Exponential stabilization of spin- $\frac{1}{2}$ systems based on switching state feedback

Jie Wen^{a,*}, Yuanhao Shi^a, Jianfang Jia^a, Jianchao Zeng^{b,*}

^a*School of Electrical and Control Engineering, North University of China, Taiyuan 030051, China*

^b*School of Data Science and Technology, North University of China, Taiyuan 030051, China*

Abstract

The exponential stabilization of eigenstates by using switching state feedback strategy for quantum spin- $\frac{1}{2}$ systems is considered in this paper. In order to obtain faster state exponential convergence, we divide the state space into two subspaces, and use two different continuous state feedback controls in the corresponding subspace. The two continuous state feedback controls form the switching state feedback, under which the state convergence is faster than that under continuous state feedback. The exponential convergence and the superiority of switching state feedback are proved in theory and verified in numerical simulations. Besides, the influence of the control parameter on the state convergence rate is also studied.

Keywords: exponential stabilization, spin- $\frac{1}{2}$ systems, state feedback, eigenstate

1. Introduction

In the second quantum technology revolution, the preparation of eigenstates is one of the basic tasks in the quantum technologies, including quantum computing, quantum communication and quantum measurement. For the high-accuracy and rapidity of preparing target eigenstate, many classical control methods, e.g. optimal control [1, 2], sliding mode control [3, 4], H_∞ control [5], and Lyapunov control [6–8], have been applied in quantum systems since the early 1980s.

On the other hand, closed-loop feedback control, which needs the measured information of the controlled systems, is recommended to use for better robustness in quantum systems control. Considering the collapse of quantum states caused by measurement, Belavkin proposed the quantum filter theory [9, 10], whose a modern introduction can be found in [11]. In quantum filter theory, a Ito stochastic differential equation, known as quantum stochastic master equation (SME) [12–14], is used to describe the quantum filter state. Based on SME, the switching control [13], continuous control [15, 16] and non-smooth state feedback control [17] had been proposed to globally stabilize the target eigenstate for quantum spin systems [12], N -level angular momentum systems [13, 15, 16] and the general finite-dimensional stochastic quantum systems [17], respectively. The main contribution of all the control strategies mentioned above is to overcome the convergence obstacle caused by the symmetric structure of state space. In order to obtain faster state convergence, Vu *et al.* proposed the coupling Lyapunov functions approach to design switching control to stabilize target state rapidly by reducing the switching times

*Corresponding authors

Email addresses: wenjie015@gmail.com (Jie Wen), zengjianchao@263.net (Jianchao Zeng)

[18]. Recently, a novel continuous state feedback control was designed by Liang *et al.* to achieve the exponential stabilization of eigenstate for spin- $\frac{1}{2}$ systems [19], based on which the robustness when the initial state and physical parameters are unknown was investigated in [20].

On the basis of the continuous state feedback in [19], we design a switching state feedback control to further improve the state convergence rate for quantum spin- $\frac{1}{2}$ systems in this paper. The state space is divided into two subspaces, and two different continuous state feedback controls are used in the different subspace, respectively. The proposed switching state feedback control consists of the two continuous control, under which the state convergence is faster than that under the control designed in [19]. To our knowledge, the switching state feedback control proposed in this paper is the state feedback control of achieving the fastest stabilization of eigenstates for quantum spin- $\frac{1}{2}$ systems. The main contributions of this paper are as follows: (1) We propose a novel switching state feedback control to exponentially stabilize eigenstates for quantum spin- $\frac{1}{2}$ systems; (2) The exponential convergence and the superiority of switching state feedback control are proved; (3) We study the influence of the control parameter on the state convergence rate.

The rest of this paper is organized as follows. The model of quantum spin- $\frac{1}{2}$ system and the exponential stabilization problem are described in Section 2. We design the switching state feedback control, and prove the exponential convergence in Section 3. In Section 4, numerical simulations are performed to verify the effectiveness and compare switching state feedback with continuous state feedback. Finally, Section 5 concludes this paper and discusses possible future directions.

Notation. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively. $A^\dagger$ is the conjugate transpose of the matrix A . $\text{tr}(A)$ denotes the trace of matrix A . I_n is the $n \times n$ identity matrix. $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ denote the Pauli matrices, where $i = \sqrt{-1}$. $[A, B] = AB - BA$ denotes the commutator of matrices A and B . $\mathcal{L}$ denotes the infinitesimal generator. $\mathbb{E}[X]$ denotes the expected value of a random variable X .

