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On A Class of Time-Varying Gaussian ISI Channels

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Abstract

This paper studies a class of stochastic and time-varying Gaussian intersymbol interference (ISI) channels. The i^{th} channel tap during time slot t is uniformly distributed over an interval of centre c_i and radius r_i . The array of channel taps is independent along both t and i . The channel state information is unavailable at both the transmitter and the receiver. Lower and upper bounds are derived on the White-Gaussian-Input (WGI) capacity C_{WGI} for arbitrary values of the radii r_i . It is shown that C_{WGI} does not scale with the average input power. The proposed lower bound is achieved by a joint-typicality decoder that is tuned to a set of candidates for the channel matrix. This set forms a net that covers the range of the random channel matrix and its resolution is optimized in order to yield the largest achievable rate. Tools in matrix analysis such as Weyl's inequality on perturbation of eigenvalues of symmetric matrices are used in order to analyze the probability of error.

Index Terms

Intersymbol Interference, Time-Varying Channels, White-Gaussian-Input Capacity, Joint-Typicality Decoding.

I. INTRODUCTION

A. Summary of prior art

Transmission beyond the Nyquist rate over a band-limited communication channel results in a phenomenon known as intersymbol interference (ISI). The time-invariant Gaussian ISI channel is modelled by a linear filter with known finite impulse-response and additive white Gaussian noise [1]. The capacity for this channel was first studied in [2], [21] where it was shown that Gaussian signalling is capacity-achieving. Addressing more practical structures for signal transmission, the Gaussian ISI channel with independent and identically distributed (i.i.d.) signalling over a fixed finite alphabet was examined in [3] and more recently in [4], [5].

The problem of universal decoding for the Gaussian ISI channel was raised for the first time in [6]. Roughly speaking, a universal decoder is one whose structure does not depend on the channel parameters and yet it achieves the same error exponent as the optimum maximum likelihood decoder tuned to the actual channel parameters. Influenced by the pioneering work of Ziv in [7] on universal decoding over channels with memory, the authors in [8] coined various notions of universal decoding such as random vs. deterministic and weak vs. strong universal decoding. In particular, [8] provided an answer to the problem raised in [6]. The authors showed that if the receiver knows the length and an upper bound on the energy of the channel impulse response, the Gaussian ISI channel admits a universal decoder in the strong sense. Later in [9], another universal decoder was proposed from the viewpoint of composite hypothesis testing. More recently, a universal decoder was presented in [10] which is easier to evaluate compared to the previously reported decoders in [8], [9].

The time-varying Gaussian ISI channel is less studied in the literature. Reference [11] presented information-theoretic considerations for multi-carrier transmission in time-varying Gaussian ISI channels. Reference [12] derived the capacity of block-memoryless channels with block-memoryless side information. Subsequently, the capacity of time-varying Gaussian ISI channels was characterized. Both [11] and [12] assume that perfect channel state information about the time-varying channel is available at the receiver side.

B. Summary of contributions

We study a class of stochastic and time-varying Gaussian ISI channels where the transmitter and the receiver only know the statistics of the array of channel taps. Consider a Gaussian ISI channel initially modelled by a sequence of filter taps $c_0, c_1, \dots, c_k$ as its impulse response. Let $\mathbf{h}_{t,i}$ be the actual i^{th} channel tap during time slot t . Then the initial model assumes $\mathbf{h}_{t,i} = c_i$ for all t and i . We ask the following question:

If $\mathbf{h}_{t,i}$ is not exactly the constant c_i , but a random variable that takes values near c_i , then how would that impact the channel capacity?

In an attempt to answer this question, it is assumed that $\mathbf{h}_{t,i}$ is uniformly distributed on the interval $[c_i - r_i, c_i + r_i]$ for all t and i where $r_i \geq 0$ is a given constant. Moreover, the array $\mathbf{h}_{t,i}$ is independent along both indices t and i . For tractability reasons, the focus of the paper is on the largest rate that can be achieved by employing white Gaussian codebooks. This is referred to as the White Gaussian Input (WGI) capacity denoted by C_{WGI} . The main contributions of the paper are lower and upper bounds on C_{WGI} . The

proposed upper bound shows that C_{WGI} does not scale with the average input power. The proposed lower bound is attained by a joint-typicality decoder that is tuned to a net of candidates for the channel matrix during the transmission of a message. This net covers the ambiguity region for the random channel matrix and its resolution is adjusted in order to yield the largest achievable rate. Tools in matrix analysis such as Weyl's inequality on perturbation of eigenvalues of symmetric matrices are used in order to analyze the probability of error.

The rest of the paper is organized as follows. We end the current section by listing the notations and presenting a lemma on the volume of Gaussian typical sets that improves upon the existing estimates in the literature. System model, the problem statement and a summary of main results appear in Section II. Section III offers examples to compare the lower and upper bounds on C_{WGI} . Section IV describes the structure of the proposed decoder. Section V is devoted to error analysis where we outline the proof of the main result (Theorem 1). The majority of details are referred to the appendices at the end of the paper. Finally, Section VI concludes the paper.

C. Notations

For a real number x , $x^+ = \max\{0, x\}$ and $[x]$ is the closest integer to x . The Euclidean space of sequences of length n whose entries are real numbers is denoted by $\mathbb{R}^n$. Vectors and sequences are identified by an underline such as $\underline{x}$. Random quantities appear in bold such as $\mathbf{x}$ and $\underline{\mathbf{x}}$ with realizations x and $\underline{x}$, respectively. An $m \times n$ matrix whose all entries are zeros is denoted by $0_{m,n}$. We use $\underline{0}_n$ to denote a column vector of length n whose all entries are zeros. The $n \times n$ identity matrix is denoted by I_n . The transpose, inverse, trace and determinant of a square matrix A are denoted by A^T , A^{-1} , $\text{tr}(A)$ and $\det(A)$, respectively. The entry at the i^{th} row and the j^{th} column of a matrix A is denoted by $A_{i,j}$ or $[A]_{i,j}$. The i^{th} entry of a vector $\underline{x}$ is denoted by x_i or $[\underline{x}]_i$. The 2-norm (Frobenius norm) and the spectral norm (operator norm) of a matrix A are defined by

$$\|A\|_2 := (\text{tr}(A^T A))^{1/2} \quad (1)$$

and

$$\|A\| := \max_{\|\underline{x}\|_2 \leq 1} \|A\underline{x}\|_2 = (\lambda_{\max}(A^T A))^{\frac{1}{2}}, \quad (2)$$

respectively, where $\lambda_{\max}(M)$ denotes the largest eigenvalue of a symmetric matrix M . The probability law of a random variable $\mathbf{x}$ (the induced probability measure on the range of $\mathbf{x}$) is denoted by $\mathcal{L}_{\mathbf{x}}(\cdot)$. The probability density function (PDF) of a continuous random variable $\underline{x}$ is denoted by $p_{\underline{x}}(\cdot)$. The PDF of a Gaussian random vector of length n with zero mean and covariance matrix Σ is denoted by

$$p_G(\underline{x}; \Sigma) := \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \underline{x}^T \Sigma^{-1} \underline{x}\right). \quad (3)$$

We write $\underline{x} \sim N(\underline{0}_n, \Sigma)$ if $p_{\underline{x}}(\underline{x}) = p_G(\underline{x}; \Sigma)$. The differential entropy of a random vector $\underline{x} \sim N(\underline{0}_n, \Sigma)$ is denoted by $h_G(\Sigma)$, i.e.,

$$h_G(\Sigma) := \frac{1}{2} \log((2\pi e)^n \det(\Sigma)), \quad (4)$$

where $\log(\cdot)$ is the logarithm function with base 2. The underlying probability measure is denoted by $\Pr(\cdot)$. Finally, $\mathbb{E}[\cdot]$ denotes the expectation operator against $\Pr(\cdot)$.

D. A lemma on the volume of Gaussian typical sets

Let us recall the definition for a typical set [13], [20]. For $\eta > 0$, positive integer n and an $n \times n$ positive-definite matrix Σ , the Gaussian typical set $\mathcal{T}_{\eta}^{(n)}(\Sigma)$ is defined by

$$\begin{aligned} \mathcal{T}_{\eta}^{(n)}(\Sigma) &:= \left\{ \underline{a} \in \mathbb{R}^n : \left| \frac{1}{n} \log p_G(\underline{a}; \Sigma) + \frac{1}{n} h_G(\Sigma) \right| < \frac{\eta}{2 \ln 2} \right\} \\ &= \left\{ \underline{a} \in \mathbb{R}^n : \left| \frac{1}{n} \underline{a}^T \Sigma^{-1} \underline{a} - 1 \right| < \eta \right\}. \end{aligned} \quad (5)$$

The following lemma on the volume of a typical set will be used later in the paper:

Lemma 1. *The volume of the typical set $\mathcal{T}_{\eta}^{(n)}(\Sigma)$ is bounded from above by*

$$\text{Vol}(\mathcal{T}_{\eta}^{(n)}(\Sigma)) \leq 2^{h_G(\Sigma) + \frac{n}{2} \log(1+\eta)}. \quad (6)$$

Proof. See Appendix A. Reference [13] reports the upper bound $2^{h_G(\Sigma) + \frac{n}{2 \ln 2} \eta}$ on $\text{Vol}(\mathcal{T}_{\eta}^{(n)}(\Sigma))$. The bound in (6) is tighter than $2^{h_G(\Sigma) + \frac{n}{2 \ln 2} \eta}$. The proposed lower bound on C_{WGI} in Theorem 1 benefits from this improvement. ■

II. SYSTEM MODEL AND SUMMARY OF MAIN RESULTS

The stochastic and time-varying Gaussian ISI channel with memory-depth $k \geq 1$ is defined by

$$\mathbf{y}_t = \sum_{i=0}^k \mathbf{h}_{t,i} \mathbf{x}_{t-i} + \mathbf{z}_t, \quad t = 1, 2, \dots, \quad (7)$$

where $\mathbf{h}_{t,0}, \mathbf{h}_{t,1}, \dots, \mathbf{h}_{t,k}$ are the channel coefficients (taps) during time slot $t = 1, 2, \dots$ and $\mathbf{x}_t, \mathbf{y}_t$ and $\mathbf{z}_t$ are the channel input, the channel output and the additive noise at time slot t , respectively. The process $\mathbf{z}_1, \mathbf{z}_2, \dots$ is white Gaussian with zero mean and unit variance. The input process $\mathbf{x}_1, \mathbf{x}_2, \dots$ is subject to the average power constraint

$$\frac{1}{n} \sum_{t=1}^n |\mathbf{x}_t|^2 \leq P, \quad (8)$$

where n is the length of a codeword.

The following assumptions are made on the array of channel coefficients:

- (i) The array of channel taps $\mathbf{h}_{t,i}$ is independent along both indices t and i with uniform marginals. More precisely, there is a sequence of numbers $\underline{c} = (c_0, c_1, \dots, c_k)$ and a sequence of nonnegative numbers $\underline{r} = (r_0, r_1, \dots, r_k)$ such that $\mathbf{h}_{t,i}$ is uniformly distributed over the interval $[c_i - r_i, c_i + r_i]$ for every t, i . We define

$$c_s := \sum_{i=0}^k |c_i|, \quad r_s := \sum_{i=0}^k r_i. \quad (9)$$

- (ii) The transmitter and the receiver only know the distribution of the random variables $\mathbf{h}_{t,i}$.
- (iii) The processes $\mathbf{h}_{t,i}$ and $\mathbf{z}_t$ are independent.

