Estimation and Equalization of residual synchronization errors in CP-OTFS

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Abstract

In this work, we analyze the effect of residual frame timing offset and carrier frequency offset along with the presence of fractional Doppler on the bit error rate performance of OFTS system. We show in this work that channel equalization with appropriate channel estimates can compensate for the combined effect of the time-varying multipath channel (TVMC) as well as residual synchronization errors. We also present a unique time-domain channel estimation method for OTFS, which can be used along with the above-mentioned channel equalization. It is seen from the extensive Monte Carlo simulation results of the LDPC coded OTFS system that the estimation and compensation methods presented here provide necessary resilience properties to OTFS against the effects of TVMC channel and residual synchronization errors. A unified signal processing flow for OTFS and OFDM is also described in this work to motivate studies on coexistence between the two as well as to encourage investigations on a seamless transition from OFDM based system to OTFS based systems for future air interface design.

I. INTRODUCTION

IMT-2020 [1], [2], aka 5G, aims to provide high spectral efficiency and reliability in high mobility scenarios, where the wireless link becomes a highly time varying multipath channel

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(TVMC) [3], [4]. The modified avatar of orthogonal frequency division multiplexing (OFDM), namely OFDM numerology [5] is the fundamental physical layer signalling technique, which is used in 5G-New radio (5G-NR) to address this operating requirement. OFDM numerology is essentially variable subcarrier bandwidth [6] along with variable guard interval [7] as described in [8]. Such modifications not only improve the immunity of OFDM to inter-carrier interference (ICI) caused by high Doppler in high mobility conditions and phase noise at high carrier frequencies but also allow a smooth transition from 4G (IMT-Advanced [9]) to 5G with minimalistic changes in the system configuration. In a recent work by the authors [10], it is shown that orthogonal time frequency space (OTFS) [11]–[13] outperforms adaptive OFDM used in 5G-NR in such high ICI conditions.

In OTFS, delay-Doppler domain data is first converted to frequency-time domain through a unitary transform, which is then converted to time domain by OFDM modulation. A cyclic prefix (CP) can be added before a block of OFDM symbols resulting in reduced CP-OTFS, namely RCP-OTFS [14], whereas in CP-OTFS a CP is added before each OFDM symbol [15], which is considered in this work.

Although OTFS is an orthogonal modulation scheme, yet when the signal passes through a TVMC the received signal in delay-Doppler domain, encounters inter symbol interference. To demodulate the interference affected signal, the works [16] and [17] describes a belief propagation receiver for OTFS. In [18] Markov chain Monte Carlo sampling based on the low-complexity OTFS signal detection scheme is presented. The work [19] describes localized search based non-linear receiver for rectangular pulse shaped OTFS. These receivers have a non-linear signal decoding structure and result in very high complexity. The work [19] also presents a linear receiver architecture however it is limited because it considers ideal pulse shape and hence is not practical. In [20] a low complexity linear receiver for rectangular orthogonal pulse shaped OTFS system is presented. In this work, we restrict ourselves to linear receiver owing to its lower complexity.

Most of the articles referred above assume the availability of ideal channel estimates for demodulation of the delay-Doppler OTFS data symbols. Channel estimation being an important element of signal demodulation, a delay-Doppler domain channel estimation algorithm is described in [14]. Authors of [18] also presents a delay-Doppler domain channel estimation. Channel estimation in delay-Doppler-angle dimension for OTFS-MIMO is shown in [21].

Delay-Doppler domain channel estimation and equalization for OTFS, described in the above-cited works, considers ideal synchronization conditions. Frame timing synchronization is necessary to delineate the incoming signal into OFDM blocks for the demodulation of OTFS signals. Carrier frequency offset (CFO) is known to arise from a mismatch between transmitter and receiver local oscillators. Frame timing offset (FTO) and CFO introduce ISI and ICI in OFDM [8]. To the best of the authors' knowledge, the effect of such errors on OTFS has not yet been explored and demands attention.

The effects of residual FTO and residual CFO on CP-OTFS, considering training sequence [22] based initial synchronization is already achieved, is evaluated in this work. We establish that the integer FTO and integer CFO create a cyclic shift in delay and Doppler dimensions respectively, thus introducing interference in the received delay-Doppler signal (Section IV). The interference gets further enhanced with fractional CFO. The expression of an equivalent channel matrix in time-domain, which includes the combined effect of the actual TVMC, FTO, and CFO is presented in Section V. This throws light on an important aspect of OTFS system that residual FTO and CFO can be compensated with appropriate time-domain channel estimation and equalization. We present the method for estimating such equivalent channel matrix in Section VI, which is necessary for equalization of received signal. A low complexity LMMSE based time-domain channel equalization, which can cancel the interference emanating from the channel as well as synchronization errors, is described in Section VII. The time-domain channel estimation and equalization described here not only includes compensation for synchronization errors but also provides an opportunity to develop a unified representation framework for multiplexing

schemes such as OTFS and OFDM, which helps in paving the path for a flexible reconfigurable air interface for future air interface design.

II. NOTATIONS

We use the following notations throughout the paper. We consider $\mathbf{x}$, $\mathbf{X}$ and x to be vectors, matrices and scalars respectively. Complex conjugate value of x is given by $\bar{x}$ whereas $j = \sqrt{-1}$. We denote the set of integers between a and b by $\mathbb{Z}[a \ b]$. For brevity, $a \bmod M$ is represented by $[a]_M$. Expectation, Ceil, and Kronecker product operations are denoted by $E\{-\}$, $\lceil - \rceil$, and $\otimes$ respectively. Matrices $\mathbf{0}_{N \times L}$, $\mathbf{I}_N$ and $\mathbf{W}_L$ are zero matrix of size $N \times L$, identity matrix with order N and L -order normalized inverse discrete Fourier transform (IDFT) matrix respectively. The superscripts $(-)^T$ and $(-)^{\dagger}$ denote transpose and conjugate transpose operators, respectively. The operator $\text{diag}\{\mathbf{x}\}$ generates a diagonal matrix with the diagonal entries of the vector $\mathbf{x}$. Circulant matrix is represented by $\text{circ}\{\mathbf{x}\}$ whose first column is $\mathbf{x}$. The operator $\text{vec}\{\mathbf{X}\}$ denotes Column-wise vectorization of matrix ($\mathbf{X}$).

III. SYSTEM MODEL

We consider a CP-OTFS system with M sub-carriers, each of Δf Hz bandwidth, N symbols of duration $T_u = \frac{1}{\Delta f}$ sec. each with T_{CP} sec. long CP. The system has bandwidth $B = M\Delta f$ Hz. and total frame duration $T_f = NT$ seconds, where $T = T_u + T_{CP}$.

A. Transmitter

QAM modulated data symbols, $d(k, l) \in \mathbb{C}$, $k \in \mathbb{Z}[0 \ N-1]$, $l \in \mathbb{Z}[0 \ M-1]$, are arranged over Doppler-delay lattice $\Lambda = \{(\frac{k}{NT_u}, \frac{l}{M\Delta f})\}$. We assume that $E[d(k, l)\bar{d}(k', l')] = \sigma_d^2 \delta(k - k', l - l')$, where δ is Dirac delta function. Doppler-delay domain data $d(k, l)$ is mapped to time-frequency domain data $X(n, m)$ on lattice $\Lambda^\perp = \{(nT, m\Delta f)\}$, $n \in \mathbb{Z}[0 \ N-1]$ and $m \in \mathbb{Z}[0 \ M-1]$ by using inverse symplectic finite Fourier transform (ISFFT). Following [23], $X(n, m)$ is expressed

as,

$$X(m, n) = \frac{1}{\sqrt{NM}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) e^{j2\pi[\frac{nk}{N} - \frac{ml}{M}]}. \quad (1)$$

$X(m, n)$ is converted to a time domain signal $s'(t)$ through a Heisenberg transform as,

$$s'(t) = \frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X(m, n) g(t - nT_u) e^{j2\pi m \Delta f (t - nT_u)}, \quad (2)$$

After appending CP to the baseband signal $s(t)$ we get,

$$s(t) = \frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X(m, n) g(t - nT) e^{j2\pi m \Delta f (t - T_{CP} - nT)}, \quad (3)$$

where, $g(t)$ is transmitter pulse of duration T . In this work, we use rectangular pulse i.e. $g(t) = 1$ if $0 \leq t \leq T$, and $g(t) = 0$, otherwise. The baseband signal $s(t)$ is up-converted to the RF carrier frequency f_c to obtain the RF signal $s_{RF}(t) = s(t) e^{j2\pi f_c t}$.

We consider a baseband time varying channel with P paths having h_p complex attenuation, τ_p delay and ν_p Doppler value for the p^{th} path where $p \in \mathbb{Z}[1 \ P]$. The delay-Doppler channel spreading function is written as,

$$h(\tau, \nu) = \sum_{p=1}^P h_p \delta(\tau - \tau_p) \delta(\nu - \nu_p). \quad (4)$$

The delay and Doppler values for p^{th} path is given as $\tau_p = \frac{l_p}{M\Delta f}$ and $\nu_p = \frac{k_p}{NT}$ where l_p and k_p are delay and Doppler bin numbers on Doppler-delay lattice Λ for p^{th} path. Let τ_{max} and ν_{max} be the maximum delay and Doppler spread respectively. Channel delay length $l_\tau = \lceil \tau_{max} M \Delta f \rceil$ and channel Doppler length, $k_\nu = \lceil \nu_{max} NT \rceil$. The RF equivalent channel can be given as, $h_{RF}(\tau, \nu) = h(\tau, \nu) e^{j2\pi f_c \tau}$.

