Exponential tracking of general references and rejection of general disturbances for nonlinear control systems with applications to the blood glucose regulation system

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Abstract

In solving the problem of exponential tracking and disturbance rejection, it has been long always assumed that the reference to be tracked and the disturbance to be rejected are generated by an exosystem such as a finite dimensional system with pure imaginary eigenvalues. The aim of this note is to show that this assumption can be removed. For any nonlinear control system subject to a general disturbance, it can be split into a linear exponentially-stable system and a dynamical regulator system. If the dynamical regulator system has a solution, then there exists a feedback and feedforward controller such that an output of the control system exponentially tracks a desired general reference. The result is applied to the blood glucose regulation system.

Keywords: Nonlinear control system, exponential tracking, disturbance rejection, general reference and disturbance, dynamical regulator system, feedback and feedforward control, blood glucose regulation.

In solving the problem of exponential tracking and disturbance rejection, it has been long always assumed that the reference to be tracked and the disturbance to be rejected are generated by an exosystem such as a finite dimensional system with pure imaginary eigenvalues (see, e.g., [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]). In fact, this assumption is sufficient, but not necessary for making the problem solvable.

The aim of this note is to show that this assumption can be removed. Consider the nonlinear control

system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}_d), \quad (1)$$

$$\mathbf{y} = \mathbf{f}_o(\mathbf{x}, \mathbf{u}, \mathbf{v}_d), \quad (2)$$

$$\mathbf{e} = \mathbf{y} - \mathbf{r}, \quad (3)$$

where $\mathbf{x}$ is a state vector in $\mathbb{R}^n$, $\mathbf{u}$ is a control vector, $\mathbf{y}$ is an output vector, $\mathbf{v}_d(t)$ is a disturbance vector, $\mathbf{r}(t)$ is a reference vector, and $\mathbf{f}$ and $\mathbf{f}_o$ are given vector functions. All functions in this note are assumed to have the required continuity and differentiability. The problem of asymptotic tracking and disturbance rejection is to design a controller $\mathbf{u}$ such that

$$\lim_{t \rightarrow \infty} \mathbf{e}(t) = \lim_{t \rightarrow \infty} [\mathbf{y}(t) - \mathbf{r}(t)] = 0. \quad (4)$$

Furthermore, if there exist positive constants C and a such that

$$\|\mathbf{e}(t)\| \leq Ce^{-at} \quad (5)$$

for all $t \geq 0$, it is called exponential tracking.

To solve this tracking problem, we introduce the variable transform

$$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{X}. \quad (6)$$

Substituting this transform into (1)-(3) gives

$$\frac{d\hat{\mathbf{x}}}{dt} + \frac{d\mathbf{X}}{dt} = \mathbf{f}(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d), \quad (7)$$

$$\mathbf{e} = \mathbf{f}_o(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d) - \mathbf{r}. \quad (8)$$

This system can be split into a linear exponentially-stable system

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{A}\hat{\mathbf{x}}, \quad (9)$$

$$\mathbf{e} = \mathbf{C}\hat{\mathbf{x}}, \quad (10)$$

and a dynamical regulator system

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d) - \mathbf{A}\hat{\mathbf{x}}, \quad (11)$$

$$0 = \mathbf{f}_o(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d) - \mathbf{C}\hat{\mathbf{x}} - \mathbf{r}, \quad (12)$$

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where $\mathbf{A}$ is a constant matrix whose eigenvalues has negative real parts, and $\mathbf{C}$ is a constant matrix.

Theorem 1. *Assume that $\mathbf{A}$ is a constant matrix whose eigenvalues has negative real parts and $\mathbf{C}$ is a constant matrix. If the dynamical regulator system (11) and (12) has a solution, then there exists a feedback and feedforward controller*

$$\mathbf{u} = \mathbf{u}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d), \quad (13)$$

such that the tracking error $\mathbf{e}(t)$ satisfies the exponential tracking estimate (5).

Proof. Let

$$\mathbf{X} = \mathbf{X}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d), \quad \mathbf{u} = \mathbf{u}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d)$$

be a solution of the dynamical regulator system (11) and (12). Then, under the controller (13), we have

$$\begin{aligned} \mathbf{e}(t) &= \mathbf{f}_o(\mathbf{x}, \mathbf{u}, \mathbf{v}_d) - \mathbf{r} \\ &= \mathbf{C}\hat{\mathbf{x}} + \mathbf{f}_o(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d) - \mathbf{C}\hat{\mathbf{x}} - \mathbf{r} \\ &= \mathbf{C}\hat{\mathbf{x}}. \end{aligned}$$

Under the assumption on the matrix $\mathbf{A}$, the system (9) is exponentially stable. So $\hat{\mathbf{x}}(t)$ converges to 0 exponentially as $t \rightarrow \infty$, and then the tracking error $\mathbf{e}(t)$ satisfies the exponential tracking estimate (5). This completes the proof.