To describe the ISI channel in matrix form, assume a codeword is transmitted during time slots $t = 1, \dots, n$. Define

$$m := n + k \quad (10)$$

and

$$\underline{\mathbf{x}} := [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n]^T, \quad \underline{\mathbf{y}} := [\mathbf{y}_1 \ \mathbf{y}_2 \ \dots \ \mathbf{y}_m]^T, \quad \underline{\mathbf{z}} := [\mathbf{z}_1 \ \mathbf{z}_2 \ \dots \ \mathbf{z}_m]^T. \quad (11)$$

Then

$$\underline{\mathbf{y}} = \mathbf{H}\underline{\mathbf{x}} + \underline{\mathbf{z}}, \quad (12)$$

where the $m \times n$ channel matrix $\mathbf{H}$ is given by

$$\mathbf{H}_{i,j} := \begin{cases} \mathbf{h}_{i,i-j} & 0 \leq i-j \leq k \\ 0 & \text{otherwise} \end{cases}. \quad (13)$$

For example, when $n = 4$ and $k = 2$, we have $m = 4 + 2 = 6$ and the matrices $\mathbf{H}$ and $\mathbb{E}[\mathbf{H}]$ are given by

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_{1,0} & 0 & 0 & 0 \\ \mathbf{h}_{2,1} & \mathbf{h}_{2,0} & 0 & 0 \\ \mathbf{h}_{3,2} & \mathbf{h}_{3,1} & \mathbf{h}_{3,0} & 0 \\ 0 & \mathbf{h}_{4,2} & \mathbf{h}_{4,1} & \mathbf{h}_{4,0} \\ 0 & 0 & \mathbf{h}_{5,2} & \mathbf{h}_{5,1} \\ 0 & 0 & 0 & \mathbf{h}_{6,2} \end{bmatrix}, \quad \mathbb{E}[\mathbf{H}] = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 \\ 0 & c_2 & c_1 & c_0 \\ 0 & 0 & c_2 & c_1 \\ 0 & 0 & 0 & c_2 \end{bmatrix}. \quad (14)$$

The message $\mathbf{W}$ is uniformly distributed over the message set $\{1, 2, \dots, 2^{nR}\}$ where R is the transmission rate. The encoder maps $\mathbf{W}$ to a codeword $\underline{\mathbf{x}}$ inside the codebook $\mathcal{C}$. This codeword is then transmitted over the channel in (12) during time slots $t = 1, \dots, n$. The decoder receives the vector $\underline{\mathbf{y}}$ and generates the estimate $\widehat{\mathbf{W}}$ for $\mathbf{W}$. The probability of decoding error is denoted by

$$p_e(\mathcal{C}) := \Pr(\widehat{\mathbf{W}} \neq \mathbf{W}). \quad (15)$$

Next, we define the notion of white-Gaussian-input capacity.

Definition- Let $\mathcal{C}$ be the ensemble of codebooks whose 2^{nR} codewords are realizations of independent $\mathcal{N}(\underline{0}_n, PI_n)$ random vectors. We say a transmission rate R is White-Gaussian-Input achievable (WGI-achievable), if

$$\lim_{n \rightarrow \infty} \langle p_e(\mathcal{C}) \rangle_{\mathcal{C}} = 0, \quad (16)$$

where $\langle p_e(\mathcal{C}) \rangle_{\mathcal{C}}$ denotes the average of $p_e(\mathcal{C})$ as the codebook $\mathcal{C}$ runs over the ensemble $\mathcal{C}$. The supremum of all WGI-achievable rates is denoted by C_{WGI} and referred to as the WGI-capacity. $\diamond$

In simple words, C_{WGI} is the largest rate that can be achieved by white Gaussian codewords. In the

special case where the ISI channel is deterministic and time-invariant, i.e., $r_i = 0$ for every i , it is well-known [2] that the WGI-capacity denoted by $C_{WGI,0}(P)$ is given by

$$C_{WGI,0}(P) := C_{WGI}|_{\forall i \geq 0, r_i=0} = \frac{1}{4\pi} \int_0^{2\pi} \log \left(1 + \left| \sum_{i=0}^k c_i e^{\sqrt{-1}i\omega} \right|^2 P \right) d\omega. \quad (17)$$

The main contributions of this paper are a lower bound (achievability result) and an upper bound (converse result) on C_{WGI} for arbitrary $\underline{r}$ and sufficiently small r_s . To describe the main results, we need to introduce several parameters. Let

$$f(\omega) = \sum_{i=0}^k c_i e^{\sqrt{-1}i\omega} \quad (18)$$

be the discrete Fourier transform of the sequence $\underline{c}$,

$$\alpha := \min_{\omega \in [0, 2\pi]} |f(\omega)|, \quad \beta := \max_{\omega \in [0, 2\pi]} |f(\omega)| \quad (19)$$

and

$$\varrho := \begin{cases} \frac{\alpha}{1+\alpha^2 P} & \frac{1}{P} \leq \alpha^2 \\ \frac{1}{2\sqrt{P}} & \alpha^2 < \frac{1}{P} < \beta^2 \\ \frac{\beta}{1+\beta^2 P} & \beta^2 \leq \frac{1}{P} \end{cases} \quad (20)$$

If there exists an (unique) index $0 \leq i_0 \leq k$ such that

$$\frac{1}{6} P r_{i_0}^2 > 1 + \frac{1}{6} P \sum_{i \neq i_0} r_i^2, \quad (21)$$

then define

$$\phi_1 := \min_{L \in \mathbb{N}} L \left(1 + \frac{1}{6} P \sum_{i \neq i_0} r_i^2 + \frac{\frac{1}{6} P r_{i_0}^2}{L^2} \right), \quad (22)$$

If there is no such index i_0 , let

$$\phi_1 := 1 + \frac{1}{6} P \|\underline{r}\|_2^2. \quad (23)$$

Also, define

$$\phi_2 := \frac{2}{\sqrt{3}} \|\underline{r}\|_2 \varrho P, \quad (24)$$

$$\phi_3 := \frac{1}{\ln 2} r_s(r_s + \beta)P, \quad \phi_4 := \min \left\{ r_s(r_s + \beta + \varrho)P, \frac{r_s(r_s + 2\beta)P}{1 + \alpha^2 P} \right\}, \quad (25)$$

and

$$\phi_5 := \frac{1}{1 + r_s(r_s + \beta + \varrho)P}. \quad (26)$$

Throughout the paper, we assume r_s is small enough such that

$$r_s(r_s + 2\beta) < \alpha^2. \quad (27)$$

This guarantees that $\phi_4 < 1$ regardless of P . We are ready to present a lower bound on C_{WGI} .

Theorem 1. C_{WGI} is bounded from below as

$$C_{WGI} \geq C_{WGI}^{(\text{lb})} := (C_{WGI,0}(P) - \delta)^+, \quad (28)$$

where

$$\delta := \log \phi_1 + \min \left\{ \phi_3, -\frac{1}{2} \log(1 - \phi_4) \right\} + \frac{1}{2 \ln 2} (1 - (1 - \phi_2)^+ \phi_5). \quad (29)$$

Proof. The proof is detailed in Section V. ■

The lower bound in (28) is achieved by a joint-typicality decoder that is tuned to a set of candidates for the random channel matrix $\mathbf{H}$. This set forms a net that covers the range (ambiguity region) of $\mathbf{H}$. The optimization in (22) over the integer $L \geq 1$ is a result of adjusting the resolution of this net in order to achieve the largest possible rate. The choice $L = 1$ simply means that the joint-typicality decoder is tuned to the *single* matrix $\mathbb{E}[\mathbf{H}]$. Throughout the paper, we denote the optimal choice of L by L_{opt} .

Next, we provide an upper bound on C_{WGI} :

Theorem 2. Let k' be the number of indices $0 \leq i \leq k$ such that $r_i > 0$. Define

$$\kappa := \max \left\{ \log \|\mathbf{r}\|_\infty, \frac{1}{2} \log k' + \frac{1}{k'} \sum_{i:r_i>0} \log r_i \right\}. \quad (30)$$

Then C_{WGI} is bounded from above as

$$\begin{aligned} C_{WGI} \leq C_{WGI}^{(\text{ub})} &:= \frac{1}{2} \log (1 + P \|\mathbf{r}\|_2^2 / 3) + C_{WGI,0} \left(\frac{P}{1 + P \|\mathbf{r}\|_2^2 / 3} \right) \\ &\quad - \left(\frac{1}{2} \log \frac{P}{\pi e} + \kappa - \frac{\gamma_e}{2 \ln 2} \right)^+, \end{aligned} \quad (31)$$

where the function $C_{WGI,0}(\cdot)$ is given in (17) and $\gamma_e \approx 0.577$ is the Euler-Mascheroni constant.

Proof. See Appendix B. ■

Both $C_{WGI}^{(\text{lb})}$ and $C_{WGI}^{(\text{ub})}$ simplify to $C_{WGI,0}(P)$ when $r_0 = r_1 = \dots = r_k = 0$.

In the special case where there is $r > 0$ such that $r_i = r$ for every $0 \leq i \leq k$, we find $\kappa = \log r + \frac{1}{2} \log k$.

The upper bound in Theorem 2 can be improved as

$$C_{WGI} \leq \frac{1}{2} \log(1 + kPr^2/3) + C_{WGI,0}\left(\frac{P}{1 + kPr^2/3}\right) - \left(\frac{1}{2} \log \frac{P}{\pi e} + \log r + \frac{1}{2} \log k + \frac{1}{2} \mathbb{E}[\log \chi_{k+1}^2] + \frac{1}{2}\right)^+, \quad (32)$$

where χ_{k+1}^2 is a chi-square random variable with $k+1$ degrees of freedom. This is also verified at the end of Appendix B.

III. SOME EXAMPLES

In this section, we consider two examples in order to compare the lower and upper bounds provided in Theorem 1 and Theorem 2.

Example 1- Let us study the bounds $C_{WGI}^{(\text{ub})}$ and $C_{WGI}^{(\text{lb})}$ in the asymptote of large P . The upper bound in (31) does not scale with P . It saturates at

$$\begin{aligned} \lim_{P \rightarrow \infty} C_{WGI}^{(\text{ub})} &= \frac{1}{2} \log \frac{\|r\|_2^2}{3} + C_{WGI,0}\left(\frac{3}{\|r\|_2^2}\right) - \kappa + \frac{1}{2} \log(\pi e) + \frac{\gamma_e}{2 \ln 2} \\ &= \frac{1}{4\pi} \int_0^{2\pi} \log \left(\frac{\|r\|_2^2}{3} + \left| \sum_{i=0}^k c_i e^{\sqrt{-1} i \omega} \right|^2 \right) d\omega - \kappa + \frac{1}{2} \log(\pi e) + \frac{\gamma_e}{2 \ln 2}. \end{aligned} \quad (33)$$

The lower bound in (28) vanishes for large values of P unless there exists a unique index $0 \leq i \leq k$ such that $r_i > 0$, i.e., only one of the taps is varying randomly from time slot to time slot. Denote this index (if it exists) by i^* and let $r = r_{i^*}$. Then one can easily check that for large enough values of P ,

$$\phi_1 = \min_{L \geq 1} L \left(1 + \frac{\frac{1}{6} Pr^2}{L^2} \right) \approx 2 \left(\frac{1}{6} Pr^2 \right)^{1/2}, \quad (34)$$

where the optimum choice of L is $L_{opt} = [(\frac{1}{6} Pr^2)^{1/2}]$ and¹

$$\delta \approx \log \phi_1 - \frac{1}{2} \log \left(1 - \frac{r(r + 2\beta)}{\alpha^2} \right) + \frac{1}{2 \ln 2}. \quad (35)$$

¹When P is large such that $\phi_2 > 1$, the term $(1 - \phi_2)^+ \phi_5$ disappears. Moreover, $\min\{\phi_3, -\frac{1}{2} \log(1 - \phi_4)\} \approx \frac{1}{2} \log(1 - \frac{r(r+2\beta)}{\alpha^2})$.

Fig. 1. Plot of $\lim_{P \rightarrow \infty} (C_{WGI}^{(ub)} - C_{WGI}^{(lb)})$ in terms of r in an ISI channel with $k = 2$, $\underline{c} = (1, 0.4, 0.1)$ and $\underline{r} = (0, 0, r)$.

