	419.8	∞	-3.7	0	373.7	∞
3	$\mathcal{M}$	524.6	∞	241.1	∞	-1.5	0	10.7	∞
4	$\mathcal{W}$	1903.7	∞	507	∞	-5.9	0	391.6	∞
5	$\mathcal{T}$	2191.7	∞	745.4	∞	-41.5	0	443.9	∞
6	FFT21	331.3	∞	91.8	∞	-0.3	0	66.9	∞

* some arbitrary % deviation from ω_0 depending on contamination level

frequency (ω) amplitudes (absolute value) from its *pristine counterpart* at a given value frequency value. The primed symbol (ω') column for a given value of ω stands for its theoretical counterpart since, any value of ω except

Fig. 6: Iterative restoration of $\hat{u}$ from u . (a) is the 0th step of the iterative sequences while, (b)-(d) are subsequent iterations with $\lambda = 1.24$ and $w = 9$. The highlighted window in each of (a)-(d) are regions where sample points from u are visually differentiable from $\hat{u}$.

Fig. 7: Relative comparison between frequency spectrums of pristine signal (filled circle), contaminated signal (filled upper triangle) and *FFT21* restored signal (filled square)

ω_0 under pristine conditions should be theoretically zero and as a results any deviation from that would tend towards ∞ . The fact that the measured values of non ω_0 frequencies are non-zero is attributed to finite sampling-size (e.g. $N = 200$) but this does not affect the arguments we are building to establish the effects of various filters on the frequency bands.

There are few surprises in col-7 of table I. a) None of the traditional filters actually improved the recovery of the pristine signal amplitudes! Although *median-filter* is the only one that doesn't make it worse, the other filters actually reduces the amplitude (e.g. degraded outcome). This is contrary to the popular **belief** that smooth looking filtered data points are actually better in information content, b) filters like $\mathcal{G}$ although makes the output signal look much smoother, e.g. Fig. 3 (b), but in reality it actually degrades the signal amplitude (while reducing the non-resonant frequency components as well) as apparent from row-5 of table I.

IV. Discussion, limitations and future directions

Signal recovery from noisy data is a very important but challenging problem that manifests ubiquitously in almost all natural systems. *FFT*, although one of the most powerful mathematical transform, relevant in nearly all signal processing contexts could be improved dramatically by carefully curating the data such that, it is transformed into more appropriate and aligned data-sets to fit into the core underlying requirements of *FFT*¹⁷. Merely feeding data into *FFT* processors doesn't guarantee accuracy or reliability of outcomes for *real-world* signals that is almost always contaminated one way or the other.

It is very important to note that, our work no way questions the validity of the *FFT* method itself, but rather illustrates the pitfalls of using *un-curated raw-data* (or even traditionally filtered data), to feed into mathematical transforms while presuming that the underlying hypothesis behind those transforms are naturally satisfied by the raw-data. Given, the immense significance that *FFT* plays in the world of *medical-diagnosis*, it is important to be aware of the limitations described here to avoid any un-verified

complacency that traditional filtering necessarily improve diagnostic outcomes that are grounded on *FFT*. General details behind the fundamental principles behind this method is described in ref. [52] and in future we would create a dedicated module to address *FFT21* specific parameters and algorithmic twists to establish a dedicated software module to make more widely available through commercial softwares like *Matlab* [53], [54], *GNU Octave* [55], [56], *Mathematica* [57], [58] etc.

Also, as a *proof of principle*, we described a simple *case-study* as an illustrative example. Although, the perturbations on the pristine signal from Eq. 1 is generated randomly in an unbiased fashion, real world signals are much more complex than the simple illustrative model described here but that discussion is beyond the scope of this work. Future work would include *case-studies* of more non-trivial, realistic signals as well as more complex simulations to capture realistic conditions.

One interesting observation is that, even the historically well established signal processing filters like *Wiener*, *Gaussian*, *Median* and *TotalVariation* are better suited only for signal-smoothing operations rather than signal-recovery processes as evident from col-7 of table I. Traditional filters fail to recover even a vanilla (or simple) category of underlying harmonic signal perturbed with a simplistic *noise-model*. One of our major insight to mitigate those limitations is the observation that, filtering process first exclude and curate the identified outlier¹⁸ data points before traditional processing can take place rather than blindly applying it to all measured raw *data-points*.

One of the known limitation of this works is that, we created a relatively simple signal and noise model to demonstrate the principles which should not be treated as representative of *real-world* complex signals like those found in realistic *EKG*, *EEG* etc. However, we'll leave that to the future work to simple it simple and rather keep the focus on the extreme importance of *raw-data pre-processing* method before *FFT* or other transforms are applied.

V. Conclusion

In conclusion, we argue that, the very important *Discrete Fourier-Transform* can be dramatically improved by pre-processing and curating *raw-data* to make it eligible for traditional processing so that the underlying assumptions behind *DFT* are minimally violated. We have demonstrated that outcomes of traditional mathematical transforms can be significantly enhanced in the context of *real-world* data if the data itself is curated to be consistent with the underlying assumptions of those transforms (e.g. *DFT*). The curation of the *real-world* data indicates that curated *raw-signal* has a close to pristine frequency amplitude (and phase as well) than that from the un-curated *raw-data*. As the world becomes more

¹⁷or any other transforms

¹⁸with respect to some arbitrary measure

and more *data-driven* the accuracy of the data (uncurated or curated) fed to the transforms, determines the accuracy of the predictions made from those data-sets. A philosophical shift is the need of the moment to develop scalable *data-curation* strategies as opposed to mere *data-collection* strategies which currently dominate the field in the context of being *data-driven*. It is to be noted that, *FFT21* is not a new *FFT* method but rather a pre-processing method to make the raw-data eligible for further processing with *FFT*. This pre-processing followed by the regular *FFT* is what makes a complete *FFT21* operation.

This work is far from complete. In future, we would develop formal case studies to determine curation strategies based on actual nature of the data (e.g. *time-series* vs. *matrix-like*) as well as targeted methods specific to the particular types of transforms like *Radon-transform* [59], [60], *Bessel-transform* [61], [62], *Hankel-transform* [63], [64] etc.

The argument we developed here is not limited to transform space scenarios. At the heart of data curation is entropy and dimensional reduction. The case study with *FFT* is a merely a mechanism to highlight the fact that curation is an essential step before processing with raw source level data. This work calls for a radical change in our approach to readily generalize the utility of pure mathematical expressions in *real-world* scenarios.

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