Improved Weighting in Particle Filters Applied to Precise State Estimation

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Abstract—This letter presents a methodology to reduce the computational burden of Particle Filters (PF) used to achieve a target state estimation accuracy in Bayesian tracking problems. A strategy, named Multiple Weighting (MW), that exploits information diversity of the input measurements for an efficient weighting of the particles is introduced. This study shows that, by relying on the prior knowledge of the state-observations relationships, it is possible to achieve a significant reduction in the number of samples required to obtain any target accuracy. An experimental assessment is provided with an application to precise positioning estimation in Global Navigation Satellite System (GNSS) receivers.

Index Terms—Bayesian estimation, global navigation satellite system, particle filter, positioning and navigation

I. INTRODUCTION

BAYESIAN state estimation methods are nowadays popular in many signal processing applications, and their effectiveness has been assessed in several fields such as tracking, localisation and navigation [1], [2]. They are used in discrete-time Hidden Markov Models (HMM) for object tracking and state estimation, leveraging on a set of observable measurements, e.g. range, bearing or heading [1], [3]. To this aim, a variety of algorithms belonging to the classes of Kalman Filter (KF) and Sequential Monte Carlo (SMC) like the Particle Filter (PF) are employed. KF estimation is widely used due to its lower computational load w.r.t. the other techniques. However, in many complex scenarios, KF solutions are sub-optimal when the errors on the measurements cannot be modelled by Normal statistics. Furthermore, when dealing with non-linear system models by using Extended Kalman Filter (EKF) estimators, performance is limited by the approximations caused by the linearization of the problem. Unlike KF-based solutions, PF can deal with any given error density as well as with non-linear system models, easing the implementation of a variety of hybridisation schemes that combine heterogeneous measurements and that can be exploited for sensor fusion and collaborative localisation [4], [5], [6], [7]. In these applications, it has been shown that PF is able to provide improved performance, but at the cost of a non-negligible computational complexity, especially for high cardinalities of the state spaces. Indeed, the number of particles needed to accurately represent the a-posteriori densities exponentially increases with the dimension of the state space [8]. Many variants of the PF have been proposed in the literature to deal with its computational requirements and main theoretical

limitations [1]. For instance, if the state model contains a linear sub-structure subject to Gaussian noise, Rao-Blackwellised PF (also denoted in the literature as Marginalised PF) allows to solve any linear sub-structure through KF, thus reducing the number of dimensions sampled by the PF [9], [10]. PF exploiting adaptive number of particles were also proposed to overcome the complexity issue [11]. Several works aimed at mitigating particles degeneracy using hybrid filtering schemes such as Unscented Particle Filter (UPF) and Auxiliary Particle Filter (APF) [12], [13]. Other contributions targeted multi-modal densities [14] or Interactive Multiple Model (IMM) to jointly exploit different state-transition and noise density models [15]. This letter presents a solution based on Sequential Importance Resampling (SIR) PF, through these contributions:

- A strategy to optimise the use of particles which leverages the information carried by different subsets of input measurements.
- An experimental assessment using real Global Navigation Satellite System (GNSS) measurements to demonstrate the accuracy improvement provided by the proposed approach in real applications.

II. PROBLEM FORMULATION

A generalised Bayesian state estimation problem can be modeled by a HMM through the following items:

- θ_k , being a stochastic discrete-time *state space vector*.
- $\theta_k = \mathbf{f}(\theta_{k-1}, \mathbf{w}_{k-1})$, modelling a discrete-time *state space transition* which accounts for process noise $\mathbf{w}_{k-1}$.
- $\mathbf{z}_k = [z_{1,k} \ \dots \ z_{M,k}]$, being a vector of M synchronous *input measurements*, also referred to *observables*.
- $\mathbf{z}_k = \mathbf{h}(\theta_k, \mathbf{v}_k)$, is referred to as *observables-state function* and represents the relationship between observations and state space also accounting for observations noise $\mathbf{v}_k$.

Bayesian filtering leverages the a-priori knowledge of $\mathbf{f}(\theta_{k-1}, \mathbf{w}_{k-1})$ to estimate the state space $\hat{\theta}$ which maximises the a-posteriori probability of the observations $\mathbf{z}_k$ [16]. The estimation is hence performed through a prediction-update approach that effectively compensates for noisy observations. Differently from other classes of Bayesian estimators, PF does not impose constraints on the items of the Bayesian estimation problem [2].