The received signal can be written as, $r_{RF}(t) = \int_{\tau=0}^{\tau_{max}} \int_{\nu=-\nu_{max}}^{\nu_{max}} (h_{RF}(\tau, \nu) s_{RF}(t - \tau) e^{j2\pi \nu (t - \tau)}) d\nu d\tau + v_{RF}(t)$, where $v_{RF}(t)$ is Gaussian noise with variance σ_v^2 . Therefore, $r_{RF}(t) = \sum_{p=1}^P \left(h_p e^{j2\pi f_c \tau_p} s(t - \tau_p) e^{j2\pi f_c (t - \tau_p)} e^{j2\pi \nu_p (t - \tau_p)} \right) + v_{RF}(t)$. The received signal after downconversion to base band is

$r(t) = r_{RF}(t)e^{-j2\pi f'_c t}$, where $f'_c = f_c - \delta f_c$ is the receiver carrier frequency with offset δf_c . The signal $r(t)$ is,

$$r(t) = \sum_{p=1}^P h_p e^{j2\pi \delta f_c \tau_p} s(t - \tau_p) e^{j2\pi (\delta f_c + \nu_p)(t - \tau_p)}. \quad (5)$$

The signal $r(t)$ sampled at $F_s = B = 1/T_s = (M + L)/T$, where $L = \lceil T_{CP} B \rceil$ is the length of sampled CP and $T = T_u + T_{CP} = (M + L)T_s$, becomes,

$$r(l') = r(l'T_s) = \sum_{p=1}^P h_p e^{j2\pi \delta f_c \tau_p} \left(\frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X(m, n) g\left([l' - \frac{\tau_p}{T_s} - n(M + L)]T_s\right) e^{j2\pi m \Delta f ([l' - \frac{\tau_p}{T_s} - n(M + L)]T_s)} \right) e^{j2\pi (\delta f_c + \nu_p)(l'T_s - \tau_p)}, \text{ where } l' \in \mathbb{Z}[0 \ N(M + L)]. \quad (6)$$

The n' th OTFS symbol with CP can be collected from the samples of received signal as $r(n'(M + L) + l)$, $\forall l = 0, 1, \dots, (M + L) - 1$ and can be written as,

$$r(n'(M + L) + l) = \sum_{p=1}^P h_p e^{j2\pi \delta f_c \tau_p} \left(\frac{1}{\sqrt{M}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X(m, n) g\left([(n' - n)(M + L) + l - \frac{\tau_p}{T_s}]T_s\right) e^{j2\pi m \Delta f ([n'(M + L) + l - \frac{\tau_p}{T_s}]T_s)} \right) e^{j2\pi (\delta f_c + \nu_p)T_s(n'(M + L) + l - \frac{\tau_p}{T_s})}. \quad (7)$$

We assume that, after initial coarse synchronization, a residual synchronization error of l_o

Fig. 1: FFT Window Matching

samples from the starting index of CP removed OTFS symbol as shown in Fig.1 exists. We also assume $(l_o T_s + \tau_{max}) \leq T_{CP}$ so that no interference is experienced from the neighbouring OTFS symbols. The first step towards decoding the signal is to perform discrete Fourier Transform

(DFT) on the CP removed samples of n' th OTFS symbol to obtain the time-frequency data and can be given as,

$$Y(m', n') = \frac{1}{\sqrt{M}} \sum_{k_t=0}^{M-1} r(n'(M+L) + L - l_0 + k_t) e^{-j \frac{2\pi m' k_t}{M}}. \quad (8)$$

Using (7) we can write,

$$Y(m', n') = \frac{1}{M} \sum_{p=1}^P h_p e^{j2\pi \delta f_c \tau_p} e^{j2\pi (\nu_p + \delta f_c) T_s (L + n'(M+L) - l_0)} e^{-j2\pi (\nu_p + \delta f_c) T_s (\frac{\tau_p}{T_s})} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X(m, n) e^{-j2\pi \frac{m(l_0 + \frac{\tau_p}{T_s} + (n' - n)(M+L))}{M}} \sum_{k_t=0}^{M-1} \left(g([(n' - n)(M+L) + L - l_0 + k_t - \frac{\tau_p}{T_s}] T_s) e^{-j2\pi \frac{k_t(m + (\nu_p + \delta f_c) T_s - m')}{M}} \right). \quad (9)$$

Since, $g(t)$ is a rectangular pulse,

$$g([(n' - n)(M+L) + L - l_0 + k_t - \frac{\tau_p}{T_s}] T_s) = \begin{cases} 1, & \text{if } n = n' \text{ \& } 0 \leq (L - l_0 + k_t - \frac{\tau_p}{T_s}) T_s \leq T \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

Then, (9) can be written as,

$$Y(m', n') = \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s (L + n'(M+L) - \frac{\tilde{\tau}_p}{T_s})} \sum_{m=0}^{M-1} X(m, n') e^{-j2\pi \frac{m \tilde{\tau}_p}{M T_s}} \Psi((m + \tilde{\nu}_p T_s - m'), M) \quad (11)$$

where $\tilde{h}_p = h_p e^{j2\pi \delta f_c \tau_p}$, $\tilde{\nu}_p = (\nu_p + \delta f_c)$, $\tilde{\tau}_p = (\tau_p + l_0 T_s)$, $\Psi(x, M) \triangleq \frac{1}{M} \sum_{k_t=0}^{M-1} e^{-j \frac{2\pi k_t x}{M}}$. When $x \in \mathbb{Z}$, $\Psi(x, M) = \delta([x]_M)$. The time-frequency signal is transformed to delay-Doppler domain using symplectic finite Fourier transform (SFFT) as,

$$y(k', l') = \frac{1}{\sqrt{NM}} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} Y(m', n') e^{-j2\pi [\frac{n' k'}{N} - \frac{m' l'}{M}]}, \quad (12)$$

which can be simplified to (as shown in appendix A)

$$y(k', l') = \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s L} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \Psi(k - k' + \frac{\tilde{\nu}_p}{\Delta \nu}, N) \Psi(l - l' + \frac{\tilde{\tau}_p}{\Delta \tau}, M) e^{j2\pi \frac{\tilde{\nu}_p (l' - \frac{\tilde{\tau}_p}{\Delta \tau})}{\Delta \nu (M+L)N}}, \quad (13)$$

where $\Delta\nu = \frac{1}{(M+L)NT_s}$ and $\Delta\tau = T_s = \frac{1}{B}$ are Doppler and delay resolution at the receiver. Let $\tilde{\tau}_p = \tilde{l}_p\Delta\tau$ and $\tilde{\nu}_p = \tilde{k}_p\Delta\nu \implies \tilde{\nu}_p T_s = \frac{\tilde{k}_p}{(M+L)N}$, where $\tilde{l}_p, \tilde{k}_p \in \mathbb{R}$, then (13) can be written as,

$$y(k', l') = \sum_{p=1}^P \left\{ \tilde{h}_p e^{j2\pi \frac{\tilde{k}_p L}{(M+L)N}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \Psi(k - k' + \tilde{k}_p, N) \Psi(l - l' + \tilde{l}_p, M) e^{j2\pi \frac{\tilde{k}_p(l' - \tilde{l}_p)}{(M+L)N}} \right\} \quad (14)$$

IV. EFFECTS OF RESIDUAL SYNCHRONIZATION ERRORS

In this section, we describe the effects of residual FTO and CFO errors in the receiver. From (13), it may be noted that $y(k', l')$ experiences ISI in both delay and Doppler dimension.

A. Integer delay and integer Doppler values

When $\tilde{l}_p$ and $\tilde{k}_p$ are integers, then (13) simplifies as,

$$y(k', l') = \sum_{p=1}^P \tilde{h}_p e^{j2\pi \frac{\tilde{k}_p L}{(M+L)N}} d([k' - \tilde{k}_p]_N, [l' - \tilde{l}_p]_M) e^{j2\pi \frac{\tilde{k}_p(l' - \tilde{l}_p)}{(M+L)N}}. \quad (15)$$

If we consider an AWGN scenario, then integer time and frequency errors result in a cyclic shift in delay and Doppler direction respectively thus cyclically shifting the origin of delay-Doppler grid to $(l_o, \frac{\delta f_c}{\Delta\nu})$. It can also be observed that, (15) resembles the received delay-Doppler signal as described in equation (24) of [3]. Thus the effect of synchronization error can be considered as a part of the channel itself, with modified channel taps $\tilde{h}_p = h_p e^{j2\pi \delta f_c \tau_p}$, $\tilde{\tau}_p = \tau_p + l_0 T_s$ & $\tilde{\nu}_p = \nu_p + \delta f_c$. Thus, it may be conjectured that channel equalization with appropriate channel coefficients may be able to equalize the effects of TVMC and residual synchronization errors.