2. Problem formulation

Under continuous measurement, two-level quantum systems can be described by the following Ito stochastic differential equation, i.e., SME [12, 13]

$$d\rho_t = \left(-i[H_0, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) - i[H_c, \rho_t]u_t \right) dt + \sqrt{\eta}(L\rho_t + \rho_t L^\dagger - \text{tr}(\rho_t(L + L^\dagger))\rho_t) dW_t \quad (1)$$

where, ρ_t is the quantum state and evolves in the state space S , i.e.

$$\rho_t \in S := \{\rho \in \mathbb{C}^{2 \times 2} : \rho = \rho^\dagger \geq 0, \text{tr}(\rho) = 1, \text{tr}(\rho^2) \leq 1\}$$

L is the measurement operator, which is a Hermitian matrix and determines the way of the system interacts with the measurement apparatus; $u_t \in \mathbb{R}$ is the control input; H_0 and H_c are also both Hermitian matrices and called free Hamiltonian and control Hamiltonian, respectively; W_t is a standard Wiener process defined on the classical complete probability space; $\eta \in (0, 1]$ is the measurement efficiency. In particular, the existence and uniqueness of the solution of SME (1) have been presented in [13, 21].

In this paper, we consider the exponential stabilization of eigenstate, i.e., the system state can converge to the target eigenstate from arbitrary initial state exponentially. To ensure the solvability of this problem, we assume that L is self-adjoint and regular, i.e., $L = L^\dagger$ and the eigenvalues of L are different, thereby the equilibria of SME (1) are the all eigenstates of the measurement operator L [17], and SME (1) becomes

$$d\rho_t = \left(-i[H_0, \rho_t] + L\rho_t L - \frac{1}{2}(L^2\rho_t + \rho_t L^2) - i[H_c, \rho_t]u_t \right) dt + \sqrt{\eta}(L\rho_t + \rho_t L - 2\text{tr}(\rho_t L)\rho_t)dW_t \quad (2)$$

Let $L = \frac{\sqrt{M}}{2}\sigma_z$ in which $M > 0$ is the strength of the interaction between the light and the atoms, $H_0 = \frac{\omega_{eg}}{2}\sigma_z$ in which ω_{eg} is the difference between the energies of the excited state and the ground state, and $H_c = \sigma_y$, (2) can be written as

$$d\rho_t = \left(-i\frac{\omega_{eg}}{2}[\sigma_z, \rho_t] + \frac{M}{4}(\sigma_z\rho_t\sigma_z - \rho_t) - i\frac{u_t}{2}[\sigma_y, \rho_t] \right) dt + \frac{\sqrt{\eta M}}{2}(\sigma_z\rho_t + \rho_t\sigma_z - 2\text{tr}(\rho_t\sigma_z)\rho_t)dW_t \quad (3)$$

which is the SME describing the dynamics of a quantum spin- $\frac{1}{2}$ system [19] and also the controlled quantum system considered in this paper.

However, when $u_t = 0$, the system state converges to an arbitrary equilibrium $\bar{\rho}$ with the probability $\text{tr}(\rho_0\bar{\rho})$ from the initial state ρ_0 [19]. Thus, the suitable controllers need to be applied to make sure that the system state converges to the target eigenstate instead of arbitrary eigenstate. For this goal, static output feedback [22–24], state feedback [12–20, 25–28] and noise-assisted feedback [29, 30] have been proposed. In particular, the continuous state feedback in [19] achieve the exponential stabilization of eigenstate for the quantum spin- $\frac{1}{2}$ system.

In order to further improve the state convergence rate, we propose the switching state feedback strategy in this paper. Therefore, the goal of this paper can be described as follows.

Exponential stabilization by switching state feedback. *For the quantum spin- $\frac{1}{2}$ system (3), design the switching state feedback control $u_t = u_t^{sf}(\rho_t)$ to exponentially render the quantum spin- $\frac{1}{2}$ system (3) from any initial state ρ_0 in the state space S to the desired target eigenstate ρ_f , and make sure that the state convergence is faster than that of using continuous state feedback in [19].*

The next main work is to design the switching state feedback control $u_t^{sf}(\rho_t)$ and prove the exponential convergence, which are the main contribution of this paper and will be presented in Section 3.

3. Design of switching state feedback

For the quantum spin- $\frac{1}{2}$ system, there are two eigenstates, i.e., $\{\rho_g, \rho_e\}$, where $\rho_g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\rho_e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The target eigenstate ρ_f is set as ρ_g in this section, the exponential stabilization of ρ_e follows the same lines.