A. Bayesian State Estimation in SIR PF algorithms

A block scheme showing a SIR PF routine is shown in Figure 1. The filter is initialised by sampling a set of

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Fig. 1. Block diagram of a SIR PF architecture implementing Bayesian state space estimation. The block devoted to weight computation is highlighted as it is the main target of this study.

N particles from a prior distribution and assigning initial importance weights [17]. Each i -th particle θ_k^i is a possible realisation of the state space vector, θ_k . At every iteration of the algorithm, prediction is performed by sampling according to a proposal distribution $q = p(\theta_k | \theta_{k-1})$. This step generalises the prediction of non-SMC estimators, i.e. KFs, which operate on a single prediction of the state space. Then, the vector of nominal measurements $\mathbf{z}^i$ is computed for each particle. Weights are computed for each particle by relying on pre-defined probability densities, $p(z_{m,k} | \theta_k^i)$, w.r.t. the input measurements. Because of our choice of proposal density q as the state space transition density, the unnormalised weights can be computed as

$$\tilde{w}_k^i = w_{k-1}^i \mathcal{L}(\mathbf{z}_k | \theta_k^i) = w_{k-1}^i \prod_{m=1}^M p(\bar{z}_{m,k}) \quad (1)$$

where $\bar{z}_{m,k} = z_{m,k} - z_{m,k}^i$. Weights are then normalised

$$w_k^i = \frac{\tilde{w}_k^i}{\sum_{i=1}^N \tilde{w}_k^i} \quad (2)$$

so that they sum up to one. The likelihood $\mathcal{L}$ represents the probability that each particle could have observed the set of input measurements. After the filter runs for a few iterations, the variance of particles will increase and many weights will take small values, thus not contributing to the final solution. This is known as the *degeneracy problem* [2], that can be prevented by resampling. This stage outputs a new set of N particles starting from the original set, where each particle has a probability of being generated equal to its weight. This means that particles with higher weights are more likely to appear in the new set and could be chosen multiple times. For this reason, the resampling step introduces a new problem referred to as *sampling impoverishment*. The simplest strategy, which we consider for the remainder of this discussion, is to perform resampling at each iteration. The weights of the resampled particles are all set to $w^i = \frac{1}{N}$, so they can be simplified from (1). Due to the computational load of this stage, many efficient resampling strategies have been proposed and analysed in the literature [18], [19], but a more detailed discussion of the resampling stage is out of the scope of this letter. The output estimate can be obtained as a weighted sum of the particles

$$\hat{\theta}_k = \sum_{i=1}^N w_k^i \theta_k^i. \quad (3)$$

III. PROPOSED APPROACH: MW-PF

A. Multiple Weighting (MW) Approach

Conventional implementations of PF compute a single weight for each particle, based on the whole set of input measurements as in (1). However, in some applications, measurements from J different types of sensors only provide information about non-overlapping sub-sets of the state space. In such cases, similar measurements can be grouped into J sub-sets of the observables $\mathbf{z}_{(j)}$, from which multiple weights $w_{(j)}$ are derived to estimate the corresponding sub-spaces of the state vector $\theta_{(j)}$. Sub-set indexes are noted using round brackets, while time indexes have been dropped for the sake of readability. In order to characterise the *information diversity* from dissimilar measurements, multiple observables-state functions can be defined

$$\mathbf{z}_{(j)} = \mathbf{h}_j(\theta_{(j)}, \mathbf{v}_j) \text{ where } \mathbf{z}_{(j)} \subset \mathbf{z}. \quad (4)$$

Equation (1) is therefore modified in order to obtain the j -th independent weight, as

$$\tilde{w}_{(j)}^i = \mathcal{L}(\mathbf{z}_{(j)} | \theta^i) = \prod_{m=1}^{M_{(j)}} p(\bar{z}_m) \quad (5)$$

where $M_{(j)}$ is the number of measurements in vector $\mathbf{z}_{(j)}$. Equivalently, the state estimate from (3) is also modified as

$$\hat{\theta}_{(j)} = \sum_{i=1}^N w_{(j)}^i \theta_{(j)}^i. \quad (6)$$

This could suggest distributing the state-space estimation to multiple filters. This approach has been addressed by distributed, multi-sensor tracking techniques [20], [21]. However, although state-observations relationships can be considered independent, states are non-independent in the state transition model and this solution turns into an ineffective prediction step [22], besides implying a more complex filter architecture. A similar problem was discussed in [23], when integrating asynchronous measurements from dissimilar sensors, and was solved by proper modifications of the resampling stage. In the problem considered here, dissimilar measurements are collected synchronously and contribute to a single weight. Therefore, the information of the likelihood of each sub-set of states is lost and only an overall likelihood of each particles is retained with the standard PF approach. Furthermore, the proposed solution differs from Rao-Blackwellised PF [9], as all subsets of the state space are estimated through PF, thus preserving the fundamental properties of SMC methods.

