Nonlinear Beamforming Based on Group-Sparsities of Periodograms for Phased Array Weather Radar

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Abstract—We propose *nonlinear* beamforming for phased array weather radars (PAWRs). Conventional beamforming is *linear* in the sense that a backscattered signal arriving from each elevation is reconstructed by a weighted sum of received signals, which can be seen as a linear transform for the received signals. For *distributed targets* such as raindrops, however, the number of scatterers is significantly large, differently from the case of *point targets* that are standard targets in array signal processing. Thus, the spatial resolution of the conventional linear beamforming is limited. To improve the spatial resolution, we exploit two characteristics of a *periodogram* of each backscattered signal from the distributed targets. The periodogram is a series of the powers of the discrete Fourier transform (DFT) coefficients of each backscattered signal and utilized as a nonparametric estimate of the *power spectral density*. Since each power spectral density is proportional to the Doppler frequency distribution, (i) major components of the periodogram are concentrated in the vicinity of the *mean Doppler frequency*, and (ii) frequency indices of the major components are similar between adjacent elevations. These are expressed as *group-sparsities* of the DFT coefficient matrix of the backscattered signals, and we propose to reconstruct the signals through convex optimization exploiting the group-sparsities. We consider two optimization problems which evaluate the group-sparsities in slightly different ways, and both problems are solved with the *alternating direction method of multipliers* including *nonlinear* mappings. Simulations using real PAWR data show that the proposed method dramatically improves the spatial resolution.

Index Terms—Phased array weather radar, distributed target, beamforming, periodogram, group-sparsity, convex optimization.

I. INTRODUCTION

PHASED array weather radars (PAWRs) [1]–[6] have been developed to quickly detect hazardous weather phenomena such as a thunderstorm with heavy rain. Classical parabolic

Doppler weather radars [7], [8] transmit a pencil beam and receive backscattered signals within a *narrow* range of elevation angles. By repeatedly transmitting and receiving the signals at several elevation angles with the mechanical vertical scan, the classical Doppler radar can observe the weather in the sky. On the other hand, PAWRs transmit a fan beam and receive backscattered signals within a *wide* range of elevation angles by an array antenna. Then, the backscattered signals within the *narrow* ranges are reconstructed from the received signals of the array antenna by digital *beamforming* [9]–[20]. This is the key technology in the PAWR because it gets rid of the mechanical vertical scan and hence the temporal resolution is significantly improved in the weather observation. For example, the PAWR developed at Osaka University [5] can observe the weather of a radius 60 kilometers in 30 seconds, while the classical Doppler radar requires 5 to 10 minutes for a similar observation [8].

Major beamforming methods [9]–[19] reconstruct the signal arriving from each elevation as a complex weighted sum of the received signals. In particular, Capon beamforming [12]–[17], also known as the minimum variance distortionless response (MVDR) beamforming [18], is widely used since it can adaptively reduce the influence of so-called *sidelobes* if a sufficient number of beams (pulses) are transmitted. However, for quick weather observations, the number of pulses should be as small as possible. To deal with such a situation, the minimum mean square error (MMSE) beamforming [19] was proposed. In this method, differently from Capon’s method, the covariance matrix of the received signals is iteratively updated, and hence the backscattered signal arriving from each elevation can be reconstructed robustly even if the number of the pulses is small.

Beamforming methods were developed originally for observations of *point targets* in array signal processing, while targets of the PAWR are *distributed targets* such as raindrops. In this case, the number of scatterers is significantly large, i.e., much greater than the number of antenna elements. As a result, the spatial resolution of the above *linear* methods [9]–[19] is limited, and fine fluctuations of the *average signal power densities* corresponding to precipitation profile have not been captured.

To overcome the limitation of the *linear* methodology, this paper proposes a *nonlinear* beamforming method. We formulate the beamforming as an ill-conditioned inverse problem. To solve it, we exploit properties of *periodograms* of the backscattered signals from the distributed targets. The periodogram is a series of the powers of the discrete Fourier transform (DFT) coefficients of each backscattered signal and often used as a nonparametric estimator for the *power spectral density* of the

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backscattered signal. Since the power spectral density is proportional to the Doppler frequency distribution [7], (i) dominant components of each periodogram are concentrated in the vicinity of the mean Doppler frequency, and (ii) frequency indices of the dominant components are similar between the periodograms of adjacent elevation angles. We express these two properties as *group-sparsities* of the DFT coefficient matrix of the backscattered signals and propose to reconstruct the DFT coefficient matrix by minimizing a cost function that consists of a data-fidelity term and two group ℓ_1 -norms evaluating the group-sparsities. We consider two optimization problems that compute the group ℓ_1 -norms in different ways, and both problems are solved by the *alternating direction method of multipliers (ADMM)* [21]–[24]. Since nonlinear mappings called the *proximity operators* are used in the ADMM iterations, the proposed beamforming is a *nonlinear* transform for the received signals. Numerical simulations based on real PAWR data show that the proposed nonlinear beamforming greatly improves the spatial resolution in comparison with the linear beamforming.

The rest of this paper is organized as follows. In Section II, after explaining mathematical notation, we introduce the basic principles of the classical Doppler weather radar, and then we formulate the observation model in the PAWR. In Section III, we describe three conventional linear beamforming [11], [12], [19]. Section IV presents the proposed nonlinear beamforming. First, we derive the data-fidelity term in the frequency domain. Second, we explain the group ℓ_1 -norms evaluating two group-sparsities of periodograms in Section IV-A and Section IV-B. Third, we propose two convex optimization problems including their ADMM iterations in Section IV-C and Section IV-D. Section V shows the effectivity of the proposed beamforming by simulations, and finally Section VI concludes this paper.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ be the sets of all real numbers, nonnegative real numbers, and complex numbers, respectively. We use $j \in \mathbb{C}$ to denote the imaginary unit, i.e., $j := \sqrt{-1}$. For any complex number $x \in \mathbb{C}$, $\bar{x}$ denotes its complex conjugate, and $|x| := \sqrt{x\bar{x}}$ denotes its absolute value. We write vectors and matrices by boldface small and capital letters, respectively. We use $\mathbf{I}_n \in \{0, 1\}^{n \times n}$ to denote the identity matrix of order n . The transpose and Hermitian transpose of a vector or a matrix are denoted by $(\cdot)^T$ and $(\cdot)^H$, respectively. The ℓ_2 -norm (or the Euclidean norm) of a vector $\mathbf{x} := (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ is defined by $\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^n |x_i|^2}$. A weighted group ℓ_1 -norm¹ of $\mathbf{x}$ is defined, with a non-overlapping partition $G := (\mathcal{G}_i)_{i=1}^{n_G}$ and a nonnegative vector $\boldsymbol{\xi} := (\xi_1, \xi_2, \dots, \xi_{n_G})^T \in \mathbb{R}_+^{n_G}$, by $\|\mathbf{x}\|_{1,\boldsymbol{\xi}}^G := \sum_{i=1}^{n_G} \xi_i \|\mathbf{x}_{\mathcal{G}_i}\|_2$, where $\mathcal{G}_i (\neq \emptyset)$ is an index set for the i th group s.t. $\bigcup_{i=1}^{n_G} \mathcal{G}_i = \{1, 2, \dots, n\}$ and $\mathcal{G}_{i_1} \cap \mathcal{G}_{i_2} = \emptyset$ ($i_1 \neq i_2$), and $\mathbf{x}_{\mathcal{G}} := (\mathbf{x}_i)_{i \in \mathcal{G}}$ is the subvector² extracted from $\mathbf{x}$ with $\mathcal{G}$. The Frobenius norm of a matrix $\mathbf{X} := (x_{i_1, i_2}) \in \mathbb{C}^{n_1 \times n_2}$ is defined by $\|\mathbf{X}\|_F := \sqrt{\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} |x_{i_1, i_2}|^2}$. We use $E[\cdot]$ to denote the expected value of some random variable.

¹We also define a weighted group ℓ_1 -norm of a matrix $\mathbf{X}$ in a similar way.

²If $\mathbf{x}_{\mathcal{G}_i} \approx \mathbf{0}$ for many groups $\mathcal{G}_i$, $\mathbf{x}$ is called *group-sparse*. In particular, if all $\mathcal{G}_i$ consist only of consecutive indices, $\mathbf{x}$ is also called *block-sparse* [25].

A. Basic Principles of Weather Observation by Doppler Radar

Classical parabolic Doppler weather radars transmit pulses for a specific elevation angle and receive backscattered signals generated by the incidence of the transmitted signals to targets (clouds and raindrops). For a fixed range r [m], the l th discrete-time sample $\hat{x}_l \in \mathbb{C}$ of the received signal is expressed as

$$\hat{x}_l := x_l + \varepsilon_l := x \left((l-1)T + \frac{2r}{c} \right) + \varepsilon_l \quad (l = 1, 2, \dots, L), \quad (1)$$

where c [m/s] is the speed of light, T [s] is the pulse repetition time satisfying $T \geq \frac{2r}{c}$, L is the number of the pulses, $x_l \in \mathbb{C}$ is the sample of the backscattered signal $x(t)$ generated by the pulse transmitted at $t = (l-1)T$, and $\varepsilon_l \in \mathbb{C}$ is additive white Gaussian noise of variance $E[|\varepsilon_l|^2] = \sigma_\varepsilon^2$. We consider $x(t)$ to be one sample realization of a continuous-time random process $X(t)$ which is *wide-sense stationary*,³ *ergodic*⁴ and zero-mean, i.e., $\mu_x := E[X(t)] = 0$, and whose autocorrelation function $R_x(\tau) := E[X(t+\tau)\bar{X}(t)]$ satisfies $\int_{-\infty}^{\infty} |\tau R_x(\tau)| d\tau < \infty$.

Strictly speaking, $x(t)$ is expressed as the sum of individual backscattered signals in the scattering resolution volume [7]:

$$x(t) = \sum_s a_s(t) e^{j(2\pi f_s t + \phi_s)},$$

where $a_s(t) > 0$ is the scattering amplitude of the s th particle, f_s [Hz] is called the Doppler frequency, which is independently generated from a *common probability density function* $q(f)$, and ϕ_s [rad] is called the phase shift, which is uniformly distributed from $[-\pi, \pi)$ under the assumption that precipitation profile is uniform within the scattering resolution volume. The *power spectral density* of the backscattered signal $x(t)$ is given by the Fourier transform of the autocorrelation function $R_x(\tau)$:

$$S(f) := \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau} d\tau. \quad (2)$$

Suppose that the amount of rainfall is constant and the shapes of all raindrops are the same, i.e., $\forall s \forall t \ a_s(t) = a$, during the observation. As a result, we have $R_x(\tau) = E[a^2 \sum_s e^{j2\pi f_s \tau}]$, and the power spectral density $S(f)$ in (2) can be expressed as

$$S(f) = E \left[a^2 \sum_s \delta(f - f_s) \right] = P q(f) = \frac{\lambda P}{2} q_v \left(\frac{\lambda f}{2} \right). \quad (3)$$

In (3), $\delta(f)$ is the delta function, $P := R_x(0) = E[|x_l|^2]$ is the *average signal power density*, λ [m] is the carrier wavelength, and $q_v : \mathbb{R} \rightarrow \mathbb{R}_+$ is a common probability density function for each Doppler velocity $v_s = \frac{\lambda f_s}{2}$ [m/s], where the direction of approach to the radar is defined as the positive direction of the Doppler velocity. From (3), we can see that the power spectral

³Let $X(t)$ be a continuous-time random process. $X(t)$ is called *wide-sense stationary* or *weak-sense stationary* if its mean and autocorrelation function do not vary by shifts in the time t , i.e., $\forall t \ E[X(t)] = \int_{\mathbb{C}} x q(x) dx = \mu_x$ and $\forall t \forall \tau \ E[X(t+\tau)\bar{X}(t)] = \int_{\mathbb{C}^2} x_1 \bar{x}_2 q(x_1, x_2 | t + \tau, t) dx_1 dx_2 = R_x(\tau)$, where $q(x|t)$ is the probability density function of $x(t)$ and $q(x_1, x_2 | t + \tau, t)$ is the joint probability density function of $x_1 := x(t + \tau)$ and $x_2 := x(t)$.