This leads to

$$\begin{aligned} \lim_{P \rightarrow \infty} C_{WGI}^{(lb)} &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{i=0}^k c_i e^{\sqrt{-1} i \omega} \right| d\omega \\ &\quad - \log r + \frac{1}{2} \log \left(1 - \frac{r(r + 2\beta)}{\alpha^2} \right) - \log \frac{2}{\sqrt{6}} - \frac{1}{2 \ln 2}. \end{aligned} \quad (36)$$

Fig. 1 depicts the limiting difference $\lim_{P \rightarrow \infty} (C_{WGI}^{(ub)} - C_{WGI}^{(lb)})$ in terms of r in an ISI channel with $k = 2$, $c_0 = 1, c_1 = 0.4, c_2 = 0.1$ and $r_0 = 0, r_1 = 0, r_2 = r$.

If $r_i > 0$ for more than one choice of $0 \leq i \leq k$, the term ϕ_1 no longer scales like $P^{1/2}$. It scales like P and hence, $(C_{WGI,0}(P) - \delta)^+$ will be eventually zero.

Example 2- Here we compare the bounds for finite values of P . Panel (a) in Fig. 2 presents plots of $C_{WGI}^{(ub)}$ and $C_{WGI}^{(lb)}$ in terms of P (in dBW) over an ISI channel with $k = 3$, $\underline{c} = (1, 0.1, 0.1, 0.1)$ and $\underline{r} = (0.01, 0.05, 0.01, 0)$. The lower bound is plotted separately for $L = L_{opt}$ and $L = 1$. In both cases, the lower bound will eventually vanish as there are more than one index i such that $r_i > 0$. Panel (b) in Fig. 2 presents plots of $C_{WGI}^{(ub)}$ and $C_{WGI}^{(lb)}$ in terms of P (in dBW) over an ISI channel with $k = 3$, $\underline{c} = (1, 0.1, 0.1, 0.1)$ and $\underline{r} = (0, 0, 0.01, 0)$. Since only one tap is random ($r_2 > 0$), the lower bound with $L = L_{opt}$ will eventually saturate at a positive value as promised in the previous example.

Fig. 2. Panel (a) presents plots of upper and lower bounds on C_{WGI} in terms of P (dBW) in an ISI channel with $k = 3$, $\underline{c} = (1, 0.1, 0.1, 0.1)$ and $\underline{r} = (0.01, 0.05, 0.01, 0)$. Panel (b) presents the same plots in an ISI channel with $k = 3$, $\underline{c} = (1, 0.1, 0.1, 0.1)$ and $\underline{r} = (0, 0, 0.01, 0)$.

IV. THE DECODER STRUCTURE

The proposed decoder is quite simple and applies joint-typicality decoding over a net that covers the ambiguity region for the random channel matrix $\mathbf{H}$. Let $\mathcal{H}$ be the range of $\mathbf{H}$, i.e., $\mathcal{H}$ is the set of all $m \times n$ matrices for which $|H_{i,j} - c_{i-j}| \leq r_{i-j}$ for all i, j such that $0 \leq i - j \leq k$ and $H_{i,j} = 0$ otherwise. Let $L_0, L_1, \dots, L_k \geq 1$ be integers to be determined and define

$$\tau_0 = \frac{2r_0}{L_0}, \quad \tau_1 = \frac{2r_1}{L_1}, \dots, \tau_k = \frac{2r_k}{L_k} \quad (37)$$

and

$$\underline{\tau} := \begin{bmatrix} \tau_0 & \tau_1 & \dots & \tau_k \end{bmatrix}^T. \quad (38)$$

Let $\mathcal{N}$ be a $\underline{\tau}$ -net over $\mathcal{H}$. By definition, $\mathcal{N}$ consists of $m \times n$ matrices N that satisfy these two conditions:

- (i) $N_{i,j} = 0$ for $i - j < 0$ or $i - j > k$.
- (ii) For every $H \in \mathcal{H}$, there exists a matrix $N \in \mathcal{N}$ such that $|H_{i,j} - N_{i,j}| \leq \frac{1}{2}\tau_{i-j}$ for every i, j such that $0 \leq i - j \leq k$.

It is clear that

$$|\mathcal{N}| = (L_0 L_1 \dots L_k)^n. \quad (39)$$

Denote the codewords by the independent random vectors $\underline{x}^{(i)} \sim \mathcal{N}(\underline{0}_n, P\mathbf{I}_n)$ for $i = 1, 2, \dots, 2^{nR}$.

Recall the definition of Gaussian typical sets in (5). The joint-typicality decoder looks for the unique index $i = 1, \dots, 2^{nR}$ for which there exists $N \in \mathcal{N}$ such that

$$\underline{\mathbf{x}}(i) \in \mathcal{T}_\epsilon^{(n)}(PI_n), \quad \underline{\mathbf{w}}(i) \in \mathcal{T}_\eta^{(m+n)}(\Sigma_N), \quad (40)$$

where $\epsilon, \eta > 0$ are constants and $\underline{\mathbf{w}}(i)$ and Σ_N are defined by

$$\underline{\mathbf{w}}(i) := [\underline{\mathbf{x}}^T(i) \quad \underline{\mathbf{y}}^T]^T, \quad \Sigma_N := \begin{bmatrix} PI_n & PN^T \\ PN & I_m + PNN^T \end{bmatrix}. \quad (41)$$

The matrix Σ_N is the covariance matrix for a vector $[\tilde{\mathbf{x}}^T \quad \tilde{\mathbf{y}}^T]^T$ where $\tilde{\mathbf{y}} = N\tilde{\mathbf{x}} + \tilde{\mathbf{z}}$ and $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{z}}$ are independent $N(\underline{0}_n, PI_n)$ and $N(\underline{0}_m, I_m)$ random vectors, respectively. To construct the set $\mathcal{T}_\eta^{(m+n)}(\Sigma_N)$, one needs to compute Σ_N^{-1} . It is easy to see that Σ_N^{-1} admits a closed form given by

$$\Sigma_N^{-1} = \begin{bmatrix} \frac{1}{P}I_n + N^TN & -N^T \\ -N & I_m \end{bmatrix}. \quad (42)$$

It is also useful to note that²

$$\det(\Sigma_N) = P^n, \quad (43)$$

regardless of the choice of $N \in \mathcal{N}$.

V. ERROR ANALYSIS

As usual, two types of error are distinguished:

- (i) The transmitted codeword does not satisfy the decoding rule in (40). We refer to this as the *type I error* and denote it by $\mathcal{E}^{(I)}$.
- (ii) A codeword different from the transmitted codeword satisfies (40). We refer to this as the *type II error* and denote it by $\mathcal{E}^{(II)}$.

Then

$$\langle p_e(\mathcal{C}) \rangle_{\mathcal{C}} \leq \Pr(\mathcal{E}^{(I)}) + \Pr(\mathcal{E}^{(II)}). \quad (44)$$

In the following we examine the two terms on the right side of (44) separately.

²This follows by the Schur's complement formula for calculating the determinant of partitioned matrices [18].

A. The probability of the type I error

Without loss of generality, assume $\underline{x}(1)$ is the transmitted codeword, i.e., $\underline{y} = \mathbf{H}\underline{x}(1) + \underline{z}$. Then

$$\Pr(\mathcal{E}^{(I)}) \leq \Pr(\underline{x}(1) \notin \mathcal{T}_\epsilon^{(n)}(PI_n)) + \Pr(\forall N \in \mathcal{N}, \underline{w}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_N)), \quad (45)$$

where $\underline{w}(1)$ and Σ_N are given in (41). For every $\epsilon > 0$, the first term on the right side of (45) tends to zero when n grows due to the weak law of large numbers. The second term on the right side (45) is studied in the next proposition:

Proposition 1. *Assume*

$$\eta > \eta_n := \frac{n}{12(m+n)} P \sum_{l=0}^k \tau_l^2. \quad (46)$$

Then

$$\Pr(\forall N \in \mathcal{N}, \underline{w}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_N)) \leq C/((m+n)(\eta - \eta_n)^2), \quad (47)$$

where

$$C := 2(r_s^2 P + r_s \sqrt{P} + 1)^2 + \frac{1}{360} P^2 \sum_{l=0}^k \tau_l^4 \quad (48)$$

is a constant.

Proof. Let us write

$$\underline{w}(1) = \mathbf{A}\underline{\nu}, \quad (49)$$

where $\mathbf{A}$ and $\underline{\nu} \sim \mathcal{N}(\underline{0}_m, I_m)$ are given by

$$\mathbf{A} := \begin{bmatrix} \sqrt{P}I_n & 0_{n,m} \\ \sqrt{P}\mathbf{H} & I_m \end{bmatrix}, \quad \underline{\nu} := \begin{bmatrix} \frac{1}{\sqrt{P}}\underline{x}^T(1) & \underline{z}^T \end{bmatrix}^T. \quad (50)$$

Then

$$\underline{w}^T(1)\Sigma_N^{-1}\underline{w}(1) = \underline{\nu}^T\Phi\underline{\nu}, \quad (51)$$

where Φ is defined by

$$\Phi := \mathbf{A}^T \Sigma_N^{-1} \mathbf{A}. \quad (52)$$

Replacing the expression for Σ_N^{-1} in (42), we find that

$$\Phi = \begin{bmatrix} I_n + P \mathbf{E}^T \mathbf{E} & \sqrt{P} \mathbf{E}^T \\ \sqrt{P} \mathbf{E} & I_m \end{bmatrix} \quad (53)$$

where the “error matrix” $\mathbf{E}$ is defined by

$$\mathbf{E} := \mathbf{H} - N, \quad (54)$$

i.e., $\mathbf{E}$ is the difference between the actual channel matrix $\mathbf{H}$ and the matrix $N \in \mathcal{N}$. Define the ball $B_\infty(N, \underline{\tau})$ to be the set of all $m \times n$ matrices H such that $|H_{i,j} - N_{i,j}| \leq \frac{1}{2}\tau_{i-j}$ for all i, j such that $0 \leq i - j \leq k$. We write

$$\begin{aligned} & \Pr(\forall N \in \mathcal{N}, \underline{\mathbf{w}}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_N)) \\ &= \sum_{N' \in \mathcal{N}} \Pr(\forall N \in \mathcal{N}, \underline{\mathbf{w}}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_N) | \mathbf{H} \in B_\infty(N', \underline{\tau})) \Pr(\mathbf{H} \in B_\infty(N', \underline{\tau})), \\ &\leq \sum_{N' \in \mathcal{N}} \Pr(\underline{\mathbf{w}}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_{N'}) | \mathbf{H} \in B_\infty(N', \underline{\tau})) \Pr(\mathbf{H} \in B_\infty(N', \underline{\tau})), \end{aligned} \quad (55)$$

where we have used the fact that $\mathcal{N}$ is a $\underline{\tau}$ -net for the range of $\mathbf{H}$. It is easy to see that the conditional probability law of the random matrix $\mathbf{H}$ given the condition $\mathbf{H} \in B_\infty(N', \underline{\tau})$ is equal to the probability law of a random matrix that we denote by $\mathbf{H}_\tau$ for which $[\mathbf{H}_\tau]_{i,j} = 0$ for $i - j < 0$ or $i - j > k$ and $[\mathbf{H}_\tau]_{i,j} - [N']_{i,j}$ for $0 \leq i - j \leq k$ are independent uniform random variables on the interval $[-\frac{1}{2}\tau_{i-j}, \frac{1}{2}\tau_{i-j}]$.