B. Integer delay and fractional Doppler values

By observing the the summation terms on the running variables l and k in (14) we can infer the following. If any Doppler or delay value of the modified channel is fractional, i.e, $\tilde{k}_p$ or $\tilde{l}_p \notin \mathbb{Z}$, then every symbol experiences interference in Doppler or delay dimension accordingly. The

interference in the Doppler axis is observed frequently because the Doppler values of channel are not usually resolved to integer values. For example, the imperfect compensation of CFO during initial synchronization, which results in δf_c , can lead to a modified fractional Doppler value even though the actual channel Doppler values (k_p) are integers. This is because $\tilde{k}_p = k_p + \frac{\delta f_c}{\Delta \nu}$. Fractional delay values are not observed since the sampling of the received signal in time domain approximates the effect of fractional channel delay to the nearest integer time bin [3]. Thus $\tilde{l}_p \in \mathbb{Z} \implies (l - l' + \tilde{l}_p) \in \mathbb{Z} \forall l' \in [0 \ N - 1] \implies \Psi(l - l' + \tilde{l}_p, M) = \delta([l - l' + \tilde{l}_p]_M)$. Therefore (14) becomes,

$$y(k', l') = \sum_{p=1}^P \left\{ \tilde{h}_p e^{j2\pi \frac{\tilde{k}_p L}{(M+L)N}} \sum_{k=0}^{N-1} d(k, [l' - \tilde{l}_p]_M) \Psi(k - k' + \tilde{k}_p, N) e^{j2\pi \frac{\tilde{k}_p (l' - \tilde{l}_p)}{(M+L)N}} \right\}. \quad (16)$$

In (16), each received symbol experiences interference from all other symbols in Doppler dimension. In the sections that follow, we describe the construction of the equivalent channel matrix, its estimation and compensation of the effects of synchronization errors described here.

V. EQUIVALENT CHANNEL MATRIX FOR OTFS INCLUDING SYNCHRONIZATION ERRORS

In this section, we derive the expression of the equivalent channel matrix in time domain, which includes the effect of residual synchronization errors and time varying channel. For this, we establish the equivalent system model in matrix-vector form for CP-OTFS. Symbols $d(k, l)$ are arranged in $M \times N$ matrix as,

$$\mathbf{D} = \begin{bmatrix} d(0, 0) & d(1, 0) & \cdots & d(N-1, 0) \\ d(0, 1) & d(1, 1) & \cdots & d(N-1, 1) \\ \vdots & \vdots & \ddots & \vdots \\ d(0, M-1) & d(1, M-1) & \cdots & d(N-1, M-1) \end{bmatrix}. \quad (17)$$

Delay-Doppler to frequency-time domain conversion (after the ISFFT) is done following $\mathbf{X} = \mathbf{W}_M^\dagger \mathbf{D} \mathbf{W}_N$, where, $\mathbf{X} = \{X(m, n) \mid \forall m \in \mathbb{Z}[0 \ M-1] \ \& \ n \in \mathbb{Z}[0 \ N-1]\}$, such that frequency is along m and time is along n . Frequency-time domain to time domain signal is obtained using

OFDM modulation, as $\mathbf{S} = \mathbf{W}_M \mathbf{X} = \mathbf{D} \mathbf{W}_N$, where $\mathbf{S} = [\mathbf{s}_0 \ \mathbf{s}_1 \ \cdots \ \mathbf{s}_{N-1}]$ is concatenation of OTFS symbol vectors $\mathbf{s}_i, \forall i \in [0 \ N-1]$. The pulse shaped samples of the signal is written as, $\mathbf{S}_{PS} = \mathbf{G}_T \mathbf{W}_M \mathbf{X}$. $\mathbf{G}_T$ is the pulse shaping matrix [24]. Since $g(t)$ is rectangular, $\mathbf{G}_T = \mathbf{I}_M$.

The CP appended signal is given as, $\mathbf{S}_{CP} = \mathbf{B}_{CP} \mathbf{S}_{PS} = \mathbf{B}_{CP} \mathbf{G}_T \mathbf{W}_M \mathbf{X} = \mathbf{B}_{CP} \mathbf{G}_T \mathbf{D} \mathbf{W}_N$, where

$\mathbf{B}_{CP} = \begin{bmatrix} \mathbf{0}_{L \times M-L} & \mathbf{I}_L \\ & \mathbf{I}_M \end{bmatrix}$ is the operator for appending CP. Thus, the transmit signal can be given as, $\mathbf{s}_{CP} = \text{vec}\{\mathbf{S}_{CP}\} = \text{vec}\{\mathbf{B}_{CP} \mathbf{G}_T \mathbf{D} \mathbf{W}_N\}$. Using the identity $\text{vec}\{\mathbf{A}_{K \times L} \mathbf{B}_{L \times M}\} = (\mathbf{I}_M \otimes \mathbf{A}) \text{vec}\{\mathbf{B}\} = (\mathbf{B} \otimes \mathbf{I}_M) \text{vec}\{\mathbf{A}\}$, we get, $\mathbf{s}_{CP} = (\mathbf{I}_N \otimes (\mathbf{B}_{CP} \mathbf{G}_T)) \text{vec}\{\mathbf{D} \mathbf{W}_N\} = (\mathbf{I}_N \otimes \mathbf{B}_{CP})(\mathbf{W}_N^T \otimes \mathbf{I}_M) \text{vec}\{\mathbf{D}\} = (\mathbf{I}_N \otimes \mathbf{B}_{CP})(\mathbf{W}_N \otimes \mathbf{I}_M) \mathbf{d}$. Therefore,

$$\mathbf{s}_{CP} = \mathbf{A}_{CP} \mathbf{d}, \quad (18)$$

where $\mathbf{d} = \text{vec}\{\mathbf{D}\}$ is data vector, $\mathbf{A}_{CP} = (\mathbf{I}_N \otimes \mathbf{B}_{CP}) \mathbf{A}$ and $\mathbf{A} = \mathbf{W}_N \otimes \mathbf{I}_M$. We also introduce the transmit signal vector $\mathbf{s} = \text{vec}\{\mathbf{S}\}$ without CP added and can be given as $\mathbf{s} = \mathbf{A} \mathbf{d}$ by putting $\mathbf{B}_{CP} = \mathbf{I}_M$ in (18). At the receiver, the noiseless received signal in discrete form [3] can be written as,

$$\mathbf{r}_{CP}(l) = \sum_{p=1}^P h_p \mathbf{s}_{CP}(l - l_p) e^{j2\pi \frac{(l-l_p)k_p}{(M+L)N}}, \quad (19)$$

where $l \in \mathbb{Z}[0 \ ((M+L)N-1)]$. We collect the samples $\mathbf{r}_{CP}(q(M+L)+l) \forall l \in \mathbb{Z}[0 \ M+L-1]$ to obtain q th OTFS symbol vector $\mathbf{r}_{CP}^q$ with CP. Then the CP removed vector $\mathbf{r}_q$ can be given as,

$$\mathbf{r}_q = \mathbf{R}_{CP} \mathbf{r}_{CP}^q, \text{ where } \mathbf{R}_{CP} = \begin{bmatrix} \mathbf{0}_{M \times L} & \mathbf{I}_M \end{bmatrix}. \quad (20)$$

Therefore,

$$\mathbf{r}_q(l) = \mathbf{r}_{CP}(q(M+L) + L + l), \ l \in [0 \ M-1]. \quad (21)$$

With the introduction of CFO at the receiver, the samples of q th received OTFS symbol $\mathbf{r}_q^f(l)$ is

$$\mathbf{r}_q^f(l) = \mathbf{r}(q(M+L) + L + l)e^{j2\pi \frac{k_0(q(M+L)+L+l)}{N(M+L)}}, \quad (22)$$

where $k_0 = \frac{\delta f_c}{\Delta \nu}$. With residual time synchronization error of l_o samples,

$$\tilde{\mathbf{r}}_q(l) = \mathbf{r}_q^f(l - l_o).$$

From (19), (21) and (22),

$$\tilde{\mathbf{r}}_q(l) = e^{j2\pi \frac{k_0(q(M+L)+L+l-l_o)}{N(M+L)}} \left(\sum_{p=1}^P h_p e^{j2\pi \frac{k_p q}{N}} e^{j2\pi \frac{(L+l-l_o-l_p)k_p}{(M+L)N}} \mathbf{s}_{CP}(q(M+L) + L + l - l_o - l_p) \right). \quad (23)$$

We assume $l_o + l_\tau < L$. Therefore,

$$\tilde{\mathbf{r}}_q(l) = \sum_{p=1}^P h_p e^{j2\pi \frac{l_p k_0}{(M+L)N}} e^{j2\pi \frac{(k_p+k_0)q}{N}} e^{j2\pi \frac{(L+l-l_o-l_p)(k_p+k_0)}{(M+L)N}} \mathbf{s}_q([l - l_p - l_o]_M). \quad (24)$$