⁴A wide-sense stationary process $X(t)$ is called *ergodic* if any sample realization $x(t)$ of the process $X(t)$ satisfies $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt = \mu_x$ and $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t + \tau) \bar{x}(t) dt = R_x(\tau)$ by the mean-square convergence.

density $S(f)$ is proportional to the Doppler frequency distribution $q(f)$ [7]. On the Doppler frequency, we call its mean

$$\mu_f := \int_{-\infty}^{\infty} f q(f) df = \int_{-\infty}^{\infty} f \frac{S(f)}{P} df$$

the *mean Doppler frequency*, and we call its standard deviation

$$\sigma_f := \sqrt{\int_{-\infty}^{\infty} (f - \mu_f)^2 q(f) df} = \sqrt{\int_{-\infty}^{\infty} (f - \mu_f)^2 \frac{S(f)}{P} df}$$

the *Doppler frequency width*. Similarly, we call $\mu_v = \frac{\lambda \mu_f}{2}$ the *mean Doppler velocity* and call $\sigma_v = \frac{\lambda \sigma_f}{2}$ the *Doppler velocity width*. The main purpose of the Doppler weather radars is to estimate P , μ_f (or μ_v) and σ_f (or σ_v) from the measurements $\hat{x}_l$ in (1), since these parameters correspond to the amount of rainfall, the mean wind speed and the variation of wind speeds.

As shown in (3), $S(f)$ has information about P and $q(f)$. However, the backscattered signal $x(t)$ is observed in discrete-time, and we have to consider the power spectral density of x_l that is given by the discrete-time Fourier transform of $R_x(lT)$:

$$S^{(d)}(f) := \sum_{l=-\infty}^{\infty} R_x(lT) e^{-j2\pi f lT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} S\left(f - \frac{n}{T}\right). \quad (4)$$

If $S(f) = 0$ for $|f| \geq \frac{1}{2T}$, $S(f)$ can be perfectly reconstructed from $S^{(d)}(f)$. To estimate $S^{(d)}(f)$ in (4) with finite samples x_l ($l = 1, 2, \dots, L$), we suppose that the number L is *even*, and we compute the normalized DFT coefficients $u_k \in \mathbb{C}$ of x_l by

$$u_k := \frac{1}{\sqrt{L}} \sum_{l=1}^L x_l e^{-j \frac{2\pi(k-1-L/2)(l-1)}{L}} \quad (k = 1, 2, \dots, L). \quad (5)$$

Then, we compute a *periodogram* as a series of the powers of the normalized DFT coefficients u_k :

$$|u_k|^2 = \frac{1}{L} \left| \sum_{l=1}^L x_l e^{-j \frac{2\pi(k-1-L/2)(l-1)}{L}} \right|^2 \quad (k = 1, 2, \dots, L). \quad (6)$$

The expected value of the periodogram ($|u_k|^2$) $_{k=1}^L$ is given by

$$\begin{aligned} E[|u_k|^2] &= \frac{1}{L} E \left[\sum_{l_1=1}^L \sum_{l_2=1}^L x_{l_1} \bar{x}_{l_2} e^{-j \frac{2\pi(k-1-L/2)(l_1-l_2)}{L}} \right] \\ &= \frac{1}{L} \sum_{l_1=1}^L \sum_{l_2=1}^L E[x_{l_1} \bar{x}_{l_2}] e^{-j \frac{2\pi(k-1-L/2)(l_1-l_2)}{L}} \\ &= \frac{1}{L} \sum_{l_1=1}^L \sum_{l_2=1}^L R_x((l_1 - l_2)T) e^{-j \frac{2\pi(k-1-L/2)(l_1-l_2)}{L}} \\ &= \sum_{l=-(L-1)}^{L-1} \left(1 - \frac{|l|}{L}\right) R_x(lT) e^{-j \frac{2\pi(k-1-L/2)l}{L}}. \quad (7) \end{aligned}$$

From (4), (7), and $\sum_{l=-\infty}^{\infty} |l R_x(lT)| < \infty$, we have

$$\lim_{L \rightarrow \infty} E[|u_k|^2] = S^{(d)}\left(\frac{k-1-L/2}{LT}\right).$$

Therefore, the periodogram is often used as a nonparametric estimator for $S^{(d)}(f)$. Moreover, the parameters P , μ_f and σ_f can be estimated from the periodogram ($|u_k|^2$) $_{k=1}^L$, e.g., by

$$\left. \begin{aligned} \hat{P} &= \frac{1}{L} \sum_{l=1}^L |x_l|^2 = \frac{1}{L} \sum_{k=1}^L |u_k|^2 \\ \hat{\mu}_f &= \frac{1}{\hat{P}L} \sum_{k=1}^L \frac{k-1-L/2}{LT} |u_k|^2 \\ \hat{\sigma}_f &= \sqrt{\frac{1}{\hat{P}L} \sum_{k=1}^L \left(\frac{k-1-L/2}{LT} - \hat{\mu}_f \right)^2 |u_k|^2} \end{aligned} \right\}. \quad (8)$$

B. Observation Model in Phased Array Weather Radar

First of all, we show the observation model for K point targets. Let a PAWR have an N -element uniform linear array with the inter-element spacing d [m]. A plane wave signal scattered from the κ th point target hits on the array antenna at an *unknown* angle $\theta_\kappa^* \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\theta_1^* < \theta_2^* < \dots < \theta_K^*$ [rad]. The l th time sample $\mathbf{y}_l \in \mathbb{C}^N$ of the received signal is given by

$$\mathbf{y}_l = \sum_{\kappa=1}^K x_{\kappa,l}^* \mathbf{a}(\theta_\kappa^*) + \boldsymbol{\varepsilon}_l \quad (l = 1, 2, \dots, L), \quad (9)$$

where $x_{\kappa,l}^* \in \mathbb{C}$ is the l th time sample of the κ th plane wave signal, $\mathbf{a}(\theta_\kappa^*) \in \mathbb{C}^N$ is the so-called *steering vector* defined by

$$\mathbf{a}(\theta) := (1, e^{-j \frac{2\pi d \sin \theta}{\lambda}}, e^{-j \frac{4\pi d \sin \theta}{\lambda}}, \dots, e^{-j \frac{(N-1)\pi d \sin \theta}{\lambda}})^T,$$

and $\boldsymbol{\varepsilon}_l \in \mathbb{C}^N$ is additive white Gaussian noise, independent of the signals $x_{\kappa,l}^*$, of covariance matrix $\mathbf{R}_\varepsilon := E[\boldsymbol{\varepsilon}_l \boldsymbol{\varepsilon}_l^H] = \sigma_\varepsilon^2 \mathbf{I}_N$.

On the other hand, our targets such as raindrops are called *distributed targets* [19], that are supposed to exist continuously (strictly speaking, a sufficient number of raindrops exist within the antenna beamwidth). In this case, the number K of scatterers (raindrops) is too large, and hence the observation model in (9) is no longer appropriate. We observe the distributed targets by dividing the whole angular interval $[\theta_{\min}, \theta_{\max}] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ into M subintervals $[\theta_m - \frac{\Delta\theta}{2}, \theta_m + \frac{\Delta\theta}{2}]$ ($m = 1, 2, \dots, M$), where $\Delta\theta := \frac{\theta_{\max} - \theta_{\min}}{M}$ and $\theta_m := \theta_{\min} + (m - \frac{1}{2})\Delta\theta$. Thus, instead of (9), we use the following observation model

$$\mathbf{y}_l = \sum_{m=1}^M x_{m,l} \mathbf{a}_m + \boldsymbol{\varepsilon}_l = \mathbf{A} \mathbf{x}_l + \boldsymbol{\varepsilon}_l \quad (l = 1, 2, \dots, L), \quad (10)$$

where $x_{m,l} \in \mathbb{C}$ is the l th sample of the sum of backscattered plane wave signals in the m th subinterval $[\theta_m - \frac{\Delta\theta}{2}, \theta_m + \frac{\Delta\theta}{2}]$, $\mathbf{x}_l := (x_{1,l}, x_{2,l}, \dots, x_{M,l})^T \in \mathbb{C}^M$, $\mathbf{a}_m := \mathbf{a}(\theta_m) \in \mathbb{C}^N$, and $\mathbf{A} := [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_M] \in \mathbb{C}^{N \times M}$. Note that the signal x_l in (1) corresponds to $x_{m,l}$ in (10) for a specific elevation angle θ_m . In the PAWR system, θ_m is the m th elevation angle, and the average signal power density vector

$$\begin{aligned} \mathbf{p} &:= (P_1, P_2, \dots, P_M)^T \\ &:= (E[|x_{1,l}|^2], E[|x_{2,l}|^2], \dots, E[|x_{M,l}|^2])^T \in \mathbb{R}_+^M \end{aligned}$$

corresponds to precipitation profile along the elevation axis.

III. CONVENTIONAL METHODS: LINEAR BEAMFORMING

The beamforming is a reconstruction problem of $\mathbf{x}_l$ in (10). Major beamforming methods [9]–[19] estimate $\mathbf{x}_l$ by multiplying complex weights $\mathbf{w}_m \in \mathbb{C}^N$ ($m = 1, 2, \dots, M$) and $\mathbf{y}_l$ as

$$\begin{aligned} \hat{\mathbf{x}}_l &:= (\hat{x}_{1,l}, \hat{x}_{2,l}, \dots, \hat{x}_{M,l})^T \\ &:= (\mathbf{w}_1^H \mathbf{y}_l, \mathbf{w}_2^H \mathbf{y}_l, \dots, \mathbf{w}_M^H \mathbf{y}_l)^T = \mathbf{W} \mathbf{y}_l. \end{aligned} \quad (11)$$

In this paper, we call the methodology based on (11) the *linear beamforming*. Note that the least squares (LS) method

$$\hat{\mathbf{x}}_{\text{LS},l} := \mathbf{W}_{\text{LS}} \mathbf{y}_l := \mathbf{A}^\dagger \mathbf{y}_l \quad (12)$$

does not necessarily work well, even if $N \geq M$, since $\mathbf{A}$ is *ill-conditioned* when $\Delta\theta$ is smaller than the antenna beamwidth that is determined by the antenna size, where $\mathbf{A}^\dagger \in \mathbb{C}^{M \times N}$ is the Moore–Penrose pseudoinverse of $\mathbf{A}$. In the following, we explain three linear methods, *Fourier (FR) beamforming* [11], *Capon (CP) beamforming* [12], and *MMSE beamforming* [19].