Let us define

$$\Phi_\tau := \begin{bmatrix} I_n + P \mathbf{E}_\tau^T \mathbf{E}_\tau & \sqrt{P} \mathbf{E}_\tau^T \\ \sqrt{P} \mathbf{E}_\tau & I_m \end{bmatrix} \quad (56)$$

where

$$\mathbf{E}_\tau := \mathbf{H}_\tau - N'. \quad (57)$$

Then

$$\begin{aligned}
& \Pr(\underline{\mathbf{w}}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_{N'}) | \mathbf{H} \in B_\infty(N', \underline{\tau})) \\
&= \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi} \underline{\boldsymbol{\nu}} - (m+n)| \geq (m+n)\eta | \mathbf{H} \in B_\infty(N', \underline{\tau})) \\
&= \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - (m+n)| \geq (m+n)\eta), \tag{58}
\end{aligned}$$

where the first step is due to (51) and the definition of $\mathcal{T}_\eta^{(m+n)}(\Sigma_{N'})$ and the second step follows the independence of $\underline{\boldsymbol{\nu}}$ and $\mathbf{H}$ and definition of $\boldsymbol{\Phi}_\tau$. We continue to further bound the term on the right side of (58). We have

$$\begin{aligned}
& \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - (m+n)| \geq (m+n)\eta) \\
& \stackrel{(a)}{\leq} \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - \mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]| + |\mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}] - (m+n)| \geq (m+n)\eta) \\
&= \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - \mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]| \geq (m+n)(\eta - |\mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]/(m+n) - 1|)) \\
& \stackrel{(b)}{=} \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - \mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]| \geq (m+n)(\eta - P\mathbb{E}[\|\mathbf{E}_\tau\|_2^2]/(m+n))), \tag{59}
\end{aligned}$$

where (a) follows by adding and subtracting $\mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]$ and applying the triangle inequality and (b) is due to

$$\mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}] = \mathbb{E}[\text{tr}(\boldsymbol{\Phi}_\tau)] = m+n + P\mathbb{E}[\|\mathbf{E}_\tau\|_2^2], \tag{60}$$

which in turn follows from the independence of $\boldsymbol{\Phi}_\tau$ and $\underline{\boldsymbol{\nu}}$. But,

$$\mathbb{E}[\|\mathbf{E}_\tau\|_2^2] = \sum_{i,j: 0 \leq i-j \leq k} \mathbb{E}[(\mathbf{E}_\tau)_{i,j}^2] = \frac{1}{12}n \sum_{l=0}^k \tau_l^2, \tag{61}$$

due to $[\mathbf{E}_\tau]_{i,j}$ being uniformly distributed over $[-\frac{1}{2}\tau_{i-j}, \frac{1}{2}\tau_{i-j}]$ for every i, j such that $0 \leq i-j \leq k$.

Putting (58), (59) and (61) together,

$$\begin{aligned}
& \Pr(\underline{\mathbf{w}}(1) \notin \mathcal{T}_\eta^{(m+n)}(\Sigma_{N'}) | \mathbf{H} \in B(N', \underline{\tau})) \\
& \leq \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}} - \mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}]| \geq (m+n)(\eta - \eta_n)) \\
& \leq \frac{\text{Var}(\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}})}{(m+n)^2(\eta - \eta_n)^2} \tag{62}
\end{aligned}$$

where η_n is defined in (46) and the last step is due to $\eta > \eta_n$ and the Chebyshev's inequality. It is shown

in Appendix C that

$$\text{Var}(\underline{\boldsymbol{\nu}}^T \boldsymbol{\Phi}_\tau \underline{\boldsymbol{\nu}}) = 2\mathbb{E}[\text{tr}(\boldsymbol{\Phi}_\tau^2)] + \text{Var}(\text{tr}(\boldsymbol{\Phi}_\tau)). \quad (63)$$

The term $\mathbb{E}[\text{tr}(\boldsymbol{\Phi}_\tau^2)]$ can be bounded from above by

$$\mathbb{E}[\text{tr}(\boldsymbol{\Phi}_\tau^2)] \leq (r_s^2 P + r_s \sqrt{P} + 1)^2 (m + n). \quad (64)$$

The proof for (64) appears in Appendix D. Moreover,

$$\begin{aligned} \text{Var}(\text{tr}(\boldsymbol{\Phi}_\tau)) &= \text{Var}(m + n + P \|\mathbf{E}_\tau\|_2^2) \\ &= P^2 \text{Var}(\|\mathbf{E}_\tau\|_2^2) \\ &= P^2 \sum_{i,j: 0 \leq i-j \leq k} \text{Var}([\mathbf{E}_\tau]_{i-j}^2) \\ &= \frac{1}{180} n P^2 \sum_{l=0}^k \tau_l^4, \end{aligned} \quad (65)$$

where the last step is due to the fact that $[\mathbf{E}_\tau]_{i,j}$ for $0 \leq i-j \leq k$ is uniformly distributed on $[-\frac{1}{2}\tau_{i-j}, \frac{1}{2}\tau_{i-j}]$.

Putting (55), (62), (63), (64) and (65) together, the proof is complete. $\blacksquare$

B. The probability of the type II error

As before, let $\underline{\mathbf{x}}(1)$ be the transmitted codeword. Define

$$\Omega_c := I_m + P H_c H_c^T, \quad (66)$$

where $H_c := \mathbb{E}[\mathbf{H}]$ is an $m \times n$ matrix given by

$$[H_c]_{i,j} = \begin{cases} c_{i-j} & 0 \leq i-j \leq k \\ 0 & \text{otherwise} \end{cases}. \quad (67)$$

Note that if $r_0 = r_1 = \dots = r_k = 0$, the channel matrix $\mathbf{H}$ reduces to H_c . Let $\eta' > 0$ to be determined. The probability of the type II error can be bounded as

$$\begin{aligned}
\Pr(\mathcal{E}^{(II)}) &= \Pr(\exists i \neq 1, \exists N \in \mathcal{N}, \underline{\mathbf{x}}(i) \in \mathcal{T}_\epsilon^{(n)}(PI_n), \underline{\mathbf{w}}(i) \in \mathcal{T}_\eta^{(m+n)}(\Sigma_N)) \\
&\leq \Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \\
&\quad + \Pr(\exists i \neq 1, \exists N \in \mathcal{N}, \underline{\mathbf{x}}(i) \in \mathcal{T}_\epsilon^{(n)}(PI_n), \underline{\mathbf{w}}(i) \in \mathcal{T}_\eta^{(m+n)}(\Sigma_N), \underline{\mathbf{y}} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \\
&\leq \Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \\
&\quad + \sum_{i=2}^{2^{nR}} \sum_{N \in \mathcal{N}} \Pr(\underline{\mathbf{x}}(i) \in \mathcal{T}_\epsilon^{(n)}(PI_n), \underline{\mathbf{w}}(i) \in \mathcal{T}_\eta^{(m+n)}(\Sigma_N), \underline{\mathbf{y}} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)), \tag{68}
\end{aligned}$$

where we have applied the union bound in the last step. The next proposition studies the first term on the right side of (68):

Proposition 2. *Recall the definition of ϱ in (20). Assume*

$$\eta' > \eta'_n := \frac{2}{\sqrt{3}} \frac{n}{m} \|\underline{\mathbf{r}}\|_2 \varrho P. \tag{69}$$

Then

$$\Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \leq C' / (m(\eta' - \eta'_n)^2), \tag{70}$$

where

$$C' := 2((c_s + r_s)^2 P + (c_s + r_s)\sqrt{P} + 1)^2 + \frac{P}{3} \|\underline{\mathbf{r}}\|_\infty^2 \tag{71}$$

is a constant.

Proof. Let us write

$$\underline{\mathbf{y}} = \mathbf{B}\underline{\mathbf{v}}, \tag{72}$$

where

$$\mathbf{B} := \begin{bmatrix} \sqrt{P}\mathbf{H} & \mathbf{I}_m \end{bmatrix} \tag{73}$$

and $\underline{\nu}$ is given in (50). Then

$$\underline{\mathbf{y}}^T \Omega_c^{-1} \underline{\mathbf{y}} = \underline{\nu}^T \Psi \underline{\nu}, \quad (74)$$

where Ψ is defined by

$$\Psi := \mathbf{B}^T \Omega_c^{-1} \mathbf{B} = \begin{bmatrix} P \mathbf{H}^T \Omega_c^{-1} \mathbf{H} & \sqrt{P} \mathbf{H}^T \Omega_c^{-1} \\ \sqrt{P} \Omega_c^{-1} \mathbf{H} & \Omega_c^{-1} \end{bmatrix}. \quad (75)$$

Following similar lines of reasoning as in (59),

$$\Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \leq \Pr(|\underline{\nu}^T \Psi \underline{\nu} - \mathbb{E}[\underline{\nu}^T \Psi \underline{\nu}]| \geq m(\eta' - |\mathbb{E}[\underline{\nu}^T \Psi \underline{\nu}]/m - 1|)). \quad (76)$$

Note that

$$\begin{aligned} \mathbb{E}[\underline{\nu}^T \Psi \underline{\nu}] &\stackrel{(a)}{=} \mathbb{E}[\text{tr}(\Psi)] \\ &\stackrel{(b)}{=} \mathbb{E}[\text{tr}(\Omega_c^{-1}(I_m + P \mathbf{H} \mathbf{H}^T))] \\ &\stackrel{(c)}{=} \mathbb{E}[\text{tr}(\Omega_c^{-1}(\Omega_c + P(\mathbf{E}_c \mathbf{H}_c^T + \mathbf{H}_c \mathbf{E}_c^T))] \\ &= m + 2P \mathbb{E}[\text{tr}(\Omega_c^{-1} \mathbf{H}_c \mathbf{E}_c^T)], \end{aligned} \quad (77)$$

where (a) is due to independence of Ψ and $\underline{\nu}$, (b) follows by the expression of Ψ in (75) and the matrix $\mathbf{E}_c$ in (c) is defined by

$$\mathbf{E}_c := \mathbf{H} - \mathbf{H}_c. \quad (78)$$

By (76) and (77),

$$\Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \leq \Pr(|\underline{\nu}^T \Psi \underline{\nu} - \mathbb{E}[\underline{\nu}^T \Psi \underline{\nu}]| \geq m(\eta' - 2P|\mathbb{E}[\text{tr}(\Omega_c^{-1} \mathbf{H}_c \mathbf{E}_c^T)]/m)). \quad (79)$$

We have the thread of inequalities

$$\begin{aligned}
|\mathbb{E}[\text{tr}(\Omega_c^{-1} H_c \mathbf{E}^T)]| &\stackrel{(a)}{\leq} \mathbb{E}[|\text{tr}(\Omega_c^{-1} H_c \mathbf{E}_c^T)|] \\
&\stackrel{(b)}{\leq} \mathbb{E}[\|\Omega_c^{-1} H_c\|_2 \|\mathbf{E}_c\|_2] \\
&= \|\Omega_c^{-1} H_c\|_2 \mathbb{E}[\|\mathbf{E}_c\|_2] \\
&\stackrel{(c)}{\leq} \|\Omega_c^{-1} H_c\|_2 (\mathbb{E}[\|\mathbf{E}_c\|_2^2])^{\frac{1}{2}} \\
&\stackrel{(d)}{=} \frac{1}{\sqrt{3}} \|\Omega_c^{-1} H_c\|_2 \sqrt{n} \|\mathbf{r}\|_2,
\end{aligned} \tag{80}$$

where (a) and (c) are due to Jensen's inequality, (b) is due to Cauchy-Schwarz inequality $|\text{tr}(XY^T)| \leq \|X\|_2 \|Y\|_2$ and (d) is due to

$$\mathbb{E}[\|\mathbf{E}_c\|_2^2] = \sum_{i,j:0 \leq i-j \leq k} \mathbb{E}[(\mathbf{h}_{i,i-j} - c_{i-j})^2] = \sum_{i,j:0 \leq i-j \leq k} \frac{1}{3} r_{i-j}^2 = \frac{1}{3} n \|\mathbf{r}\|_2^2. \tag{81}$$

The next lemma presents an upper bound on $\|\Omega_c^{-1} H_c\|_2$:

Lemma 2. *We have*

$$\|\Omega_c^{-1} H_c\|_2 \leq \sqrt{n} \varrho \tag{82}$$

where ϱ is defined in (20).