B. A simple example

In order to clarify the approach presented in this paper, a simulation is set up. Two unknown states with true value $\theta^T = [\theta_1^T \ \theta_2^T]$ (where $\theta_1^T = \theta_2^T = 4$) are estimated based on two input measurements $\mathbf{z} = [z_1 \ z_2]$ that directly represent their value, such that $z_j = \theta_j^T + \mathbf{v}_j \ \forall j \in \{1, 2\}$. It is assumed that the realisations of the input measurements are not affected by any noise or bias term ($\mathbf{v}_j = 0 \ \forall j \in \{1, 2\}$). The probability densities p used to compute the weights, as in (1), are zero mean Normal distributions with standard deviation

Fig. 2. Comparison of a state-estimation using conventional (a) and the proposed MW (b) approaches applied to a simplistic two-dimensional scenario solved by means of a SIR PF with $N = 3$ particles. Weights w^i , shown in (a), are split in independent weights $w_{1,2}^i$ and normalised for each sub-state in (b).

$\sigma = 2$. The final estimate $\hat{\theta}$ is computed according to (3) as the weighted sum of the $N = 3$ particles, whose states and computed weights are reported in the tables at the bottom right of each sub-figure. Such a scenario would not require a filter for the estimation of the two states, as their values could be directly inferred from the measurements. However, this example is presented in order to demonstrate that an error is introduced on the estimate of the standard PF when the number of samples used to represent the posterior is limited w.r.t. the number of states to be estimated. On the other hand, when computing multiple weights by associating the measurements with the corresponding states for which they provide information, the estimate is optimal and corresponds to the true value, as can be seen from Figure 2(b).

Focusing on the estimate of the first state θ_1 using the PF, it can be noticed from Figure 2(a) how the error on the estimate stems from the fact that particle θ^2 has a higher weight than θ^3 , despite both their values of $\theta_1^2 = 7$ and $\theta_1^3 = 1$ differ in equal measure from the true value $\theta_1^T = 4$, so that $p(\bar{z}_1^2) = p(\bar{z}_1^3)$ are equal since the Normal distribution is even. A higher weight is computed for θ^2 because its value of $\theta_2^2 = 4$ matches the measurement $z_2 = \theta_2^T = 4$ more closely w.r.t. to that of the other particle $\theta_2^3 = 2$. It is clear to see that the final estimate $\hat{\theta}_1$ also depends on the values of θ_2 of the particles. This happens even though the two states are not related in any way at the measurement level.

IV. APPLICATION TO GNSS POSITIONING

A practical implementation of the proposed Multi Weight Particle Filter (MW-PF) is presented in this section, applied to precise state estimation in GNSS terrestrial receivers. In this context, two classes of measurements are independently estimated through the reception of specifically designed navigation signals [24]. The state space vector to be estimated by a generic GNSS receiver is defined as

$$\theta = \left[\underbrace{x \ y \ z}_{{\theta}_{(1)}} \ \underbrace{\dot{x} \ \dot{y} \ \dot{z} \ \dot{b}}_{{\theta}_{(2)}} \right] \quad (7)$$

where the variables $[x \ y \ z]$ refers to the spatial coordinates in a given Cartesian reference system, and $[\dot{x} \ \dot{y} \ \dot{z}]$ to

the axial velocity components, while b and $\dot{b}$ are respectively the bias and drift of the local clock. The sub-sets of the state space vector $\theta_{(1)}$ and $\theta_{(2)}$ are estimated using the MW approach. Given a satellite s , a first class of observables, namely *pseudoranges*, are defined as

$$\rho_s = \sqrt{(x_s - x)^2 + (y_s - y)^2 + (z_s - z)^2} + b. \quad (8)$$

By introducing the differential vector quantities of position and velocity

$$\begin{aligned} \Delta p &= [(x_s - x) \ (y_s - y) \ (z_s - z)] \\ \Delta v &= [(\dot{x}_s - \dot{x}) \ (\dot{y}_s - \dot{y}) \ (\dot{z}_s - \dot{z})] \end{aligned} \quad (9)$$