A. FR Beamforming

FR beamforming [11] is the most basic linear method and its complex weight vector is defined by

$$\mathbf{w}_{\text{FR},m} := \frac{\mathbf{a}_m}{N} \quad (13)$$

independently of $\mathbf{y}_l$. The vector $\mathbf{w}_{\text{FR},m}$ is equal to a *matched filter* that maximizes the signal-to-noise ratio $\frac{E[|x_{m,l}|^2]}{E[|\mathbf{w}_{\text{FR},m}^H \mathbf{y}_l|^2]}$. If $N \geq M$ and there exists some positive integer n satisfying $\forall m \frac{d}{d\theta}(\sin \theta_{m+1} - \sin \theta_m) \approx \frac{n}{N}$, then FR beamforming can reconstruct the signal $\mathbf{x}_l$ in the manner of the inverse DFT, but this ideal condition is usually not met. In the typical case, the average signal power density P_m is often overestimated [19]. To explain this fact, we derive the observation model as in (9) from (10) by redefining K ($\leq M$) as the number of the sub-intervals $[\theta_m - \frac{\Delta\theta}{2}, \theta_m + \frac{\Delta\theta}{2}]$ in which scatterers (raindrops) exist, and θ_κ^* as the center of such a subinterval. Then, from

$$\hat{x}_{\text{FR},m,l} := \frac{\mathbf{a}_m^H}{N} \mathbf{y}_l = x_{m,l} + \frac{1}{N} \sum_{\theta_\kappa^* \neq \theta_m} x_{\kappa,l}^* \mathbf{a}_m^H \mathbf{a}(\theta_\kappa^*) + \frac{\mathbf{a}_m^H \boldsymbol{\varepsilon}_l}{N},$$

the signal power $P_m = E[|x_{m,l}|^2]$ is overestimated by (8):

$$\hat{P}_{\text{FR},m} = \frac{1}{L} \sum_{l=1}^L |\hat{x}_{\text{FR},m,l}|^2 \gg E[|x_{m,l}|^2] + \frac{\sigma_\varepsilon^2}{N},$$

for many elevation angles θ_m due to the influence of *sidelobes*, i.e., the influence of $\theta_\kappa^* (\neq \theta_m)$ satisfying $|\mathbf{a}_m^H \mathbf{a}(\theta_\kappa^*)| \approx 0$.

B. CP Beamforming

CP beamforming [12], that is also known as MVDR beamforming [18], is a data-dependent method, differently from FR beamforming. To avoid the overestimation, CP beamforming minimizes the signal power $\frac{1}{L} \sum_{l=1}^L |\hat{x}_{m,l}|^2 = \mathbf{w}_m^H \hat{\mathbf{R}}_y \mathbf{w}_m$ under the condition $\mathbf{w}_m^H \mathbf{a}_m = 1$, where $\hat{\mathbf{R}}_y := \frac{1}{L} \sum_{l=1}^L \mathbf{y}_l \mathbf{y}_l^H \in \mathbb{C}^{N \times N}$ denotes the sample covariance matrix of the zero-mean random vector $\mathbf{y}_l$. The weight vector $\mathbf{w}_{\text{CP},m}$ is defined as the solution to the following optimization problem

$$\text{minimize } \mathbf{w}_m^H \hat{\mathbf{R}}_y \mathbf{w}_m \quad \text{subject to } \mathbf{w}_m^H \mathbf{a}_m = 1,$$

and given by

$$\mathbf{w}_{\text{CP},m} := \frac{\hat{\mathbf{R}}_y^{-1} \mathbf{a}_m}{\mathbf{a}_m^H \hat{\mathbf{R}}_y^{-1} \mathbf{a}_m} \quad (14)$$

if $L \geq N$ (strictly speaking, if $\text{rank}(\hat{\mathbf{R}}_y) = N$). In particular, if $N \geq K + 1$ and L is sufficiently large, then we have

$$\hat{P}_{\text{CP},m} := \frac{1}{L} \sum_{l=1}^L |\hat{x}_{\text{CP},m,l}|^2 \approx E[|x_{m,l}|^2] + \sigma_\varepsilon^2 \|\mathbf{w}_{\text{CP},m}\|_2^2$$

for all m since $\forall \theta_\kappa^* \neq \theta_m |\mathbf{w}_{\text{CP},m}^H \mathbf{a}(\theta_\kappa^*)| \approx 0$ holds. However, if L is not large, each average signal power density is underestimated [19]. Moreover if $L < N$, $\hat{\mathbf{R}}_y^{-1}$ cannot be computed.

C. MMSE Beamforming

MMSE beamforming [19] was developed to robustly reconstruct the signal $\mathbf{x}_l$ in case of small L . This method approximately solves the following optimization problem

$$\text{minimize}_{\mathbf{w}_m} E[|x_{m,l} - \mathbf{w}_m^H \mathbf{y}_l|^2] \quad \text{subject to } \mathbf{w}_m^H \mathbf{a}_m = 1.$$

Suppose $E[x_{m_1,l} \bar{x}_{m_2,l}] = E[x_{m_1,l}]E[\bar{x}_{m_2,l}] = 0$ ($m_1 \neq m_2$). Then the exact solution to the optimization problem is given by

$$\mathbf{w}_{\text{MMSE},m} := \frac{\mathbf{R}_y^{-1} \mathbf{a}_m}{\mathbf{a}_m^H \mathbf{R}_y^{-1} \mathbf{a}_m}, \quad (15)$$

where $\mathbf{R}_y := E[\mathbf{y}_l \mathbf{y}_l^H]$ is the covariance matrix of $\mathbf{y}_l$. By using the covariance matrix $\mathbf{R}_x := E[\mathbf{x}_l \mathbf{x}_l^H]$ of $\mathbf{x}_l$, $\mathbf{R}_y$ is expressed as $\mathbf{R}_y = \mathbf{A} \mathbf{R}_x \mathbf{A}^H + \sigma_\varepsilon^2 \mathbf{I}_N$. Moreover, $\mathbf{R}_x = \text{diag}(\mathbf{p})$ is approximated by $\hat{\mathbf{R}}_x \odot \mathbf{I}_M := (\frac{1}{L} \sum_{l=1}^L \mathbf{x}_l \mathbf{x}_l^H) \odot \mathbf{I}_M$, where $\odot$ denotes the Hadamard product. As a result, the weight vector $\mathbf{w}_{\text{MMSE},m}$ in (15) can be approximated, from the initial estimate $\hat{\mathbf{x}}_{\text{MMSE},l}^{(0)} = \hat{\mathbf{x}}_{\text{FR},l} = \mathbf{A}^H \mathbf{y}_l / N$, by iteratively computing

$$\begin{cases} \mathbf{R}_x^{(i)} = \left(\frac{1}{L} \sum_{l=1}^L \hat{\mathbf{x}}_{\text{MMSE},l}^{(i)} \hat{\mathbf{x}}_{\text{MMSE},l}^{(i)H} \right) \odot \mathbf{I}_M \\ \mathbf{R}_y^{(i)} = \mathbf{A} \mathbf{R}_x^{(i)} \mathbf{A}^H + \sigma_\varepsilon^2 \mathbf{I}_N \\ \mathbf{w}_{\text{MMSE},m}^{(i+1)} = \frac{\mathbf{R}_y^{(i)-1} \mathbf{a}_m}{\mathbf{a}_m^H \mathbf{R}_y^{(i)-1} \mathbf{a}_m} \quad (m = 1, 2, \dots, M) \\ \hat{\mathbf{x}}_{\text{MMSE},l}^{(i+1)} = \mathbf{W}_{\text{MMSE}}^{(i+1)} \mathbf{y}_l \quad (l = 1, 2, \dots, L) \end{cases} \quad (16)$$

for $i \geq 0$ until $\eta^{(i+1)} = \sum_{m=1}^M \frac{\sum_{l=1}^L |\hat{\mathbf{x}}_{\text{MMSE},m,l}^{(i+1)} - \hat{\mathbf{x}}_{\text{MMSE},m,l}^{(i)}|^2}{M \sum_{l=1}^L |\hat{\mathbf{x}}_{\text{MMSE},m,l}^{(i)}|^2}$ becomes sufficiently small. In this method, even for small L , $\mathbf{R}_y$ and $\mathbf{x}_l$ are stably estimated. However, if K is close to N or larger than N , then the estimation accuracy degrades, i.e., fine fluctuations of the average signal power densities cannot be captured, because each vector $\mathbf{w}_m$ can only create at most $N - 1$ null directions and the influence of the sidelobes occurs.

IV. NONLINEAR BEAMFORMING VIA CONVEX OPTIMIZATION BASED ON GROUP-SPARSITIES

In this section, we propose a *nonlinear beamforming* method based on convex optimization. First, we gather $\mathbf{x}_l$ and $\mathbf{y}_l$ into $\mathbf{X} := [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_L] = [\tilde{\mathbf{x}}_1 \ \tilde{\mathbf{x}}_2 \ \dots \ \tilde{\mathbf{x}}_M]^T \in \mathbb{C}^{M \times L}$ and $\mathbf{Y} := [\mathbf{y}_1 \ \mathbf{y}_2 \ \dots \ \mathbf{y}_L] \in \mathbb{C}^{N \times L}$, respectively, where $\tilde{\mathbf{x}}_m := (x_{m,1}, x_{m,2}, \dots, x_{m,L})^T \in \mathbb{C}^L$. Then the beamforming can be

translated into a reconstruction problem of $\mathbf{X}$ from $\mathbf{Y}$, and the data-fidelity for the measurements $\mathbf{y}_l$ in (10) is evaluated by

$$\sum_{l=1}^L \|\mathbf{y}_l - \mathbf{A}x_l\|_2^2 = \|\mathbf{Y} - \mathbf{A}\mathbf{X}\|_F^2 = \|\mathbf{Y}^T - \mathbf{X}^T \mathbf{A}^T\|_F^2. \quad (17)$$

For each discrete-time backscattered signal $\tilde{x}_m$ at θ_m , define the normalized DFT coefficients $u_{k,m} \in \mathbb{C}$ ($k = 1, 2, \dots, L$) as (5). Let $\mathbf{F} := \frac{1}{\sqrt{L}} [\mathbf{f}_{-\frac{L}{2}} \ \mathbf{f}_{-\frac{L}{2}+1} \ \dots \ \mathbf{f}_{\frac{L}{2}-1}]^T \in \mathbb{C}^{L \times L}$ be the normalized discrete Fourier transform matrix, where $\mathbf{f}_i := (1, e^{-j\frac{2\pi i}{L}}, e^{-j\frac{4\pi i}{L}}, \dots, e^{-j\frac{2(L-1)\pi i}{L}})^T \in \mathbb{C}^L$. By using $\mathbf{F}$, the DFT coefficient vector $\mathbf{u}_m := (u_{1,m}, u_{2,m}, \dots, u_{L,m})^T \in \mathbb{C}^L$ is expressed as $\mathbf{u}_m = \mathbf{F}\tilde{x}_m$ ($m = 1, 2, \dots, M$). Furthermore, we define the DFT coefficient matrix $\mathbf{U} = (u_{k,m}) \in \mathbb{C}^{L \times M}$ by

$$\mathbf{U} := [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_M] = \mathbf{F}\mathbf{X}^T.$$

The reconstruction of $\mathbf{X}$ and that of $\mathbf{U}$ are essentially equivalent due to the unitarity of $\mathbf{F}$, and the weather parameters can be computed from $\mathbf{U}$ with (8). Thus in this paper, we consider the beamforming as a *reconstruction problem of $\mathbf{U}$ from $\mathbf{Y}$* . We have $\mathbf{X}^T = \mathbf{F}^H \mathbf{U}$, and (17) can be transformed into

$$\|\mathbf{Y}^T - \mathbf{X}^T \mathbf{A}^T\|_F^2 = \|\mathbf{Y}^T - \mathbf{F}^H \mathbf{U} \mathbf{A}^T\|_F^2 = \|\mathbf{F} \mathbf{Y}^T - \mathbf{U} \mathbf{A}^T\|_F^2 \quad (18)$$

from the unitarity of $\mathbf{F}$. In the following, after describing two group-sparsities of $\mathbf{U}$, we estimate $\mathbf{U}$ by solving convex optimization problems based on (18) and the two group-sparsities (see Footnote 2 for the meaning of the word “group-sparse”).