Then,

$$\tilde{\mathbf{r}}_q = \begin{bmatrix} \tilde{\mathbf{r}}_q(0) \\ \tilde{\mathbf{r}}_q(1) \\ \vdots \\ \tilde{\mathbf{r}}_q(M-1) \end{bmatrix} = \sum_{p=1}^P h_p e^{j2\pi \frac{l_p k_0}{(M+L)N}} \mathbf{\Pi}^{l_p+l_o} \mathbf{\Delta}^{k_p+k_0} e^{j2\pi \frac{(k_p+k_0)(L-l_p-l_o)}{(M+L)N}} e^{j2\pi \frac{k_p+k_0}{N}q} \mathbf{s}_q, \quad (25)$$

where $\mathbf{\Pi} = \text{circ}\{[0 \ 1 \ 0 \ \dots \ 0]_{M \times 1}^T\}$ is a circulant delay matrix and $\mathbf{\Delta} = \text{diag}\{[1 \ e^{j2\pi \frac{1}{(M+L)N}} \ \dots \ e^{j2\pi \frac{M-1}{(M+L)N}}]^T\}$ is a diagonal Doppler matrix. Let, $\tilde{l}_p = l_p + l_o$, $\tilde{k}_p = k_p + k_0$ and $\tilde{h}_p = h_p e^{j2\pi \frac{l_p k_0}{(M+L)N}}$. When we include noise,

$$\tilde{\mathbf{r}}_q = \tilde{\mathbf{H}}_q \mathbf{s}_q + \mathbf{v}_q, \quad (26)$$

where $\mathbf{v}_q$ is M length Gaussian noise vector with elemental variance σ_v^2 and

$$\tilde{\mathbf{H}}_q = \sum_{p=1}^P \tilde{h}_p \mathbf{\Pi}^{\tilde{l}_p} \mathbf{\Delta}^{\tilde{k}_p} e^{j2\pi \frac{\tilde{k}_p(L-\tilde{l}_p)}{(M+L)N}} e^{j2\pi \frac{\tilde{k}_p}{N} q}. \quad (27)$$

The above can be modified as,

$$\tilde{\mathbf{H}}_q = \sum_{l=0}^{\tilde{l}_\tau} \mathbf{\Pi}^l \sum_{k \in \tilde{k}_{\nu_l}} \text{diag}\{\tilde{h}_{l,k} e^{j2\pi \frac{k(L-l)}{(M+L)N}} e^{j2\pi \frac{k}{N} q} [1 \ e^{j2\pi \frac{k}{(M+L)N}} \ \dots \ e^{j2\pi \frac{(M-1)k}{(M+L)N}}]^T\}, \quad (28)$$

where $\tilde{l}_\tau = l_\tau + l_0$ is maximum excess delay bin value of channel. We let k_{ν_l} be the set of Doppler indices for the l th channel tap such that $P = \sum_{l=0}^{\tilde{l}_\tau} k_{\nu_l}$. Then $\tilde{k}_{\nu_l} = \{(k + k_0) | k \in k_{\nu_l}\}$ and

$$\tilde{h}_{l,k} = \begin{cases} \tilde{h}_p, & \text{if } \tilde{l}_p = l \text{ and } \tilde{k}_p = k \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the concatenation of CP removed vectors can be given as,

$$\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{\mathbf{r}}_0 \\ \tilde{\mathbf{r}}_1 \\ \vdots \\ \tilde{\mathbf{r}}_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{\mathbf{H}}_0 & & & \\ & \tilde{\mathbf{H}}_1 & & \\ & & \ddots & \\ & & & \tilde{\mathbf{H}}_{N-1} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{N-1} \end{bmatrix}}_{\mathbf{s}} + \mathbf{v}, \quad (29)$$

which can be written as, $\tilde{\mathbf{r}} = \mathbf{H}\mathbf{s} + \mathbf{v} = \mathbf{H}\mathbf{A}\mathbf{d} + \mathbf{v}$, $\mathbf{v}$ being MN length concatenated white Gaussian noise vector. Equation (26) suggests that the effect of synchronization errors can be considered as part of time domain channel matrix. Equation (27) shows that the structure of the channel matrix is invariant to the introduction of synchronization errors which is an added advantage as the number of elements in the matrix does not change with residual synchronization errors and hence the sparsity of the matrix is unaltered. Next, we propose an estimation algorithm to estimate this equivalent channel matrix, however we first give a short justification for choosing

time domain channel estimation over delay-Doppler domain processing.

$\tilde{\mathbf{H}}_q$ contains N_e number of non-zero elements. The time domain channel matrix $\mathbf{H}$ comprises of N such matrices resulting in NN_e non-zero elements. However, the delay-Doppler domain channel matrix $\mathbf{H}_{DD}$ (described in Appendix B) comprises N^2 such matrices. When channel Doppler values are resolved into integer Doppler bins, each row of $\mathbf{H}_{DD}$ contains P matrices, which results in $NP N_e$ number of non-zero elements in $\mathbf{H}_{DD}$. However, when fractional Doppler is observed, the delay-Doppler channel matrix $\mathbf{H}_{DD}$ contains $N^2 N_e$ number of non-zero elements. Thus, structure of channel matrix varies with the nature of channel Doppler values which is not observed in time domain channel matrix. Therefore, $\mathbf{H}$ is at least P times and at max N times sparse than $\mathbf{H}_{DD}$. Therefore, equalization with $\mathbf{H}$ will result in lower complexity than equalization with $\mathbf{H}_{DD}$.

VI. ESTIMATION OF EQUIVALENT CHANNEL MATRIX

In this section, we propose an algorithm to estimate the equivalent channel matrix specified in (28). From (28), we can write,

$$\tilde{\mathbf{H}}_q = \sum_{l=0}^{\tilde{l}_\tau} \mathbf{\Pi}^l \mathbf{H}_{q,l}, \text{ where,} \quad (30)$$

$$\mathbf{H}_{q,l} = \text{diag}\{[h(q(M+L)+L, l) \ h(q(M+L)+L+1, l) \ \cdots \ h(q(M+L)+M+L-1, l)]^T\}, \quad (31)$$

$$\text{where, } h(n, m) = \sum_{k \in \tilde{k}_{\nu m}} h_{m,k} e^{j2\pi \frac{k_0 m}{N(M+L)}} e^{j2\pi \frac{k(n-m)}{N(M+L)}}, \forall n \in [0 \ N(M+L) - 1], m \in [0 \ \tilde{l}_\tau]. \quad (32)$$

A. Pilot structure in delay-Doppler domain

We extend on the pilot structure described for OTFS in [14]. The pilot is a 2-dimensional (2D) impulse in delay-Doppler domain i.e. $d(k, l) = \sqrt{P_{PLT}} \delta(k - K_p, l - L_p), \forall k \in [0 \ N - 1], l \in [L_p - L \ L_p + L - 1]$, where $P_{PLT} = N_p = 2NL$ is the pilot power. At the receiver, the received

delay-Doppler signal corresponding to pilot signal, with synchronization error is,

$$y(k', l') = \sqrt{P_{PLT}} \sum_{l=0}^{\tilde{l}_\tau} \delta([l' - L_p - l]_M) \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} \Psi(K_p - k' + k, N) e^{j2\pi \frac{k(L+l'-l)}{(M+L)N}} \right\}. \quad (33)$$

From the above, one may infer that the transmitted 2D impulse pilot, after going through the channel, spreads over the entire Doppler axis while the spread in delay is limited to $(L_p + \tilde{l}_\tau)$ starting from L_p . We collect received signal $y(k', l')$, $\forall k \in [0 \ N - 1], l \in [L_p \ L_p + L - 1]$ for channel estimation. This part of the received signal contains the response of the channel to the 2D-delay-Doppler impulse pilot signal, which is not-interfered by data symbols. This is because we have assumed that $L \geq l_\tau + l_o - 1$.

B. Channel Estimation

Here we describe theorems necessary to obtain the time domain channel estimates from the received time domain signal. Theorem 1 relates the time domain channel coefficients and the received delay-Doppler signal $y(k', l')$. Theorem 2 establishes the relationship between the estimates of time domain channel coefficients and the received time domain signal $\mathbf{r}_{CP}(l)$.

Theorem 1. *The intermittent time-domain channel coefficients are directly proportional to the N point IDFT values obtained from pilot section of received delay-Doppler grid, i.e, $h(\alpha(M + L) + L + L_p + l, l) = e^{-j2\pi \frac{\alpha K_p}{N}} \left(\sum_{k'=0}^{N-1} y(k', L_p + l) e^{j2\pi \frac{\alpha k'}{N}} \right), \forall \ \alpha \in \mathbb{Z}[0 \ N - 1], l \in \mathbb{Z}[0 \ \tilde{l}_\tau - 1]$.*

Proof: $y(k', L_p + l') = \sum_{l=0}^{\tilde{l}_\tau} \delta(l' - l) \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} \Psi(K_p - k' + k, N) e^{j2\pi \frac{k_p(L+l'+L_p-l)}{(M+L)N}} \right\}$. At l th tap, i.e when $l = l'$, $y(k', L_p + l) = \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} e^{j2\pi \frac{k(L_p+L)}{(M+L)N}} \Psi(K_p - k' + k, N) \right\}$.