Proof. See Appendix E. ■

By (80) and (82),

$$|\mathbb{E}[\text{tr}(\Omega_c^{-1} H_c \mathbf{E}^T)]| \leq \frac{1}{\sqrt{3}} n \varrho \|\mathbf{r}\|_2. \tag{83}$$

Now, (79) and (83) yield

$$\Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \leq \Pr(|\underline{\boldsymbol{\nu}}^T \boldsymbol{\Psi} \underline{\boldsymbol{\nu}} - \mathbb{E}[\underline{\boldsymbol{\nu}}^T \boldsymbol{\Psi} \underline{\boldsymbol{\nu}}]| \geq m(\eta' - \eta'_n)), \tag{84}$$

where η'_n is defined in (69). Using the assumption $\eta' > \eta'_n$ and applying Chebyshev's inequality,

$$\Pr(\underline{\mathbf{y}} \notin \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \leq \frac{\text{Var}(\underline{\boldsymbol{\nu}}^T \boldsymbol{\Psi} \underline{\boldsymbol{\nu}})}{m^2(\eta' - \eta'_n)^2}. \tag{85}$$

Following similar lines of reasoning that led to (63), we can write

$$\text{Var}(\underline{\boldsymbol{\nu}}^T \boldsymbol{\Psi} \underline{\boldsymbol{\nu}}) = 2\mathbb{E}[\text{tr}(\boldsymbol{\Psi}^2)] + \text{Var}(\text{tr}(\boldsymbol{\Psi})). \quad (86)$$

An upper bound on $\mathbb{E}[\text{tr}(\boldsymbol{\Psi}^2)]$ is given by

$$\mathbb{E}[\text{tr}(\boldsymbol{\Psi}^2)] \leq ((c_s + r_s)^2 P + (c_s + r_s)\sqrt{P} + 1)^2 (m + n). \quad (87)$$

The proof is slightly different from that of (64). The details appear in Appendix F. Finally, we compute an upper bound on the second term on the right side of (86). Recall from (77) that $\text{tr}(\boldsymbol{\Psi}) = m + 2P\text{tr}(\Omega_c^{-1} H_c \mathbf{E}_c^T)$. Then

$$\begin{aligned} \text{Var}(\text{tr}(\boldsymbol{\Psi})) &= 4P^2 \text{Var}(\text{tr}(\Omega_c^{-1} H_c \mathbf{E}_c^T)) \\ &= 4P^2 \text{Var}\left(\sum_{i,j} [\Omega_c^{-1} H_c]_{i,j} [\mathbf{E}_c]_{i,j}\right) \\ &\stackrel{(a)}{=} 4P^2 \text{Var}\left(\sum_{i,j: 0 \leq i-j \leq k} [\Omega_c^{-1} H_c]_{i,j} [\mathbf{E}_c]_{i,j}\right) \\ &\stackrel{(b)}{=} 4P^2 \sum_{i,j: 0 \leq i-j \leq k} [\Omega_c^{-1} H_c]_{i,j}^2 \text{Var}([\mathbf{E}_c]_{i,j}) \\ &\stackrel{(c)}{=} \frac{4}{3} P^2 \sum_{i,j: 0 \leq i-j \leq k} [\Omega_c^{-1} H_c]_{i,j}^2 r_{i-j}^2 \\ &\stackrel{(d)}{\leq} \frac{4}{3} P^2 \|r\|_\infty^2 \|\Omega_c^{-1} H_c\|_2^2 \\ &\stackrel{(e)}{\leq} \frac{1}{3} P \|r\|_\infty^2 n, \end{aligned} \quad (88)$$

where (a) is due to $[\mathbf{E}_c]_{i,j} = 0$ for $i - j < 0$ or $i - j > k$, (b) is due to $[\mathbf{E}_c]_{i,j}$ being independent for all pairs (i, j) with $0 \leq i - j \leq k$, (c) is due to $[\mathbf{E}_c]_{i,j}$ being uniformly distributed on $[-r_{i-j}, r_{i-j}]$, (d) follows by bounding r_{i-j}^2 and $\sum_{i,j: 0 \leq i-j \leq k} [\Omega_c^{-1} H_c]_{i,j}^2$ by $\|r\|_\infty^2$ and $\sum_{i,j} [\Omega_c^{-1} H_c]_{i,j}^2 = \|\Omega_c^{-1} H_c\|_2^2$, respectively, and finally (e) is due to Lemma 2 and the fact that $\varrho \leq \frac{1}{2\sqrt{P}}$. This completes the proof. $\blacksquare$

Next, let us concentrate on the second term on the right side of (68). We write

$$\begin{aligned}
& \Pr(\underline{\mathbf{x}}(i) \in \mathcal{T}_\epsilon^{(n)}(PI_n), \underline{\mathbf{w}}(i) \in \mathcal{T}_\eta^{(m+n)}(\Sigma_N), \underline{\mathbf{y}} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)) \\
& \stackrel{(a)}{=} \int_{\mathcal{T}_\eta^{(m+n)}(\Sigma_N) \cap (\mathcal{T}_\epsilon^{(n)}(PI_n) \times \mathcal{T}_{\eta'}^{(m)}(\Omega_c))} p_{\underline{\mathbf{x}}(i)}(\underline{x}) p_{\underline{\mathbf{y}}}(y) d\underline{x} d\underline{y} \\
& \stackrel{(b)}{\leq} 2^{-h_G(PI_n) + \frac{1}{2\ln 2} n\epsilon} \times \text{Vol}(\mathcal{T}_\eta^{(m+n)}(\Sigma_N)) \sup_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} p_{\underline{\mathbf{y}}}(y) \\
& \stackrel{(c)}{\leq} 2^{-h_G(PI_n) + \frac{1}{2\ln 2} n\epsilon} \times 2^{h_G(\Sigma_N) + \frac{m+n}{2} \log(1+\eta)} \sup_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} p_{\underline{\mathbf{y}}}(y) \\
& \stackrel{(d)}{\leq} 2^{\frac{m+n}{2} \log(1+\eta) + \frac{1}{2\ln 2} n\epsilon} (2\pi e)^{\frac{m}{2}} \sup_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} p_{\underline{\mathbf{y}}}(y), \tag{89}
\end{aligned}$$

where (a) is due to independence of $\underline{\mathbf{x}}(i)$ and $\underline{\mathbf{y}}$ for $i \neq 1$, (b) is due to $p_{\underline{\mathbf{x}}(i)}(\underline{x}) = p_G(\underline{x}; PI_n) \leq 2^{-h_G(PI_n) + \frac{1}{2\ln 2} n\epsilon}$ for $\underline{x} \in \mathcal{T}_\epsilon^{(n)}(PI_n)$, (c) is due to Lemma 1 on the volume of typical sets and (d) is due to

$$h_G(\Sigma_N) - h_G(PI_n) = \frac{1}{2} \log((2\pi e)^{m+n} \det(\Sigma_N)) - \frac{1}{2} \log(2\pi e P)^n = \frac{m}{2} \log(2\pi e), \tag{90}$$

regardless of the choice of $N \in \mathcal{N}$ which in turn follows from (43). Note that $p_{\underline{\mathbf{y}}}(y) \neq p_G(y; \Omega_c)$ and hence, we can not bound $p_{\underline{\mathbf{y}}}(y)$ for $\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)$ in a similar fashion as we bounded $p_{\underline{\mathbf{x}}(i)}(\underline{x})$ for $\underline{x} \in \mathcal{T}_\epsilon^{(n)}(PI_n)$ in step (b) in (89). We proceed by presenting an upper bound on the supremum on the right side of (89). Recall we denoted the range of the random matrix $\mathbf{H}$ by $\mathcal{H}$. We write

$$\begin{aligned}
p_{\underline{\mathbf{y}}}(y) &= \int_{\mathcal{H}} p_{\underline{\mathbf{y}}}(y | \mathbf{H} = H) d\mathcal{L}_{\mathbf{H}}(H) \\
&= \int_{\mathcal{H}} \frac{1}{(2\pi)^{\frac{m}{2}} (\det(I_m + PHH^T))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \underline{y}^T (I_m + PHH^T)^{-1} \underline{y}\right) d\mathcal{L}_{\mathbf{H}}(H) \\
&\leq (2\pi)^{-\frac{m}{2}} \left(\inf_{H \in \mathcal{H}} \det(\Omega_H)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \inf_{H \in \mathcal{H}} \underline{y}^T \Omega_H^{-1} \underline{y}\right), \tag{91}
\end{aligned}$$

where $\mathcal{L}_{\mathbf{H}}$ is the probability law of $\mathbf{H}$ and we have defined

$$\Omega_H := I_m + PHH^T. \tag{92}$$

Then

$$\sup_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} p_{\underline{\mathbf{y}}}(y) \leq (2\pi)^{-\frac{m}{2}} \left(\inf_{H \in \mathcal{H}} \det(\Omega_H)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \inf_{H \in \mathcal{H}} \inf_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} \underline{y}^T \Omega_H^{-1} \underline{y}\right). \tag{93}$$

The next two lemmas provide lower bounds on the infimums on the right side of (93):

Lemma 3. Recall ϕ_3 and ϕ_4 in (25). Then

$$\inf_{H \in \mathcal{H}} \det(\Omega_H) \geq 2^{-2m\phi_3} \det(\Omega_c) \quad (94)$$

and

$$\inf_{H \in \mathcal{H}} \det(\Omega_H) \geq (1 - \phi_4)^m \det(\Omega_c). \quad (95)$$

Proof. The proof for (94) uses a perturbation argument and the Mean Value Theorem. The proof for (95) relies in part on Weyl's inequality on perturbation of eigenvalues of a symmetric matrix. See Appendix G for the details. ■

Lemma 4. Recall ϕ_5 in (26). Then

$$\inf_{H \in \mathcal{H}} \inf_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} \underline{y}^T \Omega_H^{-1} \underline{y} \geq m(1 - \eta')^+ \phi_5. \quad (96)$$

Proof. The inner optimization is a quadratically constrained quadratic convex program. See Appendix H for the details. ■

By (93), (94), (95) and (96),

$$\sup_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} p_{\underline{y}}(\underline{y}) \leq (2\pi)^{-\frac{m}{2}} 2^{-\frac{1}{2} \log \det(\Omega_c)} 2^{m \min\{\phi_3, -\frac{1}{2} \log(1 - \phi_4)\}} \exp\left(-\frac{m}{2}(1 - \eta')^+ \phi_5\right), \quad (97)$$

Putting (68), (70), (89) and (97) together, the probability of type II error is bounded by

$$\Pr(\mathcal{E}^{(II)}) \leq C' / (m(\eta' - \eta'_n)^2) + 2^{n(R - \frac{1}{2n} \log \det(\Omega_c) + \delta_n)}, \quad (98)$$

where δ_n in the exponent is given by

$$\begin{aligned} \delta_n = & \frac{1}{n} \log |\mathcal{N}| + \frac{m+n}{2n} \log(1 + \eta) + \frac{m}{n} \min\left\{\phi_3, -\frac{1}{2} \log(1 - \phi_4)\right\} \\ & + \frac{m}{2n \ln 2} (1 - (1 - \eta')^+ \phi_5) + \frac{\epsilon}{2 \ln 2}. \end{aligned} \quad (99)$$

and we have the conditions $\eta' > \eta'_n$ and $\eta > \eta_n$. The right side of (98) vanishes as n grows if the two inequalities

$$R < \lim_{n \rightarrow \infty} \frac{1}{2n} \log \det(\Omega_c) - \lim_{n \rightarrow \infty} \delta_n, \quad \eta' > \lim_{n \rightarrow \infty} \eta'_n = \phi_2 \quad (100)$$

hold and the right side of (47) vanishes as n grows if

$$\eta > \lim_{n \rightarrow \infty} \eta_n = \frac{1}{24} P \sum_{i=0}^k \tau_i^2. \quad (101)$$

We can write

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n} \log \det(\Omega_c) &= \lim_{n \rightarrow \infty} \frac{1}{2n} \log \det(I_m + PH_c H_c^T) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n} \log \det(I_n + PH_c^T H_c) \\ &= C_{WGI,0}(P), \end{aligned} \quad (102)$$

where the penultimate step is due to $\det(I_m + PH_c H_c^T) = \det(I_n + PH_c^T H_c)$ and the last step is due to Szegő's Theorem [19] after one recognizes that the matrix $H_c^T H_c$ is a banded Toeplitz matrix given by $[H_c^T H_c]_{i,j} = (\sum_{l=0}^{k-|i-j|} c_l c_{l+|i-j|}) \mathbb{1}_{|i-j| \leq k}$. Also,

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta_n &= \log(L_0 L_1 \cdots L_k) + \log(1 + \eta) + \\ &\quad \min \left\{ \phi_3, -\frac{1}{2} \log(1 - \phi_4) \right\} + \frac{1}{2 \ln 2} (1 - (1 - \eta')^+ \phi_5) + \frac{\epsilon}{2 \ln 2}, \end{aligned} \quad (103)$$

where η and η' satisfy $\eta > \frac{1}{24} P \sum_{i=0}^k \tau_i^2 = \frac{1}{6} P \sum_{i=0}^k \frac{r_i^2}{L_i^2}$ and $\eta' > \phi_2$. The smallest value for $\lim_{n \rightarrow \infty} \delta_n$ is achieved when η and η' are replaced by $\frac{1}{6} P \sum_{i=0}^k \frac{r_i^2}{L_i^2}$ and ϕ_2 , respectively. Finally, we choose the integers $L_0, L_1, \dots, L_k$ to minimize $\log(L_0 L_1 \cdots L_k) + \log(1 + \frac{1}{6} P \sum_{i=0}^k \frac{r_i^2}{L_i^2})$. Induction shows that the minimum is achieved when at least k of the variables $L_0, L_1, \dots, L_k$ are 1. Without loss of generality, let $L_1 = L_2 = \cdots = L_k = 1$. Then the objective function to be minimized is $L_0(1 + \frac{1}{6} P \sum_{i=1}^k r_i^2 + \frac{\frac{1}{6} P r_0^2}{L_0^2})$. The minimum is achieved by $L_0 = \max \left\{ 1, \left[\left(\frac{\frac{1}{6} P r_0^2}{1 + \frac{1}{6} P \sum_{i=1}^k r_i^2} \right)^{1/2} \right] \right\}$.