It looks like there are no conditions on the matrix $\mathbf{C}$. In fact, the conditions on $\mathbf{C}$ is implied in the dynamical regulator system (11) and (12). The existence of a solution of the system depends on the choice of $\mathbf{C}$. For example, if $\mathbf{C} = 0$, in general, the dynamical regulator system may have no solutions. Thus $\mathbf{C}$ should be chosen such that the system has a solution.

Note that the disturbance $\mathbf{v}_d$ is directly present in the static controller $\mathbf{u}$. Since the disturbance is unknown in reality, this is not reasonable. However, it seems that this is standard or cannot be avoided for such a static controller. For example, the disturbance is directly present in the static state feedback for an output regulation problem in Section 3.2 of [8] and in feedback control law for PDEs regulation in Problem 1.1 and Theorem 1.1 of [2]. To design a robust controller without the direct presentence of the disturbance, we have to introduce a dynamical compensator (see, e.g., [8, 13]). But this is not the topic of this short note.

Compared with the usual static regulator partial differential equations (see, e.g., [8]), the dynamical regulator ordinary differential system (11) - (12) looks simpler, but still difficult to be solved. However, it

can be solved for some important mathematical models such as the model of the blood glucose regulation system proposed by Bergman *et al.* [3]:

$$\frac{dg}{dt} = -m_1g - h_ag + J, \quad (14)$$

$$\frac{dh_a}{dt} = -m_2h_a + m_3h, \quad (15)$$

$$\frac{dh}{dt} = -m_4h + u, \quad (16)$$

$$e = g - r. \quad (17)$$

In the above equations, g and h denote concentrations of blood glucose and plasma insulin, respectively, h_a is the effect of the remote insulin on glucose, J is a rate of the exogenous glucose input from the intestine, u is a rate of insulin secreted from the endocrine system or infused externally, r is a glucose reference, and m_1, m_2, m_3, m_4 are positive rate constants. Let the exponentially-stable system be given by the simple linear system

$$\frac{d\hat{g}}{dt} = -\hat{g}, \quad (18)$$

$$\frac{d\hat{h}_a}{dt} = -\hat{h}_a, \quad (19)$$

$$\frac{d\hat{h}}{dt} = -\hat{h}, \quad (20)$$

$$e = \hat{h}. \quad (21)$$

Then the dynamical regulator system for the blood glucose regulation system is

$$\begin{aligned} \frac{dG}{dt} &= -m_1(\hat{g} + G) - (\hat{h}_a + H_a)(\hat{g} + G) \\ &\quad + \hat{g} + J, \end{aligned} \quad (22)$$

$$\frac{dH_a}{dt} = -m_2(\hat{h}_a + H_a) + m_3(\hat{h} + H) + \hat{h}_a, \quad (23)$$

$$\frac{dH}{dt} = -m_4(\hat{h} + H) + \hat{h} + u, \quad (24)$$

$$0 = \hat{g} + G - r - \hat{h}. \quad (25)$$

The solution of the system is given by

$$G = r + \hat{h} - \hat{g}, \quad (26)$$

$$H_a = \frac{J + \hat{g} - (m_1 + \hat{h}_a)(\hat{g} + G) - G'}{\hat{g} + G}, \quad (27)$$

$$H = \frac{H'_a + m_2(\hat{h}_a + H_a) - m_3\hat{h} - \hat{h}_a}{m_3}, \quad (28)$$

$$u = H' + m_4(\hat{h} + H) - \hat{h}. \quad (29)$$

Thus, we have designed a feedback and feedforward controller (29) under which the blood glucose concentration g exponentially tracks the reference $r(t)$.

Figure 1: The blood glucose concentration g exponentially tracks the reference $r(t) = 90 + 10 \cos(t)$ under the feedback and feedforward controller (29).

We conduct a numerical computation to test the controller (29). In the computation, we take $m_1 = 0.0014$ /min, $m_2 = 0.0059$ /min, $m_3 = 0.012$ /min, $m_4 = 0.00023$ /min, $J = \sin(t)$, $r = 90 + 10 \cos(t)$, and $g(0) = 180$ mg/dl. The figure 1 shows that the blood glucose concentration g exponentially tracks the reference $r(t) = 90 + 10 \cos(t)$ under the feedback and feedforward controller (29).

It is important to notice that the above idea can be directly applied to partial differential equations such as the convection diffusion equation, the wave equation, and the Burgers' equation.

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