When IFFT of $(M+L)N$ point is applied and it can be shown that,

$$\begin{aligned} \sum_{k'=0}^{N-1} y(k', L_p + l) e^{j2\pi \frac{nk'}{N(M+L)}} &= \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} e^{j2\pi \frac{k(L_p+L)}{(M+L)N}} \left(\frac{1}{N} \sum_{k_t=0}^{N-1} e^{j2\pi \frac{k_t(K_p+k)}{N}} \sum_{k'=0}^{N-1} e^{j2\pi \frac{k'(n-k_t(M+L))}{N(M+L)}} \right) \right\} \\ &= \begin{cases} \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} e^{j2\pi \frac{k(L_p+L)}{(M+L)N}} \left(e^{j2\pi \frac{n(K_p+k)}{N(M+L)}} \right) \right\}, & n = \alpha(M+L) \\ \sum_{k_t=0}^{N-1} e^{j2\pi \frac{k_t(K_p)}{N}} \Psi\left(\left(\frac{n}{(M+L)} - k_t\right), N\right) \sum_{k \in k_{\nu_l}} \left\{ \tilde{h}_{l,k} e^{j2\pi \frac{k(L_p+L)}{(M+L)N}} \right\} e^{j2\pi \frac{k_t(k)}{N}}, & \text{otherwise.} \end{cases} \end{aligned}$$

From (32),

$$h(n, m) = \begin{cases} h(n + L_p + L + l, l) e^{j2\pi \frac{n(K_p)}{N(M+L)}}, & n = \alpha(M+L) \\ \sum_{k_t=0}^{N-1} e^{j2\pi \frac{k_t(K_p)}{N}} \Psi\left(\left(\frac{n}{(M+L)} - k_t\right), N\right) h(k_t(M+L) + L_p + L + l, l), & \text{otherwise.} \end{cases}$$

$$\text{Hence } h(\alpha(M+L) + L + L_p + l, l) = e^{-j2\pi \frac{\alpha K_p}{N}} \left(\sum_{k'=0}^{N-1} y(k', L_p + l) e^{j2\pi \frac{\alpha k'}{N}} \right)$$

■

Theorem 2. *The intermittent time-domain channel coefficients are directly proportional to the samples of received signal, i.e.,*

$$\hat{h}(\alpha(M+L) + L_p + l, l) = e^{-j2\pi \frac{\alpha K_p}{N}} \mathbf{r}_{CP}(\alpha(M+L) + L + L_p + l), \quad \alpha \in [0 \ N-1].$$

Proof: We know, $\hat{h}(\alpha(M+L) + L_p + l, l) = e^{-j2\pi \frac{\alpha K_p}{N}} \left(\sum_{k'=0}^{N-1} y(k', L_p + l) e^{j2\pi \frac{\alpha k'}{N}} \right)$. We discussed in [10] that the time domain signal can be viewed as the interleaved OFDM, in which an N -point FFT is taken along the Doppler axis and then interleaved to get the time domain signal. If this is applied to the received delay-Doppler signal $y(k', l')$, then the signal without CP can be written as, $r(\alpha(M) + l') = \left(\sum_{k'=0}^{N-1} y(k', l') e^{j2\pi \frac{\alpha k'}{N}} \right)$. Due to addition of CP, $r(\alpha(M+L) + L + l') = \left(\sum_{k'=0}^{N-1} y(k', l') e^{j2\pi \frac{\alpha k'}{N}} \right)$. Therefore, $\hat{h}(\alpha(M+L) + L_p + l, l) = e^{-j2\pi \frac{\alpha K_p}{N}} r(\alpha(M+L) + L + L_p + l)$. ■

From the above theorem, we see that the estimate of channel coefficients at time instances $(\alpha(M + L) + L_p + l)T_s$, $\alpha \in [0 \ N - 1]$ for l th channel tap can directly be obtained from $\mathbf{R}(l, k) = \mathbf{r}_k(l)$, as in Algorithm 1, without going through the FFT-ISFFT path to reach delay-Doppler domain as described in [14]. This creates the opportunity for estimating time domain channel coefficients from delay-Doppler domain embedded pilot. Using the estimated channel coefficients we obtain the values at time instances nT_s , $n \in \mathbb{Z}[0 \ N(M + L) - 1]$ for each channel tap through interpolation. Since the coefficients to estimate (31) resembles a signal comprising sum of sinusoids, we considered the polynomial interpolation techniques (spline [25]) in this work. In other words,

$$h(n, l) = \text{spline_interpolate}\left([h(L_p + l, l) \ h(M + L + L_p + l, l) \ \cdots \ h((N - 1)(M + L) + L_p + l, l)]^T, \right. \\ \left. \cdots [0 \ M + L(N - 1)(M + L)]^T, [0 \ 1 \ 2 \ \cdots \ (N)(M + L) - 1]^T\right), \forall l \in [0 \ l_\tau - 1] \quad (34)$$

where `spline_interpolate` is implemented using ‘`interp1`’, a inbuilt function in Matlab[®]. Algorithm 1 describes how channel estimates for the entire time duration are generated from the received signal as well as how to estimate q th channel matrices $\hat{\mathbf{H}}_{q,i}$ & $\hat{\mathbf{H}}_q$, which are described in (31) & (30) respectively.

C. Time domain interpretation of the channel estimation

Fig. 2: Time domain CP-OTFS frame

It can be shown that the pilot-data structure in delay-Doppler as described earlier, leads to a

Algorithm 1 Estimation of $\tilde{\mathbf{H}}_q$

```

1: Given : The received signal  $r(l)$ 
2: Output :  $\hat{\tilde{\mathbf{H}}}_q$ 
3:  $\mathbf{R}(l, k) = \mathbf{r}_k(l)$ 
4: for  $l' = 0 : L$  do
5:    $\gamma = 0$ 
6:   for  $k = 0 : N - 1$  do
7:      $\gamma = \gamma + \|\mathbf{R}(L_p + l', k)\|^2$ 
8:   end for
9:   if  $\gamma > 3\sigma_n^2$  then
10:     $l = [l \ l']$ 
11:     $\hat{h}(n(M + L) + L + L_p + l', l') = \frac{1}{\sqrt{P_{PLT}}} \mathbf{R}(L_p + l', n)$ 
12:     $\hat{h}(n, l') = \text{spline\_interpolate}(\{\hat{h}(n(M + L) + L + L_p + l', l') \mid n \in [0 \ N - 1]\}, [L + L_p + l' : M + L : N(M + L)], [0 : N(M + L) - 1])$ 
13:   end if
14: end for
15: for  $q = 0 : N - 1$  do
16:    $\hat{\mathbf{H}}_q = \mathbf{0}_{M \times M}$ 
17:   for  $i \in l$  do
18:      $\hat{\mathbf{H}}_{q,i} = \text{diag}\{\{\hat{h}(q(M + L) + L + l', i) \mid l' \in [0 \ M - 1]\}\}$ 
19:      $\hat{\mathbf{H}}_q = \hat{\mathbf{H}}_q + \mathbf{\Pi}^i \hat{\mathbf{H}}_{q,i}$ 
20:   end for
21: end for

```

time domain frame as shown in Fig. 2. The samples of the time domain frame can be written as,

$$s(n) = s_p(n) + s_d(n). \quad (35)$$

Symbols $s_d(n)$ and $s_p(n)$ are the corresponding time domain equivalent signal of the the delay-Doppler data and pilot. Since pilot is a 2-D discrete impulse at location (K_p, L_p) ,

$$s_p(n) = \sqrt{\frac{P_{PLT}}{N}} e^{j2\pi \frac{K_p n}{N(M+L)}} \sum_{\alpha=0}^{N-1} \delta(n - (\alpha(M + L) + L + L_p)) \quad (36)$$

Using (19), (32), (35) and (36), the received signal is, $r(n) = \sum_{l=0}^{l_\tau-1} h(n, l) (\sqrt{\frac{P_{PLT}}{N}} e^{j2\pi \frac{K_p(n-l)}{N(M+L)}} \sum_{\alpha=0}^{N-1} \delta(n - l - (\alpha(M + L) + L + L_p))) + \sum_{l=0}^{l_\tau-1} h(n, l) s_d(n-l)$ which can be simplified as, $r(n) = \sqrt{\frac{P_{PLT}}{N}} \sum_{\alpha=0}^{N-1} h(n, n - (\alpha(M + L) + L + L_p)) e^{j2\pi \frac{K_p(\alpha(M+L)+L+L_p)}{N(M+L)}} + \sum_{l=0}^{l_\tau-1} h(n, l) s_d(n-l)$. With length of CP being

greater than τ_{max} , $r(\alpha(M+L)+L+L_p+l) = \sqrt{\frac{P_{PLT}}{N}} h(\alpha(M+L)+L_p+l, l) e^{j2\pi \frac{K_p(\alpha(M+L)+L+L_p)}{N(M+L)}}$

which implies the same result as obtained using theorem 2. From this, time domain interpretation one can visualize the time domain frame as pilots embedded in each symbol. After such a pilot passes through a channel, one can obtain the channel impulse response directly. This can be interpolated to obtain the channel coefficients at all time instants as described earlier. In the next section we propose a low complexity LMMSE equalization technique, which uses the estimated equivalent channel matrix, to compensate for the channel effects and residual synchronization errors.