VI. SUMMARY AND CONCLUDING REMARKS

We studied a stochastic and time varying Gaussian ISI channel where $\mathbf{h}_{t,i}$, the i^{th} tap during time slot t , was uniformly distributed over $[c_i - r_i, c_i + r_i]$. The array $\mathbf{h}_{t,i}$ was assumed to be independent along both indices t and i . Moreover, this array was unknown to both ends of transmission. An upper bound and a lower bound were derived on C_{WGI} , i.e., the largest achievable rate that can be achieved by white Gaussian codewords.

It was shown that the proposed upper bound saturates with the average transmission power P . As such, C_{WGI} does not scale with P . The lower bound was achieved by a joint-typicality decoder tuned to

a net of candidates for the random channel matrix $\mathbf{H}$. The net covered the range of $\mathbf{H}$ and its resolution was optimized to obtain the largest achievable rate. It was shown that if only one of the channel taps is changing randomly (e.g, the third tap in all times t), then the lower bound also saturates at a positive level as P increases. Otherwise, the lower bound vanishes for sufficiently large P . It remains unclear if this behaviour is intrinsic to joint-typicality decoding or an error analysis conducted more carefully would show that the proposed decoder always achieves positive rates regardless of how many taps are changing randomly. One direction this may be examined is to see if an upper bound more sophisticated than the naive union bound may be employed in order to bound the probability of the type II error.

An interesting problem not addressed in this paper is to examine the robustness of universal decoders in [8], [9] and [10] to variations in the unknown channel taps as modelled in this paper. It is also curious if employing such decoders may lead us to a closed expression for or a tighter lower bound on C_{WGI} .

APPENDIX A; PROOF OF LEMMA 1

For $r > 0$, positive integer n and real positive-definite matrix Σ , consider the ellipsoid

$$\mathcal{E}_r := \left\{ \underline{a} \in \mathbb{R}^n : \frac{1}{n} \underline{a}^T \Sigma^{-1} \underline{a} < r. \right\}. \quad (104)$$

Let $\Sigma = U \Lambda U^T$ be the spectral decomposition for Σ where U is a real orthogonal matrix and Λ is a diagonal matrix whose diagonal entries $\lambda_1, \dots, \lambda_N$ are the eigenvalues of Σ . The set $U^T \mathcal{E}_r := \{U^T \underline{a} : \underline{a} \in \mathcal{E}_r\}$ is an ellipsoid in standard form, i.e.,

$$U^T \mathcal{E}_r = \left\{ \underline{a} \in \mathbb{R}^n : \frac{1}{n} \sum_{i=1}^n \frac{a_i^2}{\lambda_i} < r \right\}. \quad (105)$$

The volume of the standard ellipsoid $\{(a_1, \dots, a_n) : \sum_{i=1}^n \frac{a_i^2}{c_i^2} \leq 1\}$ is $\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \prod_{i=1}^n c_i$ where $\Gamma(\cdot)$ is the Gamma function. Hence,

$$\begin{aligned}
\text{Vol}(\mathcal{E}_r) &\stackrel{(a)}{=} \text{Vol}(U^T \mathcal{E}_r) \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \prod_{i=1}^n (nr \lambda_i)^{\frac{1}{2}} \\
&\stackrel{(b)}{=} \frac{\pi^{\frac{n}{2}} n^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} (\det(\Sigma))^{\frac{1}{2}} r^{\frac{n}{2}} \\
&\stackrel{(c)}{\leq} (2\pi e)^{\frac{n}{2}} (\det(\Sigma))^{\frac{1}{2}} r^{\frac{n}{2}} \\
&= 2^{\frac{1}{2} \log((2\pi e)^n \det(\Sigma)) + \frac{n}{2} \log r} \\
&= 2^{h_G(\Sigma) + \frac{n}{2} \log r},
\end{aligned} \tag{106}$$

where (a) is due to the fact that the map $\underline{a} \mapsto U^T \underline{a}$ is an isometry, (b) is due to $\det(\Sigma) = \prod_{i=1}^n \lambda_i$ and (c) is due to the inequality $\Gamma(x+1) \geq x^x e^{-x}$ for any $x > 0$ according to Corollary 1.2 in [17]. Finally,

$$\text{Vol}(\mathcal{T}_\eta^{(n)}(\Sigma)) \leq \text{Vol}(\mathcal{E}_{1+\eta}) \leq 2^{h_G(\Sigma) + \frac{n}{2} \log(1+\eta)}. \tag{107}$$

APPENDIX B; PROOF OF THEOREM 2

Let us fix a codebook $\mathcal{C} = \{\underline{x}(1), \dots, \underline{x}(2^{nR})\}$. Then the transmitted vector $\underline{x}$ in (12) is uniformly distributed over $\mathcal{C}$. We write

$$\begin{aligned}
nR &= H(\mathbf{W}) \\
&= H(\mathbf{W}|\underline{\mathbf{y}}) + I(\mathbf{W}; \underline{\mathbf{y}}) \\
&\leq 1 + np_e(\mathcal{C})R + I(\underline{\mathbf{x}}; \underline{\mathbf{y}}),
\end{aligned} \tag{108}$$

where the last step is due to the Fano's inequality and the data processing inequality for mutual information. Averaging with respect to the codebook $\mathcal{C}$ over the Gaussian ensemble $\mathcal{C}$,

$$nR \leq 1 + nR \langle p_e(\mathcal{C}) \rangle_{\mathcal{C}} + \langle I(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \rangle_{\mathcal{C}}. \tag{109}$$

Let us write

$$\langle I(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \rangle_{\mathcal{C}} = \langle h(\underline{\mathbf{y}}) \rangle_{\mathcal{C}} - \langle h(\underline{\mathbf{y}}|\underline{\mathbf{x}}) \rangle_{\mathcal{C}}. \tag{110}$$

Next, we find an upper bound and a lower bound on the terms $\langle h(\underline{\mathbf{y}}) \rangle_{\mathcal{C}}$ and $\langle h(\underline{\mathbf{y}}|\underline{\mathbf{x}}) \rangle_{\mathcal{C}}$, respectively.

A. Upper bound on $\langle h(\underline{\mathbf{y}}) \rangle_{\mathcal{C}}$

Applying the maximum entropy lemma [20],

$$h(\underline{\mathbf{y}}) \leq \frac{1}{2} \log \left((2\pi e)^m \det(\text{Cov}(\underline{\mathbf{y}})) \right). \quad (111)$$

Define

$$\underline{\mathbf{m}}(\mathcal{C}) := \mathbb{E}[\underline{\mathbf{x}}] = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \underline{\mathbf{x}}(i), \quad \mathbf{Q}(\mathcal{C}) := \mathbb{E}[\underline{\mathbf{x}} \underline{\mathbf{x}}^T] = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \underline{\mathbf{x}}(i) \underline{\mathbf{x}}^T(i) \quad (112)$$

A simple computation shows that

$$\text{Cov}(\underline{\mathbf{y}}) = I_m + \mathbb{E}[\mathbf{H} \mathbf{Q}(\mathcal{C}) \mathbf{H}^T] - \mathbb{E}[\mathbf{H}] \underline{\mathbf{m}}(\mathcal{C}) \underline{\mathbf{m}}^T(\mathcal{C}) \mathbb{E}[\mathbf{H}^T] \quad (113)$$

By (111) and (113) and using the fact that $\log \det(\cdot)$ is nondecreasing along the cone of positive-semidefinite matrices,

$$h(\underline{\mathbf{y}}) \leq \frac{1}{2} \log \left((2\pi e)^m \det(I_m + \mathbb{E}[\mathbf{H} \mathbf{Q}(\mathcal{C}) \mathbf{H}^T]) \right). \quad (114)$$

Then

$$\begin{aligned} \langle h(\underline{\mathbf{y}}) \rangle_{\mathcal{C}} &\leq \left\langle \frac{1}{2} \log \left((2\pi e)^m \det(I_m + \mathbb{E}[\mathbf{H} \mathbf{Q}(\mathcal{C}) \mathbf{H}^T]) \right) \right\rangle_{\mathcal{C}} \\ &\stackrel{(a)}{\leq} \frac{1}{2} \log \left((2\pi e)^m \det(I_m + \langle \mathbb{E}[\mathbf{H} \mathbf{Q}(\mathcal{C}) \mathbf{H}^T] \rangle_{\mathcal{C}}) \right) \\ &\stackrel{(b)}{=} \frac{1}{2} \log \left((2\pi e)^m \det(I_m + \mathbb{E}[\mathbf{H} \langle \mathbf{Q}(\mathcal{C}) \rangle_{\mathcal{C}} \mathbf{H}^T]) \right) \\ &\stackrel{(c)}{=} \frac{1}{2} \log \left((2\pi e)^m \det(I_m + P \mathbb{E}[\mathbf{H} \mathbf{H}^T]) \right), \end{aligned} \quad (115)$$

where (a) is due to Jensen's inequality and concavity of $\log \det(\cdot)$, (b) is due to the Tonelli-Fubini Theorem enabling us to exchange the “averaging” operators $\mathbb{E}[\cdot]$ and $\langle \cdot \rangle_{\mathcal{C}}$ and (c) is due to $\langle \mathbf{Q}(\mathcal{C}) \rangle_{\mathcal{C}} = P I_n$. Note that

$$\mathbb{E}[(\mathbf{H} \mathbf{H}^T)_{i,i}] = \sum_{\substack{l=1 \\ 0 \leq i-l \leq k}}^n \mathbb{E}[\mathbf{h}_{i,i-l}^2] = \sum_{\substack{l=1 \\ 0 \leq i-l \leq k}}^n (c_{i-l}^2 + r_{i-l}^2/3) \quad (116)$$

and

$$\mathbb{E}[\mathbf{H}\mathbf{H}^T]_{i,j} = \sum_{\substack{l=1 \\ 0 \leq i-l \leq k \\ 0 \leq j-l \leq n}}^n \mathbb{E}[\mathbf{h}_{i,i-l} \mathbf{h}_{j,j-l}] = \sum_{\substack{l=1 \\ 0 \leq i-l \leq k \\ 0 \leq j-l \leq k}}^n c_{i-l} c_{j-l}, \quad i \neq j. \quad (117)$$

Let H_c be the matrix given in (67). Then we can write

$$\mathbb{E}[\mathbf{H}\mathbf{H}^T] = H_c H_c^T + D, \quad (118)$$

where D is a diagonal matrix with diagonal entries

$$D_{i,i} = \sum_{\substack{l=1 \\ 0 \leq i-l \leq k}}^n r_{i-l}^2 / 3 \leq \|\underline{r}\|_2^2 / 3 \quad (119)$$

for every $1 \leq i \leq m$. By (115), (118) and (119),

$$\begin{aligned} \langle h(\underline{\mathbf{y}}) \rangle_{\mathcal{C}} &\leq \frac{m}{2} \log(2\pi e) + \frac{1}{2} \log \det (I_m + P H_c H_c^T + P \|\underline{r}\|_2^2 / 3 I_m) \\ &\leq \frac{m}{2} \log(2\pi e) + \frac{m}{2} \log(1 + P \|\underline{r}\|_2^2 / 3) + \frac{1}{2} \log \det \left(I_m + \frac{P}{1 + P \|\underline{r}\|_2^2 / 3} H_c H_c^T \right), \end{aligned} \quad (120)$$

where in the first step once again we have used the fact that $\log \det(\cdot)$ is nondecreasing along the cone of positive-semidefinite matrices.