The continuous state feedback control in [19] is design as

$$u_t^{sf1}(\rho_t) = \alpha V^\beta(\rho_t) - \gamma \text{tr}(i[\sigma_y, \rho_t]\rho_f) \quad (4)$$

where, $\gamma \geq 0, \beta \geq 1$ and $0 < \alpha < \frac{\eta M \lambda^2}{(1-\lambda)^{\frac{\beta-1}{2}}}$ with $\lambda \in (0, 1)$. Based on (4), we design another continuous state feedback control as

$$u_t^{sf2}(\rho_t) = -\alpha \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) V^\beta(\rho_t) - \gamma \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) \quad (5)$$

i.e., replacing α with $-\alpha \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f)$.

Considering that the state convergence rate under the state feedback (4) and (5) is different, we propose the switching state feedback strategy by combining (4) and (5). The idea of switching state feedback strategy is that the state space is divided into two subspaces, which is denoted as $\{\Phi_1, \Phi_2\}$, and the state convergence under (4) is faster than that under (5) when the state is located in Φ_1 , while the state convergence under (5) is faster when the state is located in Φ_2 , so that the state convergence under switching state feedback control is faster than that under continuous state feedback controls (4) and (5) for the entire state space.

Based on the idea of switching state feedback strategy, the main result of this paper is given in Theorem 1.

Theorem 1. *For the quantum spin- $\frac{1}{2}$ system (3) under the switching state feedback control*

$$u_t^{sf}(\rho_t) = K(\rho_t) \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) V^\beta(\rho_t) - \gamma \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) \quad (6)$$

where,

$$K(\rho_t) = \begin{cases} -\alpha \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) V^\beta(\rho_t), & \text{if } \rho_t \in \Phi_1 \\ \alpha, & \text{if } \rho_t \in \Phi_2 \end{cases} \quad (7)$$

with

$$\Phi_1 := \{\rho : \operatorname{tr}^2(i[\sigma_y, \rho_t] \rho_f) + \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) > \epsilon\} \quad (8)$$

in which $\epsilon \geq 0$, and $\Phi_2 = S \setminus \Phi_1$, the Lyapunov function $V(\rho_t) = \sqrt{1 - \operatorname{tr}(\rho_t \rho_f)}$ satisfies $\mathbb{E}[V(\rho_t)] < e^{-rt} V(\rho_0)$ for all $t \geq 0$, and the target eigenstate ρ_f is globally exponentially stable with the convergence rate r that is not more than $\frac{\eta M}{2} \lambda^2 - \frac{\alpha}{2} (1 - \lambda)^{\frac{\beta-1}{2}}$. In particular, the state convergence under the switching feedback control (6) is faster than that under the state feedback controls (4) and (5) for the entire state space S .

PROOF. When $K(\rho_t) = -\alpha \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) V^\beta(\rho_t)$, the infinitesimal generator $\mathcal{L}$ associated with quantum spin- $\frac{1}{2}$ system (3) acts on V is given by

$$\begin{aligned} \mathcal{L}V(\rho_t) &= \frac{-\alpha \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) V^\beta(\rho_t) - \gamma \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f) \operatorname{tr}(i[\sigma_y, \rho_t] \rho_f)}{4} \frac{1}{V(\rho_t)} \\ &\quad - \frac{\eta M}{2} \operatorname{tr}^2(\rho_t \rho_f) V(\rho_t) \\ &= -\frac{\alpha}{4} \operatorname{tr}^2(i[\sigma_y, \rho_t] \rho_f) V^{\beta-1}(\rho_t) - \frac{\gamma \operatorname{tr}^2(i[\sigma_y, \rho_t] \rho_f)}{4V(\rho_t)} - \frac{\eta M}{2} \operatorname{tr}^2(\rho_t \rho_f) V(\rho_t) \\ &= -\frac{1}{2} \left(\eta M \operatorname{tr}^2(\rho_t \rho_f) + \frac{\gamma \operatorname{tr}^2(i[\sigma_y, \rho_t] \rho_f)}{2V^2(\rho_t)} + \frac{\alpha}{2} \operatorname{tr}^2(i[\sigma_y, \rho_t] \rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t) \end{aligned} \quad (9)$$

based on that the infinitesimal generator $\mathcal{L}$ associated with $d\rho_t = F(\rho_t)dt + G(\rho_t)dW_t$ acts on $V(\rho_t)$ in the following way,

$$\mathcal{L}V(\rho_t) = \frac{\partial V(\rho_t)}{\partial \rho_t} F(\rho_t) + \frac{1}{2} \frac{\partial^2 V(\rho_t)}{\partial \rho_t^2} (G(\rho_t), G(\rho_t)), \quad (10)$$