VII. LMMSE EQUALIZATION

In this section, we explain a low complexity LMMSE receiver for CP-OTFS based on [20]. The LMMSE equalization of $\mathbf{r}$ in (29) results in estimated data vector $\hat{\mathbf{d}}$ given as,

$$\hat{\mathbf{d}} = (\mathbf{H}\mathbf{A})^\dagger [(\mathbf{H}\mathbf{A})(\mathbf{H}\mathbf{A})^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I}]^{-1} \mathbf{r}. \quad (37)$$

When $g(t)$ is rectangular, $\mathbf{A}$ becomes unitary. Thus (37) can be written as,

$$\hat{\mathbf{d}} = \underbrace{\mathbf{A}^\dagger \mathbf{H}^\dagger [\mathbf{H}\mathbf{H}^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I}]^{-1} \mathbf{r}}_{\mathbf{r}_{ce} = \mathbf{H}_{eq} \mathbf{r}}. \quad (38)$$

Thus LMMSE equalization can be performed as a two stage equalizer. In the first stage, LMMSE channel equalization is performed to obtain $\mathbf{r}_{ce} = \mathbf{H}_{eq} \mathbf{r}$. Second stage is a OTFS matched filter receiver to obtain $\hat{\mathbf{d}} = \mathbf{A}^\dagger \mathbf{r}_{ce}$. The direct implementation of $\mathbf{r}_{ce} = \mathbf{H}_{eq} \mathbf{r}$ requires inversion of $\mathbf{\Psi} = \mathbf{H}\mathbf{H}^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I}$ and multiplication of $\mathbf{H}^\dagger$, which needs $O(M^3 N^3)$ complex multiplications (CMs). Thus, it is desired to reduce the complexity of $\mathbf{r}_{ce} = \mathbf{H}_{eq} \mathbf{r}$. It is evident from (29), that $\mathbf{H}$ matrix is a block diagonal matrix with blocks $\tilde{\mathbf{H}}_q$ of size $M \times M$. This leads to $\mathbf{\Psi} = \mathbf{H}\mathbf{H}^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I} = \text{diag}\{\mathbf{\Psi}_0, \mathbf{\Psi}_1, \dots, \mathbf{\Psi}_{N-1}\}$, which is a block diagonal matrix with blocks $\mathbf{\Psi}_q$ of size $M \times M$. It is well known that the inverse of a block diagonal matrix is also a block

diagonal matrix. In addition to that, the inverse of a block diagonal matrix can be computed using the inverse of individual blocks. Similar to the decomposition of $\mathbf{r}$ into $\mathbf{r}_q(s)$, $\mathbf{r}_{ce}$ can also be written as $\mathbf{r}_{ce} = [\mathbf{r}_{ce,0}^T \mathbf{r}_{ce,1}^T \cdots \mathbf{r}_{ce,N-1}^T]^T$, where $\mathbf{r}_{ce,q} = [\mathbf{r}_{ce}(q(M+L)+L) \mathbf{r}_{ce}(q(M+L)+L+1) \cdots \mathbf{r}_{ce}(q(M+L)+(M+L)-1)]^T$ is the q^{th} , $q \in [0 \ N-1]$ channel equalized vector. Thus we can write,

$$\mathbf{r}_{ce,q} = \tilde{\mathbf{H}}_q^\dagger [\tilde{\mathbf{H}}_q \tilde{\mathbf{H}}_q^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I}]^{-1} \mathbf{r}_q, \quad q \in \mathbb{Z}[0 \ N-1], \quad (39)$$

which can be computed using inversion and multiplication of $M \times M$ matrices. The required complexity is of $O(NM^3)$. Generally, the value of M is in the order of 100's. Although the above simplifications significantly reduce complexity, yet LMMSE processing remains a computational burden. We investigate the structure of Ψ_q involved in channel equalization described earlier, in order to reduce the complexity further.

A. Structure of $\Psi_q = [\tilde{\mathbf{H}}_q \tilde{\mathbf{H}}_q^\dagger + \frac{\sigma_v^2}{\sigma_d^2} \mathbf{I}]$

Using (27), $\tilde{\mathbf{H}}_q \tilde{\mathbf{H}}_q^\dagger$ can be expressed as, $\tilde{\mathbf{H}}_q \tilde{\mathbf{H}}_q^\dagger = \left(\sum_{p=1}^P \tilde{h}_p e^{j2\pi \frac{\tilde{k}_p(L-\tilde{l}_p)}{(M+L)N}} \Delta^{\tilde{k}_p} \Pi^{\tilde{l}_p} e^{j2\pi \frac{\tilde{k}_p}{N} q} \right) \left(\sum_{s=1}^P \tilde{h}_s e^{-j2\pi \frac{\tilde{k}_s(L-\tilde{l}_s)}{(M+L)N}} \Pi^{-\tilde{l}_s} \Delta^{-\tilde{k}_s} e^{j2\pi \frac{\tilde{k}_s}{N} (-q)} \right)$. Since Π is a circulant matrix, it can be verified that $\Pi^{l_p} = \mathbf{W} \Delta^{-l_p} \mathbf{W}^\dagger$. Therefore, $\tilde{\mathbf{H}}_q \tilde{\mathbf{H}}_q^\dagger = \sum_{p=s}^P |\tilde{h}_p|^2 \mathbf{I} + \sum_{p=1}^P \sum_{s=1, s \neq p}^P c_{p,s} \Pi^{l_p-l_s} \Delta^{k_p-k_s} e^{j2\pi q \frac{k_p-k_s}{N}}$, where $c_{p,s} = \tilde{h}_p \tilde{h}_s e^{j2\pi \frac{-\tilde{k}_p \tilde{l}_p + \tilde{k}_s \tilde{l}_s}{(M+L)N}} e^{j2\pi \frac{L(\tilde{k}_p - \tilde{k}_s)}{(M+L)N}}$. Thus, we can write $\Psi_q = \sum_{p=s}^P (|\tilde{h}_p|^2 + \frac{\sigma_v^2}{\sigma_d^2}) \mathbf{I} + \sum_{p=1}^P \sum_{s=1, s \neq p}^P c_{p,s} \Pi^{\tilde{l}_p - \tilde{l}_s} \Delta^{\tilde{k}_p - \tilde{k}_s} e^{j2\pi q \frac{\tilde{k}_p - \tilde{k}_s}{N}}$. From this, it can be concluded that the maximum shift of diagonal elements in Δ can be $\pm(\tilde{l}_\tau - 1)$. Additionally, due to the cyclic nature of the shift, Ψ_q is quasi-banded with bandwidth of $2\tilde{l}_\tau - 1$. As $\tilde{l}_\tau \ll M$, Ψ_q is also sparse for typical wireless channel. Structure of Ψ_q is similar to the channel matrix of RCP OTFS as described in equation (13) specified in [20]. Thus Ψ_q^{-1} can be computed using LU factorization of Ψ_q in a similar way as described in Sec. III B of [20], i.e. $\Psi_q = \mathbf{L}_q \mathbf{U}_q$.

B. Computation of $\hat{\mathbf{d}}$

After LU decomposition of Ψ_q , $\mathbf{r}_{ce,q}$ is simplified to, $\mathbf{r}_{ce,q} = \tilde{\mathbf{H}}_q^\dagger \underbrace{\mathbf{U}_q^{-1} \mathbf{L}_q^{-1} \mathbf{r}_q}_{\mathbf{r}_q^{(1)}}$. As $\mathbf{L}_q$ is a quasi-banded lower triangular matrix, $\mathbf{r}_q^{(1)} = \mathbf{L}_q^{-1} \mathbf{r}_q$ can be computed using low complexity forward substitution as explained in Algorithm 2 in [20]. Algorithm 3 of [20] can be used to $\mathbf{r}_q^{(2)} = \mathbf{U}_q^{-1} \mathbf{r}_q^{(1)}$. Using the definition of $\mathbf{H}_q$, $\mathbf{r}_{ce,q} = \tilde{\mathbf{H}}_q^\dagger \mathbf{r}_q^{(2)}$ can be written as, $\mathbf{r}_{ce,q} = \sum_{p=1}^P \bar{h}_p \Delta^{-k_p} \underbrace{\mathbf{\Pi}^{-l_p} \mathbf{r}_q^{(2)}}_{\text{circular shift}}$. To compute $\mathbf{r}_{ce,q}$, $\mathbf{r}_q^{(2)}$ is first circularly shifted by ‘ $-l_p$ ’ and then multiplied by $\bar{h}_p \text{diag}\{\Delta^{-k_p}\}$ using point-to-point multiplication for each path p . All vectors obtained above are summed to obtain $\mathbf{r}_{ce,q}$. Then, $\{\mathbf{r}_{ce,q}\}_{q=0}^{N-1}$ are concatenated to obtain $\mathbf{r}_{ce}$. Finally, $\hat{\mathbf{d}} = \mathbf{A}^\dagger \mathbf{r}_{ce}$ can be implemented using M number of N -point FFTs (Sec. III-C, [20]).