A. Narrow Bandwidth of Each Power Spectral Density

As mentioned in Section II-A, the power spectral density of the backscattered signal $x(t)$ is proportional to the Doppler frequency distribution. In addition, the Doppler frequency distribution and the Doppler velocity distribution are essentially the same (see (3)). Thus, unless the Doppler velocity width σ_v is extremely large, the bandwidth of the power spectral density, similar to the Doppler frequency width σ_f , is narrow compared to the sampling frequency $\frac{1}{T}$ [Hz]. In particular, when the influence of the atmospheric turbulence is large,⁵ the power spectral density can be modeled by a Gaussian function [7], [26], [27]:

$$S(f) = Pq(f) = \frac{P}{\sqrt{2\pi}\sigma_f} e^{-\frac{(f-\mu_f)^2}{2\sigma_f^2}}. \quad (19)$$

In this model, the autocorrelation function is exactly given by

$$R_x(\tau) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau} df = P e^{j2\pi\mu_f\tau} e^{-2\pi^2\sigma_f^2\tau^2}. \quad (20)$$

Figure 1 illustrates the relation between $S(f)$ in (2), $S^{(d)}(f)$ in (4), and the periodogram $(|u_k|^2)_{k=1}^L$ in (6) using (19) and (20). As shown in Fig. 1, the frequency indices k of large $E[|u_k|^2]$ are gathered in the vicinity of the mean Doppler frequency μ_f .

Since $E[|u_k|^2] \approx 0$ implies $u_k \approx 0$ and the above discussion holds for every elevation θ_m , each DFT coefficient vector $\mathbf{u}_m$ will be *group-sparse* by an appropriate *non-overlapping group*

Fig. 1. Relation between the power spectral density $S(f)$ of $x(t)$, the power spectral density $S^{(d)}(f)$ of x_l , and the periodogram $(|u_k|^2)_{k=1}^L$, where the Gaussian function in (19) is used as the Doppler frequency distribution with $P = 1$, $\mu_f = 700$ [Hz], $\sigma_f = 220$ [Hz], $T = 4 \times 10^{-4}$ [s], and $L = 20$. If we set the carrier wavelength to $\lambda = 3.181 \times 10^{-2}$ [m], then $\sigma_v \approx 3.5$ [m/s]. Black, blue, and red lines depict $\frac{1}{T}S(f)$, $S^{(d)}(f)$, and $E[|u_k|^2]$, respectively.

partition, but the appropriate partition is *unknown* in advance. Instead, we divide $\mathbf{u}_m$ into L overlapping groups of size b_f :

$$\left. \begin{aligned} \mathbf{z}_{1,m}^{(1)} &:= (u_{1,m}, u_{2,m}, \dots, u_{b_f,m})^T \in \mathbb{C}^{b_f} \\ \mathbf{z}_{2,m}^{(1)} &:= (u_{2,m}, u_{3,m}, \dots, u_{b_f+1,m})^T \in \mathbb{C}^{b_f} \\ &\vdots \\ \mathbf{z}_{L-b_f+1,m}^{(1)} &:= (u_{L-b_f+1,m}, u_{L-b_f+2,m}, \dots, u_{L,m})^T \in \mathbb{C}^{b_f} \\ \mathbf{z}_{L-b_f+2,m}^{(1)} &:= (u_{L-b_f+2,m}, \dots, u_{L,m}, u_{1,m})^T \in \mathbb{C}^{b_f} \\ &\vdots \\ \mathbf{z}_{L,m}^{(1)} &:= (u_{L,m}, u_{1,m}, \dots, u_{b_f-1,m})^T \in \mathbb{C}^{b_f} \end{aligned} \right\}$$

under the periodic boundary condition. Note that we connect $u_{L,m}$ and $u_{1,m}$ in $\mathbf{z}_{k,m}^{(1)}$ ($k = L - b_f + 2, L - b_f + 3, \dots, L$) since the frequency components of the discrete-time signal $\tilde{x}_m$ are wrapped below the Nyquist frequency as shown in Fig. 1. By arranging all $\mathbf{z}_{k,m}^{(1)}$ ($k = 1, 2, \dots, L$ and $m = 1, 2, \dots, M$), we define a matrix

$$\mathbf{Z}_1 := \begin{bmatrix} \mathbf{z}_{1,1}^{(1)} & \mathbf{z}_{1,2}^{(1)} & \dots & \mathbf{z}_{1,M}^{(1)} \\ \mathbf{z}_{2,1}^{(1)} & \mathbf{z}_{2,2}^{(1)} & \dots & \mathbf{z}_{2,M}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{z}_{L,1}^{(1)} & \mathbf{z}_{L,2}^{(1)} & \dots & \mathbf{z}_{L,M}^{(1)} \end{bmatrix} \in \mathbb{C}^{b_f L \times M}.$$

Then, $\mathbf{z}_{k,m}^{(1)} \approx \mathbf{0}$ holds for many k and m . Therefore, the matrix $\mathbf{Z}_1$ will be *group-sparse* by a *non-overlapping partition* $G_1 := (\mathcal{G}_{k,m}^{(1)})$ s.t. $\mathcal{G}_{k,m}^{(1)} := \{(b_f(k-1) + i, m)\}_{i=1}^{b_f}$. Moreover, there is a simple matrix $\mathbf{B}_f \in \{0, 1\}^{b_f L \times L}$ satisfying $\mathbf{B}_f \mathbf{U} = \mathbf{Z}_1$. This property can be evaluated by a weighted group ℓ_1 -norm

$$\|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1} := \sum_{k=1}^L \sum_{m=1}^M \xi_{k,m}^{(1)} \|\mathbf{z}_{k,m}^{(1)}\|_2, \quad (21)$$

where $\Xi_1 := (\xi_{k,m}^{(1)}) \in \mathbb{R}_+^{L \times M}$ is a nonnegative weight matrix.

⁵See, e.g., [26] for a detailed relation between the atmospheric turbulence intensity and the Doppler velocity distribution at each elevation angle.

B. Similarity between the Adjacent Power Spectral Densities

In the previous section, we focused on the narrow bandwidth of the power spectral density for each elevation angle θ_m . In this section, we explain the similarity between the power spectral densities at adjacent elevation angles θ_m and θ_{m+1} . When $\Delta\theta$ is relatively small, i.e., M is set to a relatively large value to acquire the high-resolution precipitation profile, the mean Doppler velocity μ_v and the Doppler velocity width σ_v at θ_m and those at θ_{m+1} are expected to be similar. As a result, the power spectral density at θ_m and that at θ_{m+1} are similar.

Such a similarity also applies to the adjacent periodograms $(|u_{k,m}|^2)_{k=1}^L$ and $(|u_{k,m+1}|^2)_{k=1}^L$. For a fixed frequency index k , if $E[|u_{k,m}|^2]$ is large, then $E[|u_{k,m+1}|^2]$ will be also large. If $E[|u_{k,m}|^2] \approx 0$ holds, then $E[|u_{k,m+1}|^2] \approx 0$ will also hold. As a result, a vector $\tilde{\mathbf{u}}_k := (u_{k,1}, u_{k,2}, \dots, u_{k,M})^T \in \mathbb{C}^M$ will be *group-sparse* by an appropriate *non-overlapping group partition*, but it is unknown in advance. Instead, as in the previous section, we divide $\tilde{\mathbf{u}}_k$ into M *overlapping groups* of size b_e :

$$\left. \begin{aligned} \mathbf{z}_{1,k}^{(2)} &:= (u_{k,1}, u_{k,2}, \dots, u_{k,b_e})^T \in \mathbb{C}^{b_e} \\ \mathbf{z}_{2,k}^{(2)} &:= (u_{k,2}, u_{k,3}, \dots, u_{k,b_e+1})^T \in \mathbb{C}^{b_e} \\ &\vdots \\ \mathbf{z}_{M-b_e+1,k}^{(2)} &:= (u_{k,M-b_e+1}, u_{k,M-b_e+2}, \dots, u_{k,M})^T \in \mathbb{C}^{b_e} \\ \mathbf{z}_{M-b_e+2,k}^{(2)} &:= (u_{k,M-b_e+2}, \dots, u_{k,M}, u_{k,1})^T \in \mathbb{C}^{b_e} \\ &\vdots \\ \mathbf{z}_{M,k}^{(2)} &:= (u_{k,M}, u_{k,1}, \dots, u_{k,b_e-1})^T \in \mathbb{C}^{b_e} \end{aligned} \right\}$$

under the periodic boundary condition. By arranging all $\mathbf{z}_{m,k}^{(2)}$ ($m = 1, 2, \dots, M$ and $k = 1, 2, \dots, L$), we define a matrix

$$\mathbf{Z}_2 := \begin{bmatrix} \mathbf{z}_{1,1}^{(2)} & \mathbf{z}_{1,2}^{(2)} & \dots & \mathbf{z}_{1,L}^{(2)} \\ \mathbf{z}_{2,1}^{(2)} & \mathbf{z}_{2,2}^{(2)} & \dots & \mathbf{z}_{2,L}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{z}_{M,1}^{(2)} & \mathbf{z}_{M,2}^{(2)} & \dots & \mathbf{z}_{M,L}^{(2)} \end{bmatrix} \in \mathbb{C}^{b_e M \times L}.$$

Then $\mathbf{z}_{m,k}^{(2)} \approx \mathbf{0}$ holds for many m and k . Therefore, the matrix $\mathbf{Z}_2$ will be *group-sparse* by a *non-overlapping partition* $G_2 := (\mathcal{G}_{m,k}^{(2)})$ s.t. $\mathcal{G}_{m,k}^{(2)} := \{(b_e(m-1) + i, k)\}_{i=1}^{b_e}$, and there exists a simple matrix $\mathbf{B}_e \in \{0, 1\}^{b_e M \times M}$ satisfying $\mathbf{B}_e \mathbf{U}^T = \mathbf{Z}_2$. This property can be evaluated by a weighted group ℓ_1 -norm

$$\|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2} := \sum_{m=1}^{M-b_e+1} \sum_{k=1}^L \xi_{m,k}^{(2)} \|\mathbf{z}_{m,k}^{(2)}\|_2, \quad (22)$$

where $\Xi_2 := (\xi_{m,k}^{(2)}) \in \mathbb{R}_+^{M \times L}$ is a nonnegative weight matrix s.t. $\xi_{m,k}^{(2)} := 0$ for $m = M - b_e + 2, M - b_e + 3, \dots, M$. Note that $E[|u_{k,M}|^2]$ and $E[|u_{k,1}|^2]$ are not necessarily similar, and hence we define the weights $\xi_{m,k}^{(2)}$ as zeros for the vectors $\mathbf{z}_{m,k}^{(2)}$ including both $u_{k,M}$ and $u_{k,1}$ not to evaluate these vectors.