When $K(\rho_t) = \alpha$, the infinitesimal generator $\mathcal{L}$ associated with quantum spin- $\frac{1}{2}$ system (3) acts on V is given by

$$\begin{aligned}\mathcal{L}V(\rho_t) &= \frac{\alpha V^\beta(\rho_t) - \gamma \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f)}{4} - \frac{\eta M}{2} \operatorname{tr}^2(\rho_t \rho_f) V(\rho_t) \\ &= \frac{\alpha}{4} \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) V^{\beta-1}(\rho_t) - \frac{\gamma \operatorname{tr}^2(i[\sigma_y, \rho_t]\rho_f)}{4V(\rho_t)} - \frac{\eta M}{2} \operatorname{tr}^2(\rho_t \rho_f) V(\rho_t) \\ &= -\frac{1}{2} \left(\eta M \operatorname{tr}^2(\rho_t \rho_f) + \frac{\gamma \operatorname{tr}^2(i[\sigma_y, \rho_t]\rho_f)}{2V^2(\rho_t)} - \frac{\alpha}{2} \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t)\end{aligned}\quad (11)$$

from (10).

Compare (9) and (11), if $\alpha \operatorname{tr}^2(i[\sigma_y, \rho_t]\rho_f) > -\alpha \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f)$, i.e. $\rho_t \in \Phi_1$, the state convergence is faster when $K(\rho_t) = -\alpha \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) V^\beta(\rho_t)$ than that when $K(\rho_t) = \alpha$. Thus, the state convergence is faster under the switching state feedback control (6) than that under continuous state feedback control (4) when $\rho_t \in \Phi_1$, while the convergence is the same when $\rho_t \in \Phi_2$, such that the state convergence under the switching state feedback control (6) is faster for the entire state space. The comparison of switching state feedback control (6) and continuous state feedback control (5) is similar, so we ignore it here.

Due to $\operatorname{tr}(\rho_t \rho_f) > \lambda$ as described in [19], (9) becomes

$$\begin{aligned}\mathcal{L}V(\rho_t) &\leq -\frac{1}{2} \left(\eta M \operatorname{tr}^2(\rho_t \rho_f) + \frac{\alpha}{2} \operatorname{tr}^2(i[\sigma_y, \rho_t]\rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t) \\ &\leq -\frac{\eta M}{2} \operatorname{tr}^2(\rho_t \rho_f) V(\rho_t) \\ &\leq -\frac{\eta M}{2} \lambda^2 V(\rho_t)\end{aligned}\quad (12)$$

Considering that $V(\rho_t) = 0$ if and only if $\rho_t = \rho_f$, one can obtained from (16) that

$$\mathbb{E}[V(\rho_t)] \leq V(\rho_0) e^{-\frac{\eta M}{2} \lambda^2 t} \leq V(\rho_0) e^{-rt} \quad (13)$$

based on Theorem 3 in [29], which means that ρ_f is globally exponentially stable, and the convergence rate r is not more than $\frac{\eta M}{2} \lambda^2$.

On the other hand, for the two-level quantum systems, ρ_t can also be characterized by the Bloch sphere coordinates (x_t, y_t, z_t) as

$$\rho_t = \frac{I_2 + x_t \sigma_x + y_t \sigma_y + z_t \sigma_z}{2} = \frac{1}{2} \begin{bmatrix} 1 + z_t & x_t - iy_t \\ x_t + iy_t & 1 - z_t \end{bmatrix} \quad (14)$$

where, $(x_t, y_t, z_t) \in B(\mathbb{R}^3) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. According to (14),

$$\operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) = \operatorname{tr}(\sigma_x \rho_t) = x_t \leq 2V(\rho_t) \quad (15)$$

which makes (11) become

$$\begin{aligned}\mathcal{L}V(\rho_t) &\leq -\frac{1}{2} \left(\eta M \operatorname{tr}^2(\rho_t \rho_f) - \frac{\alpha}{2} \operatorname{tr}(i[\sigma_y, \rho_t]\rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t) \\ &\leq -\frac{1}{2} (\eta M \operatorname{tr}^2(\rho_t \rho_f) - \alpha V^{\beta-1}(\rho_t)) V(\rho_t) \\ &\leq -\frac{1}{2} \left(\eta M \lambda^2 - \alpha (1 - \lambda)^{\frac{\beta-1}{2}} \right) V(\rho_t)\end{aligned}\quad (16)$$

Considering that $V(\rho_t) = 0$ if and only if $\rho_t = \rho_f$, one can obtained from (16) that

$$\mathbb{E}[V(\rho_t)] \leq V(\rho_0)e^{-\left(\frac{\eta M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}\right)t} \leq V(\rho_0)e^{-rt} \quad (17)$$

based on Theorem 3 in [29], which means that ρ_f is globally exponentially stable, and the convergence rate r is not more than $\frac{\eta M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$.