C. Computation complexity

With some effort it can be shown that the number of CMs required to implement our proposed LMMSE algorithm is $\frac{MN}{2} \log_2 N + MN[2l_\tau^2 + 2P^2k_\nu + 9l_\tau - Pk_\nu - 3] + N[\frac{2}{3}l_\tau^3 + 2l_\tau + P]$. The order of complexity achieved through our receiver is $MN \log(MN)$, which is significantly lower than the direct implementation, which is of the order of M^3N^3 . Our proposed receiver requires around 10^7 x lower CMs than the direct implementation following (37), if we consider a typical OTFS system with $\Delta f = 15 \text{ KHz}$, $f_c = 4 \text{ GHz}$, $N = 128$, $M = 512$, speed of 500 kmph and the extended vehicular A (EVA) 3GPP channel model [26] with $P = 9$ and $\tau_{max} = 2.51 \mu \text{ sec}$.

VIII. UNIFIED FRAMEWORK FOR ORTHOGONAL MULTICARRIER SYSTEMS

In this section, we describe a generalized framework for orthogonal waveforms. Modulation techniques following this framework will be able to take advantage of the channel estimation and equalization algorithms proposed earlier. Let $\mathbf{D}$ be the data matrix of size $(M) \times N$. Then the transmit signal can be of the form

$$\mathbf{S}_d = \{\mathfrak{B}\mathbf{D}\mathfrak{C}\} = \text{Avec}\{\mathbf{D}\}, \quad (40)$$

where $\mathfrak{B}$ and $\mathfrak{C}$ are modulation specific matrices. $\mathbf{A}$ in (37) can be computed using $\mathbf{A} = (\mathfrak{C} \otimes \mathbf{I})(\mathbf{I} \otimes \mathfrak{B})$. We define $\mathbf{S}_p$ as the constant pilot matrix of size $2L \times N$ as

$$\mathbf{S}_p = \begin{bmatrix} \mathbf{0}_{L \times N} \\ [1 \ 1 \ \cdots 1]_{1 \times N} \\ \mathbf{0}_{L-1 \times N} \end{bmatrix}. \quad (41)$$

The combined time domain matrix with N symbols each containing M samples can be given as, $\mathbf{S} = \begin{bmatrix} \mathbf{S}_p \\ \mathbf{S}_d \end{bmatrix}$. We append CP as described before, i.e. $\mathbf{S}_{CP} = \mathbf{B}_{CP}\mathbf{S}$. Then, the transmit vector can be given as $\mathbf{s} = \text{vec}\{\mathbf{S}_{CP}\}$. This transmit vector will resemble the time domain frame illustrated in Fig. 2. Hence, the same time domain channel estimation and low complexity LMMSE equalization proposed can be used to equalize the combined effects of channel and residual synchronization errors.

CP-OTFS presented in this work can be fitted in this framework when one sets $\mathfrak{B} = \mathbf{I}_M$ and $\mathfrak{C} = \mathbf{W}_N$ in above equations. Similarly, when $\mathfrak{B} = \mathbf{W}_M$ and $\mathfrak{C} = \mathbf{I}_N$, the above system becomes an OFDM system with no difference in the equalization techniques at the receiver. Hence, this generalized description of system can pave the way to realize a flexible communication system which can change its waveform with changing nature of channel, for e.g, the transceiver pair can use OFDM in low mobility scenarios while switching to OTFS under high speed scenario.

IX. RESULTS

In this section, we present low density parity check (LDPC) coded performance of CP-OTFS system with residual FTO and CFO errors. Since we present a unified model for constructing OTFS signal as well as OFDM signal, therefore we also present the performance of an equivalent OFDM system. The simulation parameters are mentioned in Table I. For each channel delay tap value, Doppler is generated using Jake's formula, $\nu_p = \nu_{max} \cos(\theta_p)$, where θ_p is uniformly distributed over $[-\pi \ \pi]$. CP length is chosen longer than τ_{max} of the TVMC.

TABLE I: Simulation Parameters

Parameter	Value
Carrier Frequency(f_c)	6 GHz
Bandwidth(B)	7.68 MHz
Frame Time(T_f)	8.7 ms
Subcarrier Bandwidth(Δf)	15 KHz
OTFS parameters	M=512, N=128 , $\Delta\nu = 117.18$ Hz, $\Delta\tau = 130.21$ ns
Pilot location	$K_p=0$, $L_p=10$
Channel Model	[27] TDL-A, DS=37 ns, Rural Macro
CP duration	$T_{CP} = 1.302$ μ s, $L = 10$
UE speed	500 kmph
FEC (LDPC)	Coderate 2/3, Codeword Length 648

A. BLER Performance

We begin with the block error rate (BLER) performance of CP-OTFS system through Fig. 3. The notation ‘dpt’ in the figure indicates the number of Doppler taps per channel delay tap. The curves labeled ‘Ideal’ represent evaluation with ideal channel estimate without any residual synchronization error. The legend ‘Ch est’ denotes performance with channel estimation as described in this work. The curves with l_0 and k_o identify cases with FTO and CFO. The value against l_0 and k_o conveys FTO in terms of number of samples and normalized CFO respectively.

Let us first consider the ‘Ideal’ performance . It is observed that as dpt increases, the performance of the system improves. At BLER of 10^{-1} , when dpt increases from 1 to 2 the performance improved by 4.5 dB. The improvement is 2 dB when dpt changes from 2 to 3. At BLER of 10^{-2} , the SNR gain is approximately 6 dB and 2.5 dB. This gain in performance with increasing value of dpt can be attributed to Doppler diversity. With increasing dpt, the number of independent channel paths increase. Thus resulting in increased channel diversity, which is extracted by the LMMSE receiver.

Next, we consider the performance of CP-OTFS with the proposed channel estimation algo-

rithm but without residual synchronization error, i.e. legends ‘Ch est but without l_0 and k_0 . The degradation in performance from ‘Ideal’ is limited to approximately 1 dB for all considered dpt.

Now we turn our attention to the performance of OTFS system with synchronization errors while using the estimation and compensation techniques described above. The values of l_0 is 2 samples. Normalized cfo error $k_0 = \frac{\delta f_c}{\Delta \nu} = 20$ results in $\delta f_c = k_0 \Delta \nu = 2.33$ KHz. We find that at BLER of 10^{-1} the performance degradation is less than 1 dB when compared to the situation when such residual synchronization errors are not present. However, when compared with ideal performance at the BLER threshold of 10^{-2} , the maximum SNR degradation is about 1.5, which is observed for dpt=3. Therefore, it may be stated that the performance of CP-OTFS while using the proposed estimation and compensation techniques for channel and residual synchronization errors is within acceptable limits.

Now we present the performance of OFDM system, which is described in Section VIII, in Fig. 4. We find that almost similar degradation in the performance of coded OFDM system is observed under identical test conditions. However, when we compare the ideal performance for dpt=1 of CP-OTFS and OFDM from figures 3 & 4 respectively, we find that CP-OTFS outperforms OFDM by approximately 3 dB. When dpt=2, the gap is about 4 dB. At dpt=3, the gap increases further. It can be noted from the figures that OFDM’s performance at BLER of 10^{-2} would only be significantly worse. The results with channel estimation and synchronization errors also bring out the fact that OTFS is significantly more reliable than OFDM in a TVMC. From the above discussions, it can be said that the proposed algorithms can provide sufficient resilience to OTFS against synchronization errors while compensating for TVMC, thus making OTFS a potential transmission technology candidate for use, especially in high mobility scenarios.

Having exposed the most important performance metrics of OTFS under TVMC and residual synchronization errors, we now take a look at the mean square error (MSE) of the channel estimates for the strongest tap against varying SNR for different dpt shown in Fig. 5. It can be observed that the MSE increases with dpt. Thus, one can infer that the channel coefficients

obtained from the interpolation based channel estimation deviate from actual channel coefficients as dpt increases. This is reflected in the SNR degradation observed in Fig. 3. It can also be observed that MSE saturates even when SNR increases. This indicates that the proposed estimation and compensation methods are effective only in the mid-SNR region. One may wonder that the saturation in MSE should be observed in the BLER curves as well. This is indeed true, however, the saturation in BLER occurs below the level of 10^{-3} (100 times less compared to BLER 10^{-1}), which is not captured in the figure.

Fig. 3: BLER performance with varying Doppler per tap without synch errors for OTFS

Fig. 4: BLER performance with varying Doppler per tap without synch errors for OFDM

Fig. 5: MSE of channel estimates for different number of dpt for OTFS.