Remark 1 (Difference from Our Previous Work [20]): In our previous work [20], we considered the beamforming as a reconstruction problem of $\mathbf{X}$, and used the same group ℓ_1 -norm $\|\mathbf{B}_f \mathbf{F} \mathbf{X}^T\|_{1, \Xi_1}^{G_1} = \|\mathbf{B}_f \mathbf{U}\|_{1, \Xi_1}^{G_1} = \|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1}$ as in (21) for the narrow bandwidth of each power spectral density. On the other

hand, we did not exploit the similarity of the adjacent power spectral densities but use the continuity of precipitation profile, i.e., we assumed that if $\tilde{\mathbf{x}}_m \approx \mathbf{0}$, then it is highly possible that $\tilde{\mathbf{x}}_{m+1} \approx \mathbf{0}$. This was evaluated by a weighted group ℓ_1 -norm

$$\begin{aligned} \|\mathbf{B}_e \mathbf{X}\|_{1, \xi_0}^{G_0} &= \sum_{m=1}^{M-b_e+1} \xi_m^{(0)} \|(\tilde{\mathbf{x}}_m^T, \tilde{\mathbf{x}}_{m+1}^T, \dots, \tilde{\mathbf{x}}_{m+b_e-1}^T)^T\|_2 \\ &= \sum_{m=1}^{M-b_e+1} \xi_m^{(0)} \|(\mathbf{u}_m^T, \mathbf{u}_{m+1}^T, \dots, \mathbf{u}_{m+b_e-1}^T)^T\|_2 \\ &= \sum_{m=1}^{M-b_e+1} \xi_m^{(0)} \|(\mathbf{z}_{m,1}^{(2)T}, \mathbf{z}_{m,2}^{(2)T}, \dots, \mathbf{z}_{m,L}^{(2)T})^T\|_2 = \|\mathbf{Z}_2\|_{1, \xi_0}^{G_0}, \end{aligned}$$

where $G_0 := (\mathcal{G}_m^{(0)})$ s.t. $\mathcal{G}_m^{(0)} := \{(b_e(m-1) + i, k)\}_{i=1,2,\dots,b_e}^{k=1,2,\dots,L}$ and $\xi_0 := (\xi_m^{(0)}) \in \mathbb{R}_+^M$ is some vector s.t. $\xi_m^{(0)} = 0$ for $m = M - b_e + 2, M - b_e + 3, \dots, M$. Therefore, the current work evaluates the group-sparsity of $\mathbf{Z}_2$ in detail for each frequency by (22), while the previous work roughly evaluated it by collecting all frequency components. In addition, if $\|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1}$ and $\|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2}$ are small, then $\|\mathbf{Z}_2\|_{1, \xi_0}^{G_0}$ will automatically be small. Thus, we do not use our previous cost $\|\mathbf{Z}_2\|_{1, \xi_0}^{G_0}$ in this paper.

C. The Proposed Nonlinear Beamforming (Formulation I)

Based on the data-fidelity term in (18) and the two weighted group ℓ_1 -norms (21) and (22), we reconstruct $\mathbf{U}$ from $\mathbf{Y}$ by solving a convex optimization problem

$$\begin{aligned} \underset{\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2}{\text{minimize}} \quad & \frac{1}{2} \|\mathbf{F} \mathbf{Y}^T - \mathbf{U} \mathbf{A}^T\|_F^2 + \|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1} + \|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2} \\ \text{subject to} \quad & \mathbf{Z}_1 = \mathbf{B}_f \mathbf{U} \text{ and } \mathbf{Z}_2 = \mathbf{B}_e \mathbf{U}^T, \end{aligned} \quad (23)$$

with the use of ADMM [21]–[24] (see Appendix for details of ADMM). By defining two convex functions g and h in (40) as

$$g(\mathbf{Z}_1, \mathbf{Z}_2) := \|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1} + \|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2}$$

and

$$h(\mathbf{U}) := \frac{1}{2} \|\mathbf{F} \mathbf{Y}^T - \mathbf{U} \mathbf{A}^T\|_F^2,$$

the problem in (23) is expressed as an ADMM-form:

$$\begin{aligned} \underset{\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2}{\text{minimize}} \quad & g(\mathbf{Z}_1, \mathbf{Z}_2) + h(\mathbf{U}) \\ \text{subject to} \quad & (\mathbf{Z}_1, \mathbf{Z}_2) = \mathcal{L}(\mathbf{U}) := (\mathbf{B}_f \mathbf{U}, \mathbf{B}_e \circ \mathcal{T}(\mathbf{U})), \end{aligned} \quad (24)$$

where $\mathcal{T}$ is the transpose operator, $\circ$ denotes the composition of two mappings, and $\mathcal{L} : \mathbb{C}^{L \times M} \rightarrow \mathbb{C}^{b_f L \times M} \times \mathbb{C}^{b_e M \times L}$ is a linear mapping that maps $\mathbf{U}$ to $(\mathbf{Z}_1, \mathbf{Z}_2)$ with $\mathbf{B}_f$ and $\mathbf{B}_e$.

The ADMM iterations for (24) are given as follows. On the first line in (41), since the function $g(\mathbf{Z}_1, \mathbf{Z}_2)$ is divided into two parts $\|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1}$ and $\|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2}$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are updated by

$$\left. \begin{aligned} \mathbf{Z}_1^{(i)} &= \text{prox}_{\gamma \|\cdot\|_{1, \Xi_1}^{G_1}} (\mathbf{B}_f \mathbf{U}^{(i)} + \mathbf{V}_1^{(i)}) \\ \mathbf{Z}_2^{(i)} &= \text{prox}_{\gamma \|\cdot\|_{1, \Xi_2}^{G_2}} (\mathbf{B}_e \mathbf{U}^{(i)T} + \mathbf{V}_2^{(i)}) \end{aligned} \right\} \quad (25)$$

with the use of the *proximity operators* of the group ℓ_1 -norms in (42), where $\gamma > 0$ and $\mathbf{V}_1 \in \mathbb{C}^{b_f L \times M}$ and $\mathbf{V}_2 \in \mathbb{C}^{b_e M \times L}$ are dual variables. Then, on the second line in (41), we have

$$\left. \begin{aligned} \tilde{\mathbf{V}}_1^{(i)} &= \mathbf{V}_1^{(i)} + \mathbf{B}_f \mathbf{U}^{(i)} - \mathbf{Z}_1^{(i)} \\ \tilde{\mathbf{V}}_2^{(i)} &= \mathbf{V}_2^{(i)} + \mathbf{B}_e \mathbf{U}^{(i)T} - \mathbf{Z}_2^{(i)} \end{aligned} \right\}. \quad (26)$$

On the third line in (41), since $\mathbf{U}$ is updated as the solution to a least squares problem, the solution $\mathbf{U}^{(i+1)}$ has to satisfy

$$\begin{aligned} & \gamma \mathbf{U}^{(i+1)} \mathbf{A}^T \bar{\mathbf{A}} + \mathcal{L}^* \circ \mathcal{L}(\mathbf{U}^{(i+1)}) \\ &= \gamma \mathbf{F} \mathbf{Y}^T \bar{\mathbf{A}} + \mathcal{L}^*(\mathbf{Z}_1^{(i)} - \tilde{\mathbf{V}}_1^{(i)}, \mathbf{Z}_2^{(i)} - \tilde{\mathbf{V}}_2^{(i)}), \end{aligned} \quad (27)$$

with the adjoint operator $\mathcal{L}^*: \mathbb{C}^{b_f L \times M} \times \mathbb{C}^{b_e M \times L} \rightarrow \mathbb{C}^{L \times M}$

$$\mathcal{L}^*(\mathbf{Z}_1, \mathbf{Z}_2) := \mathbf{B}_f^T \mathbf{Z}_1 + \mathcal{T}(\mathbf{B}_e^T \mathbf{Z}_2) = \mathbf{B}_f^T \mathbf{Z}_1 + \mathbf{Z}_2^T \mathbf{B}_e.$$

Moreover, the composite mapping $\mathcal{L}^* \circ \mathcal{L}$ can be computed by

$$\begin{aligned} \mathcal{L}^* \circ \mathcal{L}(\mathbf{U}) &= \mathcal{L}^*(\mathbf{B}_f \mathbf{U}, \mathbf{B}_e \mathbf{U}^T) \\ &= \mathbf{B}_f^T \mathbf{B}_f \mathbf{U} + \mathcal{T}(\mathbf{B}_e^T \mathbf{B}_e \mathbf{U}^T) \\ &= b_f \mathbf{I}_L \mathbf{U} + \mathcal{T}(b_e \mathbf{I}_M \mathbf{U}^T) = (b_f + b_e) \mathbf{U} \end{aligned} \quad (28)$$

By substituting (28) into (27), we have

$$\begin{aligned} & \mathbf{U}^{(i+1)} (\gamma \mathbf{A}^T \bar{\mathbf{A}} + (b_f + b_e) \mathbf{I}_M) \\ &= \gamma \mathbf{F} \mathbf{Y}^T \bar{\mathbf{A}} + \mathbf{B}_f^T (\mathbf{Z}_1^{(i)} - \tilde{\mathbf{V}}_1^{(i)}) + (\mathbf{Z}_2^{(i)} - \tilde{\mathbf{V}}_2^{(i)})^T \mathbf{B}_e, \end{aligned} \quad (29)$$

and $\mathbf{U}^{(i+1)}$ is obtained by applying $(\gamma \mathbf{A}^T \bar{\mathbf{A}} + (b_f + b_e) \mathbf{I}_M)^{-1}$ to (29) from the right side. Finally, on the fourth line in (42), $\mathbf{V}_1$ and $\mathbf{V}_2$ are updated, with the use of $\rho^{(i+1)} \in [0, 2]$, by

$$\begin{aligned} \mathbf{V}_1^{(i+1)} &= \tilde{\mathbf{V}}_1^{(i)} + (\rho^{(i+1)} - 1) (\mathbf{B}_f \mathbf{U}^{(i+1)} - \mathbf{Z}_1^{(i)}) \\ \mathbf{V}_2^{(i+1)} &= \tilde{\mathbf{V}}_2^{(i)} + (\rho^{(i+1)} - 1) (\mathbf{B}_e \mathbf{U}^{(i+1)T} - \mathbf{Z}_2^{(i)}) \end{aligned} \quad (30)$$

By repeating (25), (26), (29) and (30) until a convergence condition is met, the solution to (24) is obtained as an estimate $\hat{\mathbf{U}}$.

D. The Proposed Nonlinear Beamforming (Formulation II)

In addition to the optimization problem in (23), we propose another optimization problem in which the relation between $\mathbf{U}$ and $(\mathbf{Z}_1, \mathbf{Z}_2)$ is modified to further promote group-sparsities. To increase almost-zero vectors $\mathbf{z}_{k,m}^{(1)} \approx \mathbf{0}$ and $\mathbf{z}_{m,k}^{(2)} \approx \mathbf{0}$, i.e., to make $\mathbf{Z}_1$ and $\mathbf{Z}_2$ more group-sparse, we use the *latent group lasso* formulation [28]–[31]. Specifically, we propose to solve

$$\begin{aligned} & \underset{\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2}{\text{minimize}} \quad \frac{1}{2} \|\mathbf{F} \mathbf{Y}^T - \mathbf{U} \mathbf{A}^T\|_F^2 + \|\mathbf{Z}_1\|_{1, \Xi_1}^{G_1} + \|\mathbf{Z}_2\|_{1, \Xi_2}^{G_2} \\ & \text{subject to} \quad \mathbf{B}_f^T \mathbf{Z}_1 = \mathbf{U} \text{ and } \mathbf{B}_e^T \mathbf{Z}_2 = \mathbf{U}^T, \end{aligned} \quad (31)$$

where the weight matrix $\Xi_2 = (\xi_{m,k}^{(2)})$ satisfies $\xi_{m,k}^{(2)} = \infty$ for $m = M - b_e + 2, M - b_e + 3, \dots, M$, i.e., $\mathbf{z}_{m,k}^{(2)} = \mathbf{0}$ has to hold for $m = M - b_e + 2, M - b_e + 3, \dots, M$. Differently from the problem in (23), $\mathbf{U}$ and $\mathbf{U}^T$ in (31) are constructed by applying $\mathbf{B}_f^T$ and $\mathbf{B}_e^T$ to $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively. Such a latent group lasso formulation can provide more accurate evaluations of the group-sparsity for *fixed-size overlapping groups* [30].