Under the switching state feedback control 6, the convergence rate r is not more than $\frac{\eta M}{2}\lambda^2$ from (12) when $\rho_t \in \Phi_1$, while the convergence rate r is not more than $\frac{\eta M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$ from (17) when $\rho_t \in \Phi_2$, so that the convergence rate r is not more than $\frac{\eta M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$ for the entire state space S due to $\frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}} > 0$. The proof is complete.

Theorem 2. For the quantum spin- $\frac{1}{2}$ system (3) under the continuous state feedback control (5), the Lyapunov function $V(\rho_t) = \sqrt{1 - \text{tr}(\rho_t \rho_f)}$ satisfies $\mathbb{E}[V(\rho_t)] < e^{-rt}V(\rho_0)$ for all $t \geq 0$, and the target eigenstate ρ_f is globally exponentially stable with the convergence rate r that is not more than $\frac{\eta M}{2}\lambda^2$.

PROOF. According to (12) and Theorem 3 in [29], the conclusion is easy to obtain.

Remark 1. Although the minimum exponential convergence rate $\frac{\eta M}{2}\lambda^2$ under continuous state feedback control (5) is larger than that $\frac{\eta M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$ under continuous state feedback control (4) and switching state feedback control (6) from Theorem 4.3 in [19], Theorem 1 and Theorem 2, the comparison conclusion of real state convergence rate is not consistent with that of the minimum convergence rate, i.e. the state convergence under (4) and (6) is not necessarily slower than that under (5). Actually, the state convergence under (6) is faster than under (5), as described in Theorem 1.

4. Numerical simulations

In this section, we use numerical simulations to verify the effectiveness of switching state feedback control, and compare switching state feedback control with the two continuous state feedback controls to present the superiority of switching state feedback control.

4.1. Exponential stabilization of ρ_e by switching state feedback

In this subsection, the eigenstate ρ_e is selected as the target state, while the initial state is set as $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which corresponds to the point $(1, 0, 0)$ in the Bloch sphere. Let the physical parameters $\omega_{eg} = 0, \eta = 0.5, M = 1$, and the control parameters $\alpha = 0.5, \beta = 8, \gamma = 5, \epsilon = 0.3$ in switching state feedback control (6). According to (8), we can obtain from (14) that the Φ_1 in the Bloch sphere is the points set of satisfying the conditions $x^2 + x > 0.3$ and $-1 \leq x \leq 1$, while $\Phi_2 = \{x : x^2 + x \leq 0.3, -1 \leq x \leq 1\}$. It is obvious that $\rho_0 \in \Phi_1$ and $\rho_f \in \Phi_2$. The results of 10 experiments with $\lambda = 0.9$ are shown in Fig. 1, from which one can see that the quantum state ρ_t converges to the target eigenstate ρ_e from $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which means the switching feedback control (6) is available.

Moreover, we also study the influence of the control parameter ϵ on the convergence rate. Let $\epsilon = 0.3, 0.5$ and 0.7 , respectively, the results of 10 experiments are shown in

Figure 1: Exponential stabilization of eigenstate ρ_e for the quantum spin- $\frac{1}{2}$ system (3) by using the switching feedback control (6) starting at $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $\alpha = 0.5$, $\beta = 8$, $\gamma = 5$ and $\epsilon = 0.3$. (a) The curves of $V(\rho_t)$ in 10 arbitrary sample trajectories, and the black curve represents the mean value of 10 arbitrary sample trajectories, while the red curve represents the exponential references with exponent $\frac{\eta M}{2} \lambda^2 - \frac{\alpha}{2} (1 - \lambda)^{\frac{\beta-1}{2}}$; (b) The semi-log version of (a); (c) The curve of $V(\rho_t)$ under the switching feedback control (6) in a sample trajectory; (d) The curve of the switching feedback control $u(\rho_t)$ in a sample trajectory. The pink background presents the time domain of $\rho_t \in \Phi_1$, while the white background represents the time domain of $\rho_t \in \Phi_2$.