a second class of measurements, known as *range rate*, is therefore defined as

$$\Delta \rho_s = \Delta v \frac{\Delta p}{\|\Delta p\|} + \dot{b}. \quad (10)$$

In the investigated application, range measurements do not provide any knowledge about the receiver velocity, but only about its position, as can be seen from (8). Using range information to compute a weight which is used to estimate also the velocity is hence not only incorrect but leads to sub-optimal use of particles. It is worth noting that from (10) the range rate measurement has a limited dependency on the particle position. A key assumption introduced here is that the difference in position between the particles has a negligible contribution to the computation of the nominal range rate. In other words, we assume that if particles all had the same velocity, they would measure the same range rate. Since the distance between satellites and particle is much greater than the distance between any two particles, all the vectors Δp pointing from the particles to the satellite are approximately parallel to each other. Equation (10) computes the normalised projection of the relative velocity Δv on vector Δp . Since the latter contribution is approximated to be the same for all particles, the range rate measurement depends only on the velocity and clock drift of the particle. This fundamental observation allows for the split of the input measurements as

$$\mathbf{z} = \left[\underbrace{\rho_1 \ \dots \ \rho_{M(1)}}_{{\mathbf{z}}_{(1)}} \ \underbrace{\Delta \rho_1 \ \dots \ \Delta \rho_{M(2)}}_{{\mathbf{z}}_{(2)}} \right] \quad (11)$$

Fig. 3. Weight computation stage of the proposed MW-PF architecture, based on two sub-sets of observables according to (5).

Fig. 4. Comparison of PF and MW-PF solutions applied to the position estimation of a static GNSS receiver. Error ellipses identify the 3σ confidence interval of the state space estimation.

where $M_{(1)} = M_{(2)}$ is the number of visible satellites. Figure 3 provides a scheme of the integration of measurements and the computation of the weights.

A. Experimental Assessment

Although Bayesian estimation is primarily designed for kinematic state estimation, accurate static state estimation is still of interest as it can temporarily occur in any real trajectory. Therefore, a first assessment of a static position estimation is performed. Figure 4 shows a remarkable reduction of the estimation uncertainty, as reported in Table I. For a second experiment, positioning solutions are computed over a real kinematic trajectory of 20 minutes, travelled by a vehicle in an urban scenario. Input measurements are collected at a rate of 10 Hz. The 75-th percentile (PCTL) of the positioning error is extracted from the Empirical Cumulative Density Function (ECDF) of the position estimation w.r.t. a ground-truth trajectory. The experiment is repeated by varying the number of particles in the range $[500 - 20000]$. As shown in Figure 5 and quantified in Table II, the reduction in error is relevant for small numbers of particles. The MW-PF can hence achieve any target accuracy with a lower computational load.

TABLE I
COMPARISON OF THE STATIC POSITION ERROR IN METERS FOR MW-PF AND CANONICAL SIR PF AT DIFFERENT PERCENTILES ($N = 10000$).

Alg.	ECDF 50th PCTL	ECDF 75th PCTL	ECDF 95th PCTL
PF	0.966	1.222	1.577
MW-PF	0.773	0.941	1.203

Fig. 5. Comparison of 75th percentile of the ECDFs for the position estimation of a dynamic GNSS receiver using canonical and MW approaches.

TABLE II
IMPROVEMENT PERCENTAGE OF KINEMATIC POSITIONING SOLUTIONS FOR MW-PF W.R.T. TO CANONICAL SIR PF AT THE 75TH PERCENTILE OF THE ECDF.

N	Imp.(%)	N	Imp.(%)	N	Imp.(%)
500	53.3	2000	33.0	8000	18.5
750	50.7	3000	28.9	10000	11.9
1000	41.7	4000	23.1	15000	7.5
1500	38.4	6000	18.2	20000	9.3

V. CONCLUSION

Experimental results have shown a considerable reduction in number of particles needed to obtain any target accuracy when multiple weights are employed in the state space estimation via SIR PF. Indeed, the MW approach allows for a more efficient use of particles, as multiple weights represent the likelihood of the different sub-sets of the state vector for each particle, as opposed to a single weight giving only the overall likelihood across the full state space vector. In this way, each particle retains more information about the unknown states to be estimated, and an accurate representation of the posterior density is possible with fewer particles, thus exploiting information diversity and reducing computational effort. Furthermore, it should be added that the split of the state space allows for a straightforward parallel implementation of the resampling step. It is worth remarking that due to its generality, the MW technique is suitable to different SIR PF architectures. Also, while in the proposed example the state vector was divided into two sub-sets, it is in principle possible to define any number of sub-sets of any dimension, according to the type of problem and available measurements.

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