To solve the problem in (31) with ADMM, we define five auxiliary variables $\mathbf{\Lambda}_1 \in \mathbb{C}^{b_f L \times M}$, $\mathbf{\Lambda}_2 \in \mathbb{C}^{b_e M \times L}$, and $\mathbf{\Gamma}_n \in \mathbb{C}^{L \times M}$ ($n = 1, 2, 3$). Then, define functions g and h in (40) as

$$g(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) := \|\mathbf{\Lambda}_1\|_{1, \Xi_1}^{G_1} + \|\mathbf{\Lambda}_2\|_{1, \Xi_2}^{G_2} + \iota(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3)$$

and

$$h(\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2) := \frac{1}{2} \|\mathbf{F} \mathbf{Y}^T - \mathbf{U} \mathbf{A}^T\|_F^2,$$

with the use of an *indicator function* ι s.t. $\iota(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) := 0$ if $\mathbf{\Gamma}_1 = \mathbf{\Gamma}_2 = \mathbf{\Gamma}_3$ and $\iota(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) := \infty$ otherwise. Moreover, by defining an linear mapping $\mathcal{L}$ in (40) as

$$\mathcal{L}(\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2) := (\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{U}, \mathbf{B}_f^T \mathbf{Z}_1, \mathcal{T}(\mathbf{B}_e^T \mathbf{Z}_2)),$$

the problem in (31) is expressed as an ADMM-form:

$$\begin{aligned} & \underset{\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3}{\text{minimize}} \quad g(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) + h(\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2) \\ & \text{subject to} \quad (\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) = \mathcal{L}(\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2). \end{aligned} \quad (32)$$

Note that although the function $h(\mathbf{U}, \mathbf{Z}_1, \mathbf{Z}_2)$ actually depends only on $\mathbf{U}$, we added $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as arguments to make it easier to understand that the problem in (32) as the ADMM-form.

The ADMM iterations for (32) are given as follows. On the first line in (41), since $g(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3)$ is divided into three parts $\|\mathbf{\Lambda}_1\|_{1, \Xi_1}^{G_1}$, $\|\mathbf{\Lambda}_2\|_{1, \Xi_2}^{G_2}$, and $\iota(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3)$, the variables $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are updated, with the use of some $\gamma > 0$, by

$$\begin{aligned} \mathbf{\Lambda}_1^{(i)} &= \text{prox}_{\gamma \|\cdot\|_{1, \Xi_1}^{G_1}} (\mathbf{Z}_1^{(i)} + \mathbf{V}_1^{(i)}) \\ \mathbf{\Lambda}_2^{(i)} &= \text{prox}_{\gamma \|\cdot\|_{1, \Xi_2}^{G_2}} (\mathbf{Z}_2^{(i)} + \mathbf{V}_2^{(i)}) \end{aligned} \quad (33)$$

and $\mathbf{\Gamma}_n$ ($n = 1, 2, 3$) are updated, with the use of the proximity operators of the indicator function in (43), by

$$\mathbf{\Gamma}_n^{(i)} = \frac{1}{3} (\mathbf{U}^{(i)} + \mathbf{B}_f^T \mathbf{Z}_1^{(i)} + \mathbf{Z}_2^{(i)T} \mathbf{B}_e + \mathbf{V}_3^{(i)} + \mathbf{V}_4^{(i)} + \mathbf{V}_5^{(i)}), \quad (34)$$

where $\mathbf{V}_1 \in \mathbb{C}^{b_f L \times M}$, $\mathbf{V}_2 \in \mathbb{C}^{b_e M \times L}$, and $\mathbf{V}_n \in \mathbb{C}^{L \times M}$ ($n = 3, 4, 5$) are dual variables. On the second line in (41), we have

$$\left. \begin{aligned} \tilde{\mathbf{V}}_1^{(i)} &= \mathbf{V}_1^{(i)} + \mathbf{Z}_1^{(i)} - \mathbf{\Lambda}_1^{(i)} \\ \tilde{\mathbf{V}}_2^{(i)} &= \mathbf{V}_2^{(i)} + \mathbf{Z}_2^{(i)} - \mathbf{\Lambda}_2^{(i)} \\ \tilde{\mathbf{V}}_3^{(i)} &= \mathbf{V}_3^{(i)} + \mathbf{U}^{(i)} - \mathbf{\Gamma}_1^{(i)} \\ \tilde{\mathbf{V}}_4^{(i)} &= \mathbf{V}_4^{(i)} + \mathbf{B}_f^T \mathbf{Z}_1^{(i)} - \mathbf{\Gamma}_2^{(i)} \\ \tilde{\mathbf{V}}_5^{(i)} &= \mathbf{V}_5^{(i)} + \mathbf{Z}_2^{(i)T} \mathbf{B}_e - \mathbf{\Gamma}_3^{(i)} \end{aligned} \right\}. \quad (35)$$

On the third line in (41), $\mathbf{U}$, $\mathbf{Z}_1$, and $\mathbf{Z}_2$ can be updated as the solutions to three different least squares problems since the linear mapping $\mathcal{L}$ does not mix $\mathbf{U}$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$. Specifically, from

$$\mathcal{L}^*(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3) := (\mathbf{\Gamma}_1, \mathbf{\Lambda}_1 + \mathbf{B}_f \mathbf{\Gamma}_2, \mathbf{\Lambda}_2 + \mathbf{B}_e \mathbf{\Gamma}_3^T),$$

$\mathbf{U}$ is updated by

$$\mathbf{U}^{(i+1)} = (\gamma \mathbf{F} \mathbf{Y}^T \bar{\mathbf{A}} + \mathbf{\Gamma}_1^{(i)} - \tilde{\mathbf{V}}_3^{(i)}) (\gamma \mathbf{A}^T \bar{\mathbf{A}} + \mathbf{I}_M)^{-1}. \quad (36)$$

$\mathbf{Z}_1$ and $\mathbf{Z}_2$ are respectively updated by

$$\mathbf{Z}_1^{(i+1)} = (\mathbf{I}_{b_f L} + \mathbf{B}_f \mathbf{B}_f^T)^{-1} (\mathbf{\Lambda}_1^{(i)} - \tilde{\mathbf{V}}_1^{(i)} + \mathbf{B}_f (\mathbf{\Gamma}_2^{(i)} - \tilde{\mathbf{V}}_4^{(i)})), \quad (37)$$

and

$$\mathbf{Z}_2^{(i+1)} = (\mathbf{I}_{b_e M} + \mathbf{B}_e \mathbf{B}_e^T)^{-1} (\mathbf{\Lambda}_2^{(i)} - \tilde{\mathbf{V}}_2^{(i)} + \mathbf{B}_e (\mathbf{\Gamma}_3^{(i)} - \tilde{\mathbf{V}}_5^{(i)})^T). \quad (38)$$

Note that, if we use the *matrix inversion lemma* [32], [33], then $(\mathbf{I}_{b_f L} + \mathbf{B}_f \mathbf{B}_f^T)^{-1}$ in (37) can be computed by

$$\begin{aligned} (\mathbf{I}_{b_f L} + \mathbf{B}_f \mathbf{B}_f^T)^{-1} &= \mathbf{I}_{b_f L} - \mathbf{B}_f (\mathbf{I}_L + \mathbf{B}_f^T \mathbf{B}_f)^{-1} \mathbf{B}_f^T \\ &= \mathbf{I}_{b_f L} - \frac{1}{b_f + 1} \mathbf{B}_f \mathbf{B}_f^T, \end{aligned}$$

Fig. 2. Real PAWR data observed at Osaka University on March 30th, 2014, and a flowchart on the average signal power density used in our simulations. Note that this real PAWR data is based on the results of FR beamforming.

and $(\mathbf{I}_{b_e M} + \mathbf{B}_e \mathbf{B}_e^T)^{-1}$ in (38) can also be computed similarly. Finally, on the fourth line in (41), $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ and $\mathbf{V}_5$ are updated, with the use of $\rho^{(i+1)} \in [0, 2]$, by

$$\left. \begin{aligned} \mathbf{V}_1^{(i+1)} &= \tilde{\mathbf{V}}_1^{(i)} + (\rho^{(i+1)} - 1)(\mathbf{Z}_1^{(i+1)} - \mathbf{\Lambda}_1^{(i)}) \\ \mathbf{V}_2^{(i+1)} &= \tilde{\mathbf{V}}_2^{(i)} + (\rho^{(i+1)} - 1)(\mathbf{Z}_2^{(i+1)} - \mathbf{\Lambda}_2^{(i)}) \\ \mathbf{V}_3^{(i+1)} &= \tilde{\mathbf{V}}_3^{(i)} + (\rho^{(i+1)} - 1)(\mathbf{U}^{(i+1)} - \mathbf{\Gamma}_1^{(i)}) \\ \mathbf{V}_4^{(i+1)} &= \tilde{\mathbf{V}}_4^{(i)} + (\rho^{(i+1)} - 1)(\mathbf{B}_f^T \mathbf{Z}_1^{(i+1)} - \mathbf{\Gamma}_2^{(i)}) \\ \mathbf{V}_5^{(i+1)} &= \tilde{\mathbf{V}}_5^{(i)} + (\rho^{(i+1)} - 1)(\mathbf{Z}_2^{(i+1)T} \mathbf{B}_e - \mathbf{\Gamma}_3^{(i)}) \end{aligned} \right\}. \quad (39)$$

By repeating (33)–(39) until a convergence condition is met, the solution to (32) is obtained as an estimate $\hat{\mathbf{U}}$.

V. NUMERICAL SIMULATIONS

A. Simulation Settings

To show the effectiveness of the proposed nonlinear beamforming, we conducted simulations based on the real reflection intensity, observed by the PAWR at Osaka University, in Fig. 2. We conducted the following simulations on the basis of [34]. At the range $r = 7.5$ [km], we extracted 55 real reflection intensity data between $\theta_{\min} = -15$ [deg] and $\theta_{\max} = 30$ [deg]. The target reflection intensity $\hat{Z}_m$ [dBZ] at each θ_m was generated by cubic spline interpolation of these 55 data followed by adding Gaussian random numbers of variance 9. Note that the random numbers were needed to create the high-resolution precipitation profile. Since the minimum value of the original data in Fig. 2 is 10 [dBZ] in the area where raindrops exist, we defined the average signal power density P_m at each θ_m by

$$P_m = \begin{cases} \frac{55}{M} \times 10^{\hat{Z}_m/10} & \text{if } \hat{Z}_m \geq 10, \\ \frac{55}{M} \times 10^{-9} & \text{if } \hat{Z}_m < 10. \end{cases}$$

We used $\lambda = 31.81$ [mm], $d = 16.5$ [mm], and $T = 0.4$ [ms]. As a result, the Nyquist frequency was $\frac{1}{2T} = 1250$ [Hz] and the Nyquist Doppler velocity was $\frac{\lambda}{4T} \approx 19.88$ [m/s]. We used the Gaussian function in (19) as the power spectral density $S_m(f)$ at each θ_m , and we defined the mean Doppler frequency by

$$\mu_{f_m} = -\frac{2}{\lambda} \left(20 \sin\left(\frac{2\pi m}{M}\right) + n_m \right) [\text{Hz}],$$

where $n_m \in \mathbb{R}$ is a Gaussian random number of variance 1. The Doppler frequency width at each θ_m was simply fixed to $\sigma_{f_m} = \frac{4}{\lambda} \approx 125.7$ [Hz] which corresponds to $\sigma_{v_m} = 2$ [m/s].

The exact autocorrelation function $R_{x_m}(\tau)$ of the backscattered signal $x_m(t)$ at each θ_m can be computed as in (20), and the covariance matrix of each random vector $\tilde{\mathbf{x}}_m$ is given by

$$\mathbf{R}_m := E[\tilde{\mathbf{x}}_m \tilde{\mathbf{x}}_m^H] = \begin{bmatrix} R_{x_m}(0) & \overline{R_{x_m}(T)} & \cdots & \overline{R_{x_m}((L-1)T)} \\ R_{x_m}(T) & R_{x_m}(0) & \cdots & \overline{R_{x_m}((L-2)T)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{x_m}((L-1)T) & R_{x_m}((L-2)T) & \cdots & R_{x_m}(0) \end{bmatrix}.$$

Based on [7], [35], each target signal $\tilde{\mathbf{x}}_m$ was generated from a circularly-symmetric complex Gaussian distribution⁶

$$\tilde{q}_m(\tilde{\mathbf{x}}) = \frac{1}{\pi^L \det(\mathbf{R}_m)} e^{-\tilde{\mathbf{x}}^H \mathbf{R}_m^{-1} \tilde{\mathbf{x}}}.$$

In (10), the variance of the noise ϵ_l was set to $\sigma_\epsilon^2 = 2.5$.