B. Lower bound on $\langle h(\underline{\mathbf{y}}|\underline{\mathbf{x}}) \rangle_{\mathcal{C}}$

Fix a realization $\underline{x} \in \mathcal{C}$ for $\underline{\mathbf{x}}$. Then

$$\begin{aligned} h(\underline{\mathbf{y}}|\underline{\mathbf{x}} = \underline{x}) &= h(\mathbf{H}\underline{x} + \mathbf{z}|\underline{\mathbf{x}} = \underline{x}) \\ &\stackrel{(a)}{=} h(\mathbf{H}\underline{x} + \underline{\mathbf{z}}) \\ &\stackrel{(b)}{=} h(\mathbf{H}\underline{x} - H_c \underline{x} + \underline{\mathbf{z}}) \\ &\stackrel{(c)}{=} h(\mathbf{E}_c \underline{x} + \underline{\mathbf{z}}) \\ &\stackrel{(d)}{=} \sum_{i=1}^m h([\mathbf{E}_c \underline{x}]_i + \mathbf{z}_i) \\ &\stackrel{(e)}{\geq} \sum_{i=k+1}^n h([\mathbf{E}_c \underline{x}]_i + \mathbf{z}_i), \end{aligned} \quad (121)$$

where (a) is due to independence of $\underline{x}$ and the pair $(\mathbf{H}, \mathbf{z})$, (b) is due to $H_c \underline{x}$ being a deterministic vector, the matrix $\mathbf{E}_c = \mathbf{H} - H_c$ in (c) was originally defined in (78), (d) follows from independence of

the entries of $\mathbf{E}_c \underline{x} + \underline{z}$ and in step (e) we have discarded the indices $1 \leq i \leq k$ and $n+1 \leq i \leq m$. For every $k+1 \leq i \leq n$, the random variable $[\mathbf{E}_c \underline{x}]_i = \sum_{j=i-k}^i [\mathbf{E}_c]_{i,j} x_j$ is the sum of (a constant number of) $k+1$ independent random variables. By entropy power inequality [20],

$$\begin{aligned}
h([\mathbf{E}_c \underline{x}]_i + \mathbf{z}_i) &\geq \frac{1}{2} \log \left(\sum_{j=i-k}^i 2^{2h([\mathbf{E}_c]_{i,j} x_j)} + 2^{2h(\mathbf{z}_i)} \right) \\
&\stackrel{(a)}{=} \frac{1}{2} \log \left(\sum_{j=i-k}^i 2^{2 \log(2r_{i-j}|x_j|)} + 2\pi e \right) \\
&= \frac{1}{2} \log \left(\sum_{j=i-k}^i 4r_{i-j}^2 x_j^2 + 2\pi e \right) \\
&= \frac{1}{2} \log \left(\sum_{j=0}^k 4r_j^2 x_{i-j}^2 + 2\pi e \right) \\
&\stackrel{(b)}{\geq} \frac{1}{2} \max \left\{ \log \sum_{j=0}^k 4r_j^2 x_{i-j}^2, \log(2\pi e) \right\} \\
&\stackrel{(c)}{=} \frac{1}{2} \log(2\pi e) + \frac{1}{2} \left(\log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} \right)^+
\end{aligned} \tag{122}$$

where (a) is due to $h(\mathbf{z}_i) = \frac{1}{2} \log(2\pi e)$ and the fact that $[\mathbf{E}_c]_{i,j} x_j = (\mathbf{h}_{i,i-j} - c_{i-j})x_j$ is uniformly distributed over the interval $[-r_{i-j}|x_j|, r_{i-j}|x_j|]$ for every i, j where $0 \leq i-j \leq k$, (b) is due to the trivial inequality $\log(a+b) \geq \max\{\log a, \log b\}$ and (c) follows by adding and subtracting $\frac{1}{2} \log(2\pi e)$. By (121) and (122) and averaging over $\underline{x} \in \mathcal{C}$,

$$\begin{aligned}
\langle h(\underline{\mathbf{y}}|\underline{\mathbf{x}}) \rangle_{\mathcal{C}} &\geq \frac{1}{2}(n-k) \log(2\pi e) + \frac{1}{2} \sum_{i=k+1}^n \left\langle \left(\log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} \right)^+ \right\rangle_{\mathcal{C}} \\
&\geq \frac{1}{2}(n-k) \log(2\pi e) + \frac{1}{2} \sum_{i=k+1}^n \left(\left\langle \log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} \right\rangle_{\mathcal{C}} \right)^+,
\end{aligned} \tag{123}$$

where the second step uses Jensen's inequality and convexity of the function $u^+ = \max\{u, 0\}$ in terms of u . The term on the right side of (123) is an expectation involving a “weighted chi-square” random variable whose density does not admit a closed form. We further bound this term from below as follows. Let $\mathcal{J} = \{j : r_j > 0\}$ and $k' = |\mathcal{J}|$. Using arithmetic-geometric inequality,

$$\begin{aligned}
\log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} &\geq \log \left(k' \left(\prod_{j \in \mathcal{J}} \frac{2r_j^2 x_{i-j}^2}{\pi e} \right)^{\frac{1}{k'}} \right) \\
&= \log k' + \frac{1}{k'} \sum_{j \in \mathcal{J}} \log \frac{2r_j^2 x_{i-j}^2}{\pi e}.
\end{aligned} \tag{124}$$

Then

$$\begin{aligned}
\left\langle \log \sum_{j \in \mathcal{J}} \frac{2r_j^2 x_{i-j}^2}{\pi e} \right\rangle_{\mathcal{C}} &\geq \log k' + \frac{1}{k'} \sum_{j \in \mathcal{J}} \left\langle \log \frac{2r_j^2 x_{i-j}^2}{\pi e} \right\rangle_{\mathcal{C}} \\
&= \log k' + \frac{1}{k'} \sum_{j \in \mathcal{J}} \left(\log \frac{2r_j^2}{\pi e} + \langle \log x_{i-j}^2 \rangle_{\mathcal{C}} \right).
\end{aligned} \tag{125}$$

But,

$$\begin{aligned}
\langle \log x_{i-j}^2 \rangle_{\mathcal{C}} &= \int_{-\infty}^{\infty} p_G(x; P) \log x^2 dx \\
&= \frac{2}{\sqrt{2\pi P}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2P}} \log |x| dx \\
&= \log(2P) + \frac{4}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} \log x dx,
\end{aligned} \tag{126}$$

where we have changed the variable x to $\sqrt{2P}x$. It is an easy exercise to show that

$$\int_0^{\infty} e^{-x^2} \log x dx = -\frac{\sqrt{\pi}}{2} \left(\frac{1}{2 \ln 2} \gamma_e + 1 \right), \tag{127}$$

where $\gamma_e \approx 0.577$ is the Euler-Mascheroni constant. Putting (125), (126) and (127) together,

$$\left\langle \log \sum_{j \in \mathcal{J}} \frac{2r_j^2 x_{i-j}^2}{\pi e} \right\rangle_{\mathcal{C}} \geq \log \frac{P}{\pi e} - \frac{1}{\ln 2} \gamma_e + \log k' + \frac{2}{k'} \sum_{j \in \mathcal{J}} \log r_j. \tag{128}$$

By (123) and (128) and denoting $\frac{1}{2} \log k' + \frac{1}{k'} \sum_{j \in \mathcal{J}} \log r_j$ by κ' , we get

$$\langle h(\underline{\mathbf{y}}|\underline{\mathbf{x}}) \rangle_{\mathcal{C}} \geq \frac{1}{2} (n-k) \log(2\pi e) + (n-k) \left(\frac{1}{2} \log \frac{P}{\pi e} + \kappa' - \frac{1}{2 \ln 2} \gamma_e \right)^+. \tag{129}$$

Putting (120) and (129) together,

$$\begin{aligned}
\langle I(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \rangle_{\mathcal{C}} &\leq k \log(2\pi e) + \frac{m}{2} \log(1 + P \|\underline{\mathbf{r}}\|_2^2/3) + \frac{1}{2} \log \det \left(I_m + \frac{P}{1 + \|\underline{\mathbf{r}}\|_2^2 P/3} H_c H_c^T \right) \\
&\quad - (n-k) \left(\frac{1}{2} \log \frac{P}{\pi e} + \kappa' - \frac{1}{2 \ln 2} \gamma_e \right)^+.
\end{aligned} \tag{130}$$

The desired upper bound in (31) follows from (109) and (130) and the facts that $\lim_{n \rightarrow \infty} \langle p_e(\mathcal{C}) \rangle_{\mathcal{C}} = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{2m} \log \det \left(I_m + \frac{P}{1 + P \|\underline{\mathbf{r}}\|_2^2/3} H_c H_c^T \right) = C_{WGI,0} \left(\frac{P}{1 + P \|\underline{\mathbf{r}}\|_2^2/3} \right)$.

We end this appendix by two remarks:

(i) The constant κ' in (130) can be replaced by $\kappa = \max\{\kappa', \log \|\underline{\mathbf{r}}\|_{\infty}\}$ defined in (30). This follows

from a different lower bound on the term on the right side of (124) given by

$$\log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} \geq \log \frac{2\|r\|_\infty^2 x_{i-j_0}^2}{\pi e} = \log \frac{2\|r\|_\infty^2}{\pi e} + \log x_{i-j_0}^2, \quad (131)$$

where $0 \leq j_0 \leq k$ is such that $r_{j_0} = \|r\|_\infty$. The term $\langle \log x_{i-j_0}^2 \rangle_{\mathcal{C}}$ can be computed as in (126).