Figure 2: Exponential stabilization of eigenstate ρ_e for the quantum spin- $\frac{1}{2}$ system (3) by using the switching feedback control (6) starting at $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $\alpha = 0.5, \beta = 8, \gamma = 5$ and $\epsilon = 0.3, 0.5, 0.7$, respectively. (a) The black, blue and purple curves represent the mean value of 10 arbitrary sample trajectories for $\alpha = 0.3, 0.5, 0.7$, respectively, while the red curve represents the exponential references with exponent $\frac{\eta M}{2} \lambda^2 - \frac{\alpha}{2} (1 - \lambda)^{\frac{\beta-1}{2}}$; (b) The semi-log version of (a).

Fig. 2, from which one can see that the smaller the α , the faster the state convergence, which means the smaller ϵ helps to improve the state convergence rate, so the smaller ϵ is recommended to be used in switching state feedback control when $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

4.2. Comparison of the two continuous state feedback controls

In this subsection, we compare the two continuous state feedback controls (4) and (5) via numerical simulations, in which the physical parameters ω_{eg}, η, M and the control parameters α, β, γ as well as the initial state ρ_0 and target eigenstate ρ_f are the same as that in Fig. 1 of subsection 4.1. The results of 10 experiments under (4) and (5) are shown in Fig. 3, from which one can see that both of (4) and (5) can achieve the exponential stabilization of ρ_e , and the state convergence is faster under continuous state feedback control (5) in most of the time domain $[0, 8]$ and $[16, 20]$, while the state convergence is faster under continuous state feedback control (4) in most of the time domain $[8, 16]$. Thus, the two continuous state feedback controls (4) and (5) have the advantage of state convergence rate in different time domains for the entire control time domain. In order to give full play to their advantages in different time domains, which is the starting point of this paper, we propose the switching state feedback strategy to further improve the state convergence rate.

4.3. Comparison of switching state feedback and continuous state feedback

In this subsection, we compare the switching state feedback control (6) with the two continuous state feedback controls (4) and (5). Put the results of 10 experiments by using (6), (4) and (5) respectively in one figure, we can obtain the Fig. 4. One can see from Fig. 4 that the state convergence is faster under the switching state feedback control (6) than that under the two continuous state feedback controls (4) and (5), which is consistent with the theoretical results as described in Theorem 1, and presents the superiority of switching state feedback control (6).

5. Conclusions

In this paper, we propose the switching state feedback to exponentially stabilize the target eigenstate for the quantum spin- $\frac{1}{2}$ system. The exponential convergence under

Figure 3: Exponential stabilization of eigenstate ρ_e for the quantum spin- $\frac{1}{2}$ system (3) by using the two continuous state feedback controls (4) and (5) starting at $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $\alpha = 0.5$, $\beta = 8$ and $\gamma = 5$, respectively. (a) The green and yellow curves represent the mean value of 10 arbitrary sample trajectories under (4) and (5), respectively, while the red curve represents the exponential references with exponent $\frac{\eta_M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$; (b) The semi-log version of (a).

Figure 4: Exponential stabilization of eigenstate ρ_e for the quantum spin- $\frac{1}{2}$ system (3) by using switching state feedback control (6) and continuous state feedback controls (4) and (5) starting at $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $\alpha = 0.5$, $\beta = 8$, $\gamma = 5$ and $\epsilon = 0.3$, respectively. (a) The black, green and yellow curves represent the mean value of 10 arbitrary sample trajectories under switching state feedback control (6) and continuous state feedback controls (4) and (5), respectively, while the red curve represents the exponential references with exponent $\frac{\eta_M}{2}\lambda^2 - \frac{\alpha}{2}(1-\lambda)^{\frac{\beta-1}{2}}$; (b) The semi-log version of (a).

switching state feedback control and the superiority relative to continuous state feedback controls is proved, and also verified in numerical simulations. We also study the influence of the control parameter on the state convergence rate, and give the suggestion of using switching state feedback control. The further works can be considered as follows: (1) The improvement and optimization of the switching state feedback strategies, include the design of control parameters and the partition of state space; (2) Extension of the controlled quantum systems, i.e., considering the exponential stabilization of N -level quantum systems or multi-qubit quantum systems; (3) Exponential stabilization of time-delay stochastic quantum systems, specially the case of time-delay control, include constant time-delay and time-varying time-delay.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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