We set the number of antenna elements to $N = 128$. For $M = 110$ and $M = 166$, we compared the proposed nonlinear beamforming, NL-I in (23) and NL-II in (31), with LS in (12),⁷ FR in (13), and MMSE beamforming in (16), in cases of $L = 20$ and $L = 60$. Note that CP beamforming in (14) cannot be used since $L < N$ holds in both cases. When $M = 110$, the number of subintervals where signals exist⁸ was $K = 93 < N$. On the other hand, when $M = 166$, that was $K = 142 > N$, and hence it is difficult to estimate $\mathbf{X}$ or $\mathbf{U}$. Moreover, in both cases, the condition number of $\mathbf{A}^H \mathbf{A}$, denoted by $\text{cn}(\mathbf{A}^H \mathbf{A})$ in Table I, is extremely large, and the beamforming problem based on the observation model in (10) is very ill-conditioned.

The weights of NL-I in (23) were set to $\xi_{k,m}^{(1)} = 1.5 \frac{N}{\sqrt{b_f}} \times 10^{-3}$ and $\xi_{m,k}^{(2)} = 2.5 \frac{N}{\sqrt{b_e}} \times 10^{-4}$, the group size b_f was set to $b_f = 3$ for $L = 20$ and $b_f = 8$ for $L = 60$, and the group size b_e was set to $b_e = 5$ for $M = 110$ and $b_e = 7$ for $M = 166$. The weights of NL-II in (31) were set to $\xi_{k,m}^{(1)} = 6.5N\sqrt{b_f} \times 10^{-4}$ and $\xi_{m,k}^{(2)} = 7.5N\sqrt{b_e} \times 10^{-5}$, the group size b_f was set to $b_f = 5$ for $L = 20$ and $b_f = 13$ for $L = 60$, and the group size b_e was set to $b_e = 6$ for $M = 110$ and $b_e = 7$ for $M = 166$.

We evaluated the estimate $\hat{\mathbf{X}}$ or $\hat{\mathbf{U}} = \mathbf{F} \hat{\mathbf{X}}^T$ of each method by the following normalized mean square error (NMSE).

- NMSE for $x_{m,l}$ or $u_{k,m}$

$$\begin{aligned} \frac{\|\mathbf{X} - \hat{\mathbf{X}}\|_F^2}{\|\mathbf{X}\|_F^2} &= \frac{\sum_{m=1}^M \sum_{l=1}^L |x_{m,l} - \hat{x}_{m,l}|^2}{\sum_{m=1}^M \sum_{l=1}^L |x_{m,l}|^2} \\ &= \frac{\|\mathbf{U} - \hat{\mathbf{U}}\|_F^2}{\|\mathbf{U}\|_F^2} = \frac{\sum_{k=1}^L \sum_{m=1}^M |u_{k,m} - \hat{u}_{k,m}|^2}{\sum_{k=1}^L \sum_{m=1}^M |u_{k,m}|^2}. \end{aligned}$$

Moreover, we evaluated the estimated periodograms ($|\hat{u}_{k,m}|^2$) by the following normalized mean absolute errors (NMAEs).

⁶In this model, the variance $V[|u_{k,m}|^2] := E[|u_{k,m}|^2] - E[|u_{k,m}|^2]^2$ of the periodogram satisfies $V[|u_{k,m}|^2] = (E[|u_{k,m}|^2])^2$, which leads to

$$\lim_{L \rightarrow \infty} V[|u_{k,m}|^2] = \left(S_m^{(d)} \left(\frac{k-1-L/2}{LT} \right) \right)^2,$$

i.e., the variance does not converge to zero when L approaches infinity [36].

⁷To avoid the numerical instability in the computation of the pseudoinverse $\mathbf{A}^\dagger$, we truncated the singular values of $\mathbf{A}$ that are smaller than 0.005.

⁸We defined K as the number of the indices m s.t. $\hat{Z}_m \geq 10$.

TABLE I
AVERAGES OF THE NORMALIZED ERRORS OF THE ESTIMATED SIGNALS AND THE ESTIMATED PERIODOGRAMS OF EACH METHOD IN 100 TRIALS

Simulation Settings			Evaluation Metric \ Method	LS	FR	MMSE	NL-I	NL-II
$N = 128$ $\lambda = 31.81$ [mm] $d = 16.5$ [mm] $T = 0.4$ [ms] $r = 7.5$ [km] $c = 300000$ [km/s] $\theta_{\min} = -15$ [deg] $\theta_{\max} = 30$ [deg]	$M = 110$ $K = 93$ $\Delta\theta \approx 0.4091$ [deg] $\text{cn}(\mathbf{A}^H \mathbf{A}) > 10^{30}$ (we used the same $S_m(f)$ for $L = 20$ and $L = 60$)	$L = 20$	NMSE for $u_{k,m}$	0.6377	1.1060	0.4189	0.2608	0.2556
			NMAE for $ u_{k,m} ^2$	0.9231	1.3744	0.6372	0.5066	0.4931
			NMAE for $E[u_{k,m} ^2]$	0.9185	1.6742	0.9726	0.7223	0.7143
			NMAE for $S_m^{(d)}(\frac{k-1-L/2}{LT})$	0.9546	1.6975	1.0039	0.7428	0.7366
		$L = 60$	NMSE for $u_{k,m}$	0.6324	1.1057	0.4271	0.2328	0.2265
			NMAE for $ u_{k,m} ^2$	0.9263	1.3621	0.6549	0.4787	0.4669
			NMAE for $E[u_{k,m} ^2]$	0.9222	1.6341	1.0041	0.6845	0.6808
			NMAE for $S_m^{(d)}(\frac{k-1-L/2}{LT})$	0.9338	1.6407	1.0131	0.6899	0.6866
	$M = 166$ $K = 142$ $\Delta\theta \approx 0.2711$ [deg] $\text{cn}(\mathbf{A}^H \mathbf{A}) = \infty$ (we used the same $S_m(f)$ for $L = 20$ and $L = 60$)	$L = 20$	NMSE for $u_{k,m}$	0.7324	2.1567	1.4603	0.6012	0.6078
			NMAE for $ u_{k,m} ^2$	0.8670	2.4415	1.6916	0.7816	0.7813
			NMAE for $E[u_{k,m} ^2]$	0.8184	2.4541	1.7487	0.7859	0.7828
			NMAE for $S_m^{(d)}(\frac{k-1-L/2}{LT})$	0.8639	2.4797	1.7845	0.8069	0.8060
		$L = 60$	NMSE for $u_{k,m}$	0.7261	2.1812	1.5514	0.5694	0.5664
			NMAE for $ u_{k,m} ^2$	0.8733	2.4994	1.8467	0.7600	0.7556
			NMAE for $E[u_{k,m} ^2]$	0.8292	2.5310	1.9364	0.7487	0.7454
			NMAE for $S_m^{(d)}(\frac{k-1-L/2}{LT})$	0.8473	2.5390	1.9474	0.7559	0.7530

- NMAE for $|u_{k,m}|^2$

$$\frac{\sum_{k=1}^L \sum_{m=1}^M ||u_{k,m}|^2 - |\hat{u}_{k,m}|^2|}{\sum_{k=1}^L \sum_{m=1}^M |u_{k,m}|^2}.$$

- NMAE for $E[|u_{k,m}|^2]$

$$\frac{\sum_{k=1}^L \sum_{m=1}^M |E[|u_{k,m}|^2] - |\hat{u}_{k,m}|^2|}{\sum_{k=1}^L \sum_{m=1}^M E[|u_{k,m}|^2]}.$$

- NMAE for $S_m^{(d)}(\frac{k-1-L/2}{LT})$

$$\frac{\sum_{k=1}^L \sum_{m=1}^M |S_m^{(d)}(\frac{k-1-L/2}{LT}) - |\hat{u}_{k,m}|^2|}{\sum_{k=1}^L \sum_{m=1}^M S_m^{(d)}(\frac{k-1-L/2}{LT})}.$$

B. Simulation Results

Table I summarizes the average, for each situation and each method, of the normalized errors in 100 trials, where the power spectral density $S_m(f)$ at each θ_m was fixed throughout all the 100 trials. From Table I, we can see that the proposed methods, NL-I and NL-II, reduced NMSE by 59–64% from LS, 76–80% from FR, and 38–47% from MMSE when $M = 110$, while by 17–22% from LS, 72–74% from FR, and 58–63% from MMSE when $M = 166$. Thus, the proposed methods greatly improved the estimation accuracy of $\mathbf{U}$. In addition, we find that MMSE beamforming estimated $\mathbf{X}$ with relatively high accuracy when $M = 110$ while the estimation accuracy significantly degraded due to the occurrence of sidelobes when $M = 166$.

On the estimated periodograms ($|\hat{u}_{k,m}|^2$), we can see that the proposed methods reduced NMAE for $|u_{k,m}|^2$ by 45–50% from LS, 63–66% from FR, and 20–29% from MMSE when $M = 110$, while by 10–13% from LS, 68–70% from FR, and 54–59% from MMSE when $M = 166$. Moreover, the proposed methods reduced NMAEs for $E[|u_{k,m}|^2]$ and $S_m^{(d)}(\frac{k-1-L/2}{LT})$ by 21–26% from LS, 56–58% from FR, and 26–32% from MMSE when $M = 110$, while by 4–11% from LS, 67–71% from FR, and 55–62% from MMSE when $M = 166$. Thus, the proposed methods obtained the most accurate periodograms.

On the number of pulses, all the linear methods, LS, FR, and MMSE, did not necessarily reduce NMSE and NMAEs, even if L was changed from 20 to 60. On the other hand, the proposed methods, NL-I and NL-II, could reduce NMSE and NMAEs by changing L from 20 to 60. This is because, as L increases, $E[|u_{k,m}|^2]$ approaches $S_m^{(d)}(\frac{k-1-L/2}{LT})$ and $\mathbf{U}$ becomes more group-sparse, which is ideal for the proposed methods.

Figures 3, 4, 5, and 6 show examples of beamforming results for $(M, L) = (110, 20), (110, 60), (166, 20), (166, 60)$, where (a) shows the power spectral densities, (b) shows the expected values of periodograms, (c) shows target periodograms, and (d), (e), (f), (g) and (h) show estimated periodograms by LS, FR, MMSE, NL-I and NL-II beamforming, respectively. From Figs. 3(a)–3(c), 4(a)–4(c), 5(a)–5(c), and 6(a)–6(c), we can see that $E[|u_{k,m}|^2]$ in (b) approached $S_m^{(d)}(\frac{k-1-L/2}{LT})$ in (a), and the target periodograms ($|u_{k,m}|^2$) in (c) became more group-sparse as the number L of pulses increased from 20 to 60.

When $M = 110$, from Figs. 3(d) and 4(d), LS beamforming diffused the energy of the backscattered signals to all elevation angles, and these results cannot be used for the weather observation. From Figs. 3(e) and 4(e), FR beamforming failed in the estimation particularly from 23 [deg] to 27 [deg] since the signal power should be almost 0 in this interval. From Figs. 3(f) and 4(f), MMSE beamforming obtained the periodograms with relatively high accuracy but the energy of the target signals was diffused mainly in the elevation direction. On the other hand, from Figs. 3(g), 3(h), 4(g), and 4(h), we can confirm that the proposed nonlinear methods, NL-I and NL-II, could obtain the accurate periodograms with less energy diffusion. Though the results by NL-I and NL-II were very similar, NL-II could give the slightly higher-resolution periodograms than NL-I.