B. Limit of CFO tolerance

Since we are concerned with the effect of residual synchronization errors, of which the CFO has more significant effect on the received signal, we present the BLER performance against normalized CFO ($\frac{\delta f_c}{\Delta \nu}$) for different dpt at SNR of 15 dB in Fig. 6. It can be observed that BLER increases with increase in CFO. This increase in BLER can be attributed to increase in MSE with CFO observed in Fig. 7, which shows the MSE of channel estimates at SNR of 15 dB for the strongest tap. The increase in MSE is due to an increasing mismatch between the estimates of channel coefficients obtained using the proposed method and the actual value of channel coefficients. An increase in CFO, which has a similar effect as increase in Doppler, results in a higher rate of channel fluctuations. Increasing the frequency of the number of time-domain pilots may help in improving the situation but at the cost of reduced spectral efficiency, the detailed analysis of which is beyond the scope of this work. Therefore it may be said that, with a limited allowable SNR degradation from the ideal performance, our proposed channel estimation method can compensate normalized CFO value up to 20. A normalized CFO value of 20 leads to an offset value of $\delta f_c = 20\Delta \nu = 2.33kHz$, which is 15% of the sub-carrier bandwidth and is 0.4 ppm of carrier frequency. This value can be considered as a significantly high limit of residual

frequency error, which can be addressed by the methods described in this work. It is important to mention that the performance is evaluated considering channel conditions where the maximum Doppler frequency is 2.76kHz, which is additive in nature with CFO as discussed in Section IV. This results in a significantly high value of equivalent maximum Doppler of 5.09 kHz.

Fig. 6: BLER performance for 16QAM with varying CFO at SNR 15dB

Fig. 7: MSE of channel estimates for different number of dpt with varying CFO at SNR 15 dB

X. CONCLUSIONS

In this work, we have described the system model of a rectangular pulse shaped CP-OTFS system with residual synchronization errors. We exposed that integer time and frequency errors

result in a cyclic shift in delay and Doppler dimensions. We also show that fractional Doppler or delay causes interference in Doppler or delay dimension respectively. We have brought out that the effect of synchronization errors can be considered as a part of the channel itself, however with modified channel taps. Initially we began with the conjecture that channel equalization with appropriate channel coefficients may compensate for the effects of synchronization errors along and TVMC. Accordingly we presented methods to estimate and equalize this effective channel matrix and the conjecture was finally shown to be valid. Using the estimates of the effective channel matrix, we derive a low complexity LMMSE receiver to cancel the interference, which grows log-linearly and has about 10^7 times lower complexity than the direct implementation. With the proposed method, the maximum SNR loss at BLER 10^{-2} is around 1 dB in the absence of residual synchronization errors. It is also shown that the proposed compensation method shows promise to compensate for residual CFO of up to 15% of subcarrier bandwidth or 0.4 ppm of carrier frequency with a loss in SNR up to 1.5 dB at BLER 10^{-2} . We have shown that OTFS with practical channel estimation and residual synchronization errors significantly outperforms OFDM (3 dB or more) especially when such errors are high as encountered in high ICI conditions. Considering that OFDM with frequency-domain adaptive modulation and coding is known for providing high spectral efficiency especially in low mobility conditions and that OTFS provides much superior reliability in high mobility conditions, the unified framework for representing both modulations presented in this work can help pave the path for a flexible and reconfigurable future air interface.

APPENDIX A

PROOF: DELAY-DOPPLER INPUT-OUTPUT RELATION

$$y(k', l') = \frac{1}{\sqrt{NM}} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} Y(n', m') e^{-j2\pi[\frac{n'k'}{N} - \frac{m'l'}{M}]}$$

Using (1) and (11), with some effort it can be simplified as,

$$y(k', l') = \frac{1}{NM} \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s (L - \frac{\tilde{\tau}_p}{T_s})} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} e^{-j2\pi \frac{m \tilde{\tau}_p}{MT_s}} e^{j2\pi \frac{m' l'}{MT_s}} \Psi(m + \tilde{\nu}_p T_s - m', M)$$

$$\sum_{n'=0}^{N-1} e^{j2\pi \frac{n'(\tilde{\nu}_p T_s N(M+L) + k - k')}{N}}. \text{ Substituting } \Psi(m + \tilde{\nu}_p T_s - m', M) = \frac{1}{M} \sum_{k_t=0}^{M-1} e^{-j2\pi \frac{k_t(m + \tilde{\nu}_p T_s - m')}{M}},$$

$$y(k', l') = \frac{1}{NM^2} \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s (L - \frac{\tilde{\tau}_p}{T_s})} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \left(\sum_{n'=0}^{N-1} e^{j2\pi \frac{n'(\tilde{\nu}_p T_s N(M+L) + k - k')}{N}} \right) \quad (42)$$

$$\sum_{k_t=0}^{M-1} e^{-j2\pi k_t(\tilde{\nu}_p T_s)} \sum_{m=0}^{M-1} e^{-j2\pi \frac{m}{M}(k_t - l - \frac{\tilde{\tau}_p}{T_s})} \sum_{m'=0}^{M-1} e^{j2\pi \frac{m'(l' - k_t)}{M}}$$

Since $(l' - k_t) \in \mathbb{Z}$, we substitute $e^{j2\pi \frac{m'(l' - k_t)}{M}} = M\delta[(l' - k_t)_M]$, $\frac{1}{N} \sum_{n'=0}^{N-1} e^{j2\pi \frac{n'(\tilde{\nu}_p T_s N(M+L) + k - k')}{N}} = \Psi(\tilde{\nu}_p T_s N(M+L) + k - k', N)$ and $\frac{1}{M} \sum_{m=0}^{M-1} e^{-j2\pi \frac{m}{M}(k_t - l - \frac{\tilde{\tau}_p}{T_s})} = \Psi(k_t - l - \frac{\tilde{\tau}_p}{T_s}, M)$,

$$y(k', l') = \frac{1}{M} \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s (L - \frac{\tilde{\tau}_p}{T_s})} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \Psi(\tilde{\nu}_p T_s N(M+L) + k - k', N) \sum_{k_t=0}^{M-1} e^{-j2\pi k_t(\tilde{\nu}_p T_s)}$$

$$\Psi(k_t - l - \frac{\tilde{\tau}_p}{T_s}, M) \left(M\delta[(l' - k_t)_M] \right). \text{ Using } \Delta\tau = \frac{1}{B} = T_s, \Delta\nu = \frac{1}{N(M+L)T_s} \text{ and } l' = k_t,$$

Using $\Delta\tau = \frac{1}{B} = T_s$ $\Delta\nu = \frac{1}{N(M+L)T_s}$ and substituting $l' = k_t$,

$$y(k', l') = \sum_{p=1}^P \tilde{h}_p e^{j2\pi \tilde{\nu}_p T_s (L)} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} d(k, l) \Psi\left(\frac{\tilde{\nu}_p}{\Delta\nu} + k - k', N\right) \Psi\left(l + \frac{\tilde{\tau}_p}{\Delta\tau} - l', M\right) e^{-j2\pi(l' - \frac{\tilde{\tau}_p}{\Delta\tau})(\frac{\tilde{\nu}_p}{\Delta\nu N(M+L)})}$$

APPENDIX B

OTFS CHANNEL MATRICES

Delay-Doppler channel matrix $\mathbf{H}_{DD}$ and time domain channel matrix $\mathbf{H}$ are related as [24],

$\mathbf{H}_{DD} = \mathbf{A}^\dagger \mathbf{H} \mathbf{A}$. Using the definition of $\mathbf{A} = \mathbf{W}_N \otimes \mathbf{I}_M$, $\mathbf{H}_{DD}$ can be simplified as a block matrix

with blocks of size $M \times M$ and can be written as, $\mathbf{H}_{DD} = \begin{bmatrix} \mathbf{H}_{dd}^{0,0} & \dots & \mathbf{H}_{dd}^{(N-1),0} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{dd}^{0,(N-1)} & \dots & \mathbf{H}_{dd}^{(N-1),(N-1)} \end{bmatrix},$

whose $(l, k)^{\text{th}}$ block can be given as, $\mathbf{H}_{dd}^{l,k} = \sum_{q=0}^{N-1} \bar{w}_{q,l} w_{q,k} \tilde{\mathbf{H}}_q$, where, $w_{l,k} = \frac{1}{\sqrt{N}} e^{j2\pi \frac{lk}{N}}$.

Finally, using the definition of $\tilde{\mathbf{H}}_q$ in (29), $\mathbf{H}_{dd}^{l,k}$ can be further simplified as,

$$\mathbf{H}_{dd}^{l,k} = \begin{cases} \sum_{p=1}^P \tilde{h}_p \mathbf{\Pi}^{\tilde{l}_p} \mathbf{\Delta}^{\tilde{k}_p} e^{j2\pi \frac{\tilde{k}_p(L-\tilde{l}_p)}{(M+L)N}} \delta([k-l+\tilde{k}_p]_N), & \tilde{k}_p \in \mathbb{Z} \\ \sum_{p=1}^P \tilde{h}_p \mathbf{\Pi}^{\tilde{l}_p} \mathbf{\Delta}^{\tilde{k}_p} e^{j2\pi \frac{\tilde{k}_p(L-\tilde{l}_p)}{(M+L)N}} \Psi(k-l+\tilde{k}_p, N), & \tilde{k}_p \notin \mathbb{Z} \end{cases}$$

We can infer the following from the above equation. When channel Doppler values are resolved in integer Doppler bins, each row of $\mathbf{H}_{DD}$ contains P matrices where $l-k = \tilde{k}_p$, $\forall p = 1, \dots, P$ which results in $NP N_e$ number of non-zero elements in $\mathbf{H}_{DD}$. However, when fractional Doppler is observed, the delay-Doppler channel matrix $\mathbf{H}_{DD}$ contains $N^2 N_e$ number of elements.

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