When $M = 166$, each signal power P_m decreases compared to the case of $M = 110$ since the width $\Delta\theta$ of each subinterval $[\theta_m - \frac{\Delta\theta}{2}, \theta_m + \frac{\Delta\theta}{2}]$ becomes narrower as M increases. Hence, if we increase the number M of the subintervals, then not only the number of variables to be estimated increases, but also the signal-to-noise ratio decreases. Even in such a severe situation,

Fig. 3. Comparison of the power spectral densities, true periodograms, and each beamforming result in case of $M = 110$ and $L = 20$: (a) $S_m^{(d)}(\frac{k-1-L/2}{LT})$, (b) $E[|u_{k,m}|^2]$, (c) $|u_{k,m}|^2$, (d) $|\hat{u}_{k,m}|^2$ by LS, (e) $|\hat{u}_{k,m}|^2$ by FR, (f) $|\hat{u}_{k,m}|^2$ by MMSE, (g) $|\hat{u}_{k,m}|^2$ by NL-I, and (h) $|\hat{u}_{k,m}|^2$ by NL-II beamforming.

Fig. 4. Comparison of the power spectral densities, true periodograms, and each beamforming result in case of $M = 110$ and $L = 60$: (a) $S_m^{(d)}(\frac{k-1-L/2}{LT})$, (b) $E[|u_{k,m}|^2]$, (c) $|u_{k,m}|^2$, (d) $|\hat{u}_{k,m}|^2$ by LS, (e) $|\hat{u}_{k,m}|^2$ by FR, (f) $|\hat{u}_{k,m}|^2$ by MMSE, (g) $|\hat{u}_{k,m}|^2$ by NL-I, and (h) $|\hat{u}_{k,m}|^2$ by NL-II beamforming.

the proposed nonlinear methods, NL-I and NL-II, obtained the most accurate periodograms as shown in Figs. 5(g), 5(h), 6(g), and 6(h). As in the case of $M = 110$, NL-I and NL-II obtained the high-resolution periodograms without the energy diffusion, while the conventional linear methods, LS, FR and MMSE, led to the blurring and the overestimation. Thus, the proposed nonlinear beamforming greatly improved the spatial resolution.

VI. CONCLUSION

In this paper, we proposed a nonlinear beamforming method for a PAWR. Differently from a standard radar which observes point targets, the PAWR receives a lot of backscattered signals from distributed targets, and thus the spatial resolution of the linear beamforming methods is limited. To acquire the high-resolution precipitation profile, we used two characteristics of periodograms of the backscattered signals. One is the narrow bandwidth of each periodogram, and the other is the similarity between the adjacent periodograms. Both characteristics were expressed as group-sparsities of the DFT coefficient matrix of

the backscattered signals. We proposed to reconstruct the DFT coefficient matrix by minimizing a convex cost function which consists of one data-fidelity term and two group ℓ_1 -norms with ADMM. Numerical simulations based on the real data showed that, compared to the linear methods, the proposed nonlinear beamforming obtains the high-resolution precipitation profile.

APPENDIX

ALTERNATING DIRECTION METHOD OF MULTIPLIERS

Let us consider the following convex optimization problem

$$\underset{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z}}{\text{minimize}} \quad g(\mathbf{z}) + h(\mathbf{x}) \quad \text{subject to} \quad \mathbf{z} = \mathcal{L}(\mathbf{x}), \quad (40)$$

where $\mathcal{X}$ and $\mathcal{Z}$ are finite-dimensional real Hilbert spaces with the standard inner products, $\mathcal{L} : \mathcal{X} \rightarrow \mathcal{Z}$ is a linear mapping, and both functions $g : \mathcal{Z} \rightarrow \mathbb{R} \cup \{\infty\}$ and $h : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ are *proper*, *lower semicontinuous*, and *convex*.⁹ The alternating

⁹A function $h : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ is called *proper*, *lower semicontinuous*, and *convex* if $\text{dom}(h) := \{\mathbf{x} \in \mathcal{X} \mid h(\mathbf{x}) < \infty\}$ is nonempty, $\text{lev}_{\leq \alpha}(h) := \{\mathbf{x} \in \mathcal{X} \mid h(\mathbf{x}) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$, and $h(\beta \mathbf{x}_1 + (1 - \beta) \mathbf{x}_2) \leq \beta h(\mathbf{x}_1) + (1 - \beta) h(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and all $\beta \in (0, 1)$, respectively.

Fig. 5. Comparison of the power spectral densities, true periodograms, and each beamforming result in case of $M = 166$ and $L = 20$: (a) $S_m^{(d)}(\frac{k-1-L/2}{LT})$, (b) $E[|u_{k,m}|^2]$, (c) $|u_{k,m}|^2$, (d) $|\hat{u}_{k,m}|^2$ by LS, (e) $|\hat{u}_{k,m}|^2$ by FR, (f) $|\hat{u}_{k,m}|^2$ by MMSE, (g) $|\hat{u}_{k,m}|^2$ by NL-I, and (h) $|\hat{u}_{k,m}|^2$ by NL-II beamforming.

Fig. 6. Comparison of the power spectral densities, true periodograms, and each beamforming result in case of $M = 166$ and $L = 60$: (a) $S_m^{(d)}(\frac{k-1-L/2}{LT})$, (b) $E[|u_{k,m}|^2]$, (c) $|u_{k,m}|^2$, (d) $|\hat{u}_{k,m}|^2$ by LS, (e) $|\hat{u}_{k,m}|^2$ by FR, (f) $|\hat{u}_{k,m}|^2$ by MMSE, (g) $|\hat{u}_{k,m}|^2$ by NL-I, and (h) $|\hat{u}_{k,m}|^2$ by NL-II beamforming.

direction method of multipliers (ADMM) [21]–[24] solves the problem in (40), together with the corresponding dual problem, from any $(\mathbf{x}^{(0)}, \mathbf{v}^{(0)}) \in \mathcal{X} \times \mathcal{Z}$ by iteratively computing

$$\begin{cases} \mathbf{z}^{(i)} = \text{prox}_{\gamma g}(\mathcal{L}(\mathbf{x}^{(i)}) + \mathbf{v}^{(i)}) \\ \tilde{\mathbf{v}}^{(i)} = \mathbf{v}^{(i)} + \mathcal{L}(\mathbf{x}^{(i)}) - \mathbf{z}^{(i)} \\ \mathbf{x}^{(i+1)} = \underset{\mathbf{x} \in \mathcal{X}}{\text{argmin}} \gamma h(\mathbf{x}) + \frac{1}{2} \|\mathcal{L}(\mathbf{x}) - \mathbf{z}^{(i)} + \tilde{\mathbf{v}}^{(i)}\|_{\mathcal{X}}^2 \\ \mathbf{v}^{(i+1)} = \tilde{\mathbf{v}}^{(i)} + (\rho^{(i+1)} - 1)(\mathcal{L}(\mathbf{x}^{(i+1)}) - \mathbf{z}^{(i)}) \end{cases} \quad (41)$$

for $i \geq 0$. In (41), $\gamma > 0$, $\mathbf{v} \in \mathcal{Z}$ and $\tilde{\mathbf{v}} \in \mathcal{Z}$ are dual variables, $\|\cdot\|_{\mathcal{X}}$ is the Euclidean norm of $\mathcal{X}$ induced by the standard inner product, $\text{prox}_{\gamma g} : \mathcal{Z} \rightarrow \mathcal{Z}$ is the *proximity operator* defined by

$$\text{prox}_{\gamma g}(\zeta) := \underset{\mathbf{z} \in \mathcal{Z}}{\text{argmin}} \gamma g(\mathbf{z}) + \frac{1}{2} \|\zeta - \mathbf{z}\|_{\mathcal{Z}}^2,$$

and $\rho^{(i+1)} \in [0, 2]$ satisfies $\sum_{i=0}^{\infty} \rho^{(i+1)}(2 - \rho^{(i+1)}) = \infty$. In particular, if $\rho^{(i+1)} = 1$ for all i , which is the most commonly used setting, then we can replace $\tilde{\mathbf{v}}^{(i)}$ with $\mathbf{v}^{(i+1)}$ and remove the fourth line in (41). The sequence $(\mathbf{x}^{(i)}, \mathbf{z}^{(i)})_{i=0}^{\infty}$ defined by

the iterations in (41) converges to a solution to (40). Moreover, the sequences $(\mathbf{v}^{(i)}/\gamma)_{i=0}^{\infty}$ and $(\tilde{\mathbf{v}}^{(i)}/\gamma)_{i=0}^{\infty}$ both converge to a solution to the dual problem. As described in [24], the over-relaxation setting, i.e., $\rho^{(i+1)} > 1$ for all i , can accelerate the convergence, compared to the standard setting s.t. $\rho^{(i+1)} = 1$.

In the following, we introduce the proximity operators used in this paper. When g is a weighted group ℓ_1 -norm with a non-overlapping group partition $G := (\mathcal{G}_i)_{i=1}^{n_G}$ and a weight vector $\xi := (\xi_1, \xi_2, \dots, \xi_{n_G})^T \in \mathbb{R}_+^{n_G}$, from $\gamma g(\mathbf{z}) + \frac{1}{2} \|\zeta - \mathbf{z}\|_{\mathcal{Z}}^2 = \sum_{i=1}^{n_G} (\gamma \xi_i \|\mathbf{z}_{\mathcal{G}_i}\|_2 + \frac{1}{2} \|\zeta_{\mathcal{G}_i} - \mathbf{z}_{\mathcal{G}_i}\|_2^2)$, the computation of $\text{prox}_{\gamma g}$ is divided into those of $\text{prox}_{\gamma \xi_i \|\cdot\|_2}$. Specifically, $\text{prox}_{\gamma g}(\zeta) = \text{prox}_{\gamma \|\cdot\|_{1,\xi}}(\zeta)$ can be computed for each sub-vector $\zeta_{\mathcal{G}_i}$ by

$$\text{prox}_{\gamma \xi_i \|\cdot\|_2}(\zeta_{\mathcal{G}_i}) = \begin{cases} \frac{\|\zeta_{\mathcal{G}_i}\|_2 - \gamma \xi_i}{\|\zeta_{\mathcal{G}_i}\|_2} \zeta_{\mathcal{G}_i} & \text{if } \|\zeta_{\mathcal{G}_i}\|_2 > \gamma \xi_i, \\ \mathbf{0} & \text{if } \|\zeta_{\mathcal{G}_i}\|_2 \leq \gamma \xi_i. \end{cases} \quad (42)$$

Let $\zeta_i \in \mathcal{Z}$ ($i = 1, 2, \dots, n$) be n variables of the same size. Let ι be the *indicator function* of the consensus set [37] for the n -tuple $(\zeta_1, \zeta_2, \dots, \zeta_n)$, i.e., $\iota(\zeta_1, \zeta_2, \dots, \zeta_n) := 0$ if $\zeta_1 =$

$\zeta_2 = \dots = \zeta_n$ and $\iota(\zeta_1, \zeta_2, \dots, \zeta_n) := \infty$ otherwise. The proximity operators of ι can be easily computed by

$$\text{prox}_{\gamma\iota}(\zeta_1, \zeta_2, \dots, \zeta_n) = \frac{1}{n} \left(\sum_{i=1}^n \zeta_i, \sum_{i=1}^n \zeta_i, \dots, \sum_{i=1}^n \zeta_i \right). \quad (43)$$

As can be seen from (43), the proximity operator of the indicator function is equal to the projection onto the consensus set and does not depend on the value of γ differently from (42).

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