(ii) Assume there is $r > 0$ such that $r_i = r$ for every $0 \leq i \leq k$. Then

$$\left\langle \log \sum_{j=0}^k \frac{2r_j^2 x_{i-j}^2}{\pi e} \right\rangle_{\mathcal{C}} = \log \frac{2r^2}{\pi e} + \left\langle \log \sum_{j=0}^k x_{i-j}^2 \right\rangle_{\mathcal{C}} = \log \frac{2r^2}{\pi e} + \log P + \mathbb{E}[\log \chi_{k+1}^2]. \quad (132)$$

APPENDIX C; PROOF OF (63)

For a random variable $\mathbf{x}$ and a measurable function f , we adopt the notation

$$\mathbb{E}_{\mathbf{x}}[f(\mathbf{x})] = \mathbb{E}[f(\mathbf{x})]. \quad (133)$$

For notational simplicity, we denote Φ_τ by Φ through this appendix. By definition,

$$\text{Var}(\underline{\nu}^T \Phi \underline{\nu}) = \mathbb{E}[(\underline{\nu}^T \Phi \underline{\nu})^2] - (\mathbb{E}[\underline{\nu}^T \Phi \underline{\nu}])^2. \quad (134)$$

But,

$$\begin{aligned} \mathbb{E}[(\underline{\nu}^T \Phi \underline{\nu})^2] &\stackrel{(a)}{=} \mathbb{E}_{\Phi} \left[\mathbb{E}[(\underline{\nu}^T \Phi \underline{\nu})^2 | \Phi = \Phi] \right] \\ &\stackrel{(b)}{=} \mathbb{E}_{\Phi} \left[\mathbb{E}[(\underline{\nu}^T \Phi \underline{\nu})^2] \right] \\ &\stackrel{(c)}{=} \mathbb{E}_{\Phi} \left[\text{Var}(\underline{\nu}^T \Phi \underline{\nu}) + (\mathbb{E}[\underline{\nu}^T \Phi \underline{\nu}])^2 \right] \\ &\stackrel{(d)}{=} \mathbb{E}_{\Phi} [2\text{tr}(\Phi^2) + (\text{tr}(\Phi))^2] \\ &\stackrel{(e)}{=} 2\mathbb{E}[\text{tr}(\Phi^2)] + \mathbb{E}[(\text{tr}(\Phi))^2], \end{aligned} \quad (135)$$

where (a) is due to the tower property of expectation, i.e., $\mathbb{E}[\mathbf{x}] = \mathbb{E}_{\mathbf{y}}[\mathbb{E}[\mathbf{x}|\mathbf{y} = y]]$, (b) is due to independence of $\underline{\nu}$ and Φ , (c) is due to the definition of variance, (d) is due to the well-known formula $\text{Var}(\underline{\nu}^T Q \underline{\nu}) = 2\text{tr}(Q^2)$ for arbitrary square matrix Q and (e) is due to the notation adopted in (133). Moreover, we have seen earlier in (60) that $\mathbb{E}[\underline{\nu}^T \Phi \underline{\nu}] = \mathbb{E}[\text{tr}(\Phi)]$. Using this together with (135), the proof of (63) is complete.

APPENDIX D; PROOF OF (64)

We need the following lemma:

Lemma 5. *For a matrix M , define the maximum row-sum norm by*

$$\|M\|_r := \max_i \sum_j |M_{i,j}|. \quad (136)$$

Then $\|\cdot\|_r$ becomes a matrix norm. In particular, it is sub-multiplicative, i.e.,

$$\|M_1 M_2\|_r \leq \|M_1\|_r \|M_2\|_r, \quad (137)$$

for every two matrices M_1 and M_2 of proper sizes. Moreover,

$$\|M\|^2 \leq \|M^T\|_r \|M\|_r. \quad (138)$$

for every matrix M .

Proof. See [18]. The statement in (138) is a special case of the fact that the spectral radius of a square matrix is bounded from above by every matrix-norm of that matrix. ■

Fix a realization Φ_τ of Φ . We proceed to find an upper bound on $\text{tr}(\Phi_\tau^2)$. We write

$$\text{tr}(\Phi_\tau^2) \leq (m+n)\lambda_{\max}(\Phi_\tau^2) = (m+n)\|\Phi\|^2 \leq (m+n)\|\Phi_\tau\|_r^2, \quad (139)$$

where the last step is due to (138) in Lemma 5. Looking at the structure of Φ in (53),

$$\begin{aligned} \|\Phi\|_r &= \max \left\{ \left\| \begin{bmatrix} P E_\tau^T E_\tau & \sqrt{P} E_\tau^T \end{bmatrix} \right\|_r, \left\| \begin{bmatrix} \sqrt{P} E_\tau & I_n \end{bmatrix} \right\|_r \right\} \\ &\leq \max \{ P \|E_\tau^T E_\tau\|_r + \sqrt{P} \|E_\tau^T\|_r, 1 + \sqrt{P} \|E_\tau\|_r \} \\ &\leq \max \{ P \|E_\tau^T\|_r \|E_\tau\|_r + \sqrt{P} \|E_\tau^T\|_r, 1 + \sqrt{P} \|E_\tau\|_r \}, \end{aligned} \quad (140)$$

where the penultimate step is due to the fact that $\|M\|_r \leq \|M_1\|_r + \|M_2\|_r$ for a block matrix $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ and the last step is due to (137) in Lemma 5. The matrix E_τ is banded, i.e., $[E_\tau]_{i,j} = 0$ for $i - j < 0$ or $i - j > k$ and $|[E_\tau]_{i,j}| \leq \frac{1}{2}\tau_{i-j}$ for $0 \leq i - j \leq k$. It follows that

$$\|E_\tau\|_r \leq \frac{1}{2} \sum_{l=0}^k \tau_l, \quad \|E_\tau^T\|_r \leq \frac{1}{2} \sum_{l=0}^k \tau_l. \quad (141)$$

It is clear that $\frac{1}{2} \sum_{l=0}^k \tau_l \leq \sum_{l=0}^k r_l = r_s$. Putting (140) and (141) together,

$$\|\Phi_\tau\|_r \leq \max \{r_s^2 P + r_s \sqrt{P}, 1 + r_s \sqrt{P}\} \leq r_s^2 P + r_s \sqrt{P} + 1. \quad (142)$$

By (139) and (142),

$$\text{tr}(\Phi_\tau^2) \leq (r_s^2 P + r_s \sqrt{P} + 1)^2 (m + n). \quad (143)$$

APPENDIX E; PROOF OF LEMMA 2

We write

$$\begin{aligned} \|\Omega_c^{-1} H_c\|_2^2 &= \text{tr}((\Omega_c^{-1} H_c)^T \Omega_c^{-1} H_c) \\ &\leq n \lambda_{\max}((\Omega_c^{-1} H_c)^T \Omega_c^{-1} H_c) \\ &= n \lambda_{\max}(H_c^T \Omega_c^{-2} H_c). \end{aligned} \quad (144)$$

Let

$$H_c = U \Lambda V^T \quad (145)$$

be the singular value decomposition for H_c . Here, U and V are $m \times m$ and $n \times n$ real orthogonal matrices and

$$\Lambda = \begin{bmatrix} \Lambda_c \\ 0_{k,n} \end{bmatrix}, \quad (146)$$

where Λ_c is an $n \times n$ diagonal matrix whose diagonal entries are the nonnegative square roots of the eigenvalues of $H_c^T H_c$. Then

$$H_c^T \Omega_c^{-2} H_c = V \Lambda^T (I_m + P \Lambda \Lambda^T)^{-2} \Lambda V^T \quad (147)$$

and we get

$$\lambda_{\max}(H_c^T \Omega_c^{-2} H_c) = \lambda_{\max}(\Lambda^T (I_m + P \Lambda \Lambda^T)^{-2} \Lambda). \quad (148)$$

By (146),

$$\Lambda^T (I_m + P \Lambda \Lambda^T)^{-2} \Lambda = \Lambda_c^2 (I_n + P \Lambda_c^2)^{-2} \quad (149)$$

As such,

$$\lambda_{\max}(H_c^T \Omega_c^{-2} H_c) = \max \left\{ \frac{\lambda}{(1 + P\lambda)^2} : \lambda \text{ is an eigenvalue of } H_c^T H_c \right\}. \quad (150)$$

We need the following lemma which is a restatement of Lemma 4.1 in [19]:

Lemma 6. *Let T be an $n \times n$ symmetric Toeplitz matrix with real entries given by $T_{i,j} = t_{i-j}$ where $t_{-l} = t_l$ for every $0 \leq l \leq n-1$. Moreover, define the function $f_T(\cdot)$ by*

$$f_T(\omega) = \sum_{l=-(n-1)}^{n-1} t_l e^{\sqrt{-1} l \omega}, \quad 0 \leq \omega \leq 2\pi. \quad (151)$$

Then every eigenvalue of T lies in the interval $[\min_{\omega} f_T(\omega), \max_{\omega} f_T(\omega)]$.

The matrix $H_c^T H_c$ is a symmetric Toeplitz matrix given by $[H_c^T H_c]_{i,j} = \sum_{l=0}^{k-|i-j|} c_l c_{l+|i-j|} \mathbb{1}_{|i-j| \leq k}$. Then $f_{H_c^T H_c}(\omega) = |\sum_{i=0}^k c_i e^{\sqrt{-1} i \omega}|^2$. By Lemma 6, every eigenvalue λ of $H_c^T H_c$ satisfies

$$\alpha^2 \leq \lambda \leq \beta^2, \quad (152)$$

where α and β are given in (19). The function $\frac{x}{(1+Px)^2}$ is increasing for $0 < x \leq \frac{1}{P}$ and decreasing for $x \geq \frac{1}{P}$. Using this fact together with (150) and (152),

$$\lambda_{\max}(H_c^T \Omega_c^{-2} H_c) \leq \begin{cases} \frac{\beta^2}{(1+P\beta^2)^2} & \frac{1}{P} \geq \beta^2 \\ \frac{\alpha^2}{(1+P\alpha^2)^2} & \frac{1}{P} \leq \alpha^2 \end{cases}. \quad (153)$$

For values of P satisfying $\alpha^2 < \frac{1}{P} < \beta^2$, we simply let

$$\lambda_{\max}(H_c^T \Omega_c^{-2} H_c) \leq \sup_{\lambda \geq 0} \frac{\lambda}{(1 + P\lambda)^2} = \frac{1}{4P}. \quad (154)$$

By (144), (153) and (154) and recalling the definition of ϱ in (20), we get $\|\Omega_c^{-1} H_c\|_2 \leq \sqrt{n} \varrho$.

APPENDIX F; PROOF OF (87)

We compute an upper bound on $\text{tr}(\Psi^2)$ where the positive semidefinite matrix Ψ is a fixed realization of Ψ in (75). It appears useless to start as in (139) due to the fact that we do not have an upper bound on $\|\Omega_c^{-1}\|_r$. The following lemma proves helpful:

Lemma 7. *Let the partitioned symmetric matrix M be given by*

$$M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}. \quad (155)$$

Then

$$\|M\| \leq \max \{ \|A\| + \|B\|, \|C\| + \|B\| \}. \quad (156)$$

Proof. Let us write the matrix M in (155) as

$$M = M_1 + M_2, \quad (157)$$

with

$$M_1 = \begin{bmatrix} A & O_1^T \\ O_1 & C \end{bmatrix}, \quad M_2 = \begin{bmatrix} O_2 & B^T \\ B & O_3 \end{bmatrix}, \quad (158)$$

where O_1, O_2 and O_3 are zero matrices of appropriate sizes. It is well-known that

$$\|M_1\| = \max \{ \|A\|, \|C\| \}, \quad \|M_2\| = \|B\|. \quad (159)$$

Hence,

$$\|M\| \leq \|M_1\| + \|M_2\| = \max \{ \|A\| + \|B\|, \|C\| + \|B\| \}. \quad (160)$$

■

Let us write

$$\Psi = \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} \\ \Psi_{1,2}^T & \Psi_{2,2} \end{bmatrix}, \quad (161)$$

where

$$\Psi_{1,1} := PH^T\Omega_c^{-1}H \quad \Psi_{1,2} := \sqrt{P}H^T\Omega_c^{-1}, \quad \Psi_{2,2} := \Omega_c^{-1} \quad (162)$$

Then

$$\text{tr}(\Psi^2) \leq (m+n)\lambda_{\max}(\Psi^2) = (m+n)\|\Psi\|^2 \leq (m+n)\left(\max\{\|\Psi_{1,1}\| + \|\Psi_{1,2}\|, \|\Psi_{2,2}\| + \|\Psi_{1,2}\|\}\right)^2, \quad (163)$$

where last step is due to Lemma 7. But,

$$\|\Psi_{1,1}\| = \|PH^T\Omega_c^{-1}H\| \leq P\|H^T\|\|\Omega_c^{-1}\|\|H\| \leq P\|H\|^2, \quad (164)$$

where we have used the facts that $\|H^T\| = \|H\|$ and $\|\Omega_c^{-1}\| \leq 1$. Similarly,

$$\|\Psi_{1,2}\| \leq \sqrt{P}\|H\| \quad (165)$$

and

$$\|\Psi_{2,2}\| \leq 1. \quad (166)$$

By (163), (164), (165) and (166),

$$\begin{aligned} \text{tr}(\Psi^2) &\leq (m+n)\left(\max\{P\|H\|^2 + \sqrt{P}\|H\|, 1 + \sqrt{P}\|H\|\}\right)^2 \\ &\leq (m+n)(P\|H\|^2 + \sqrt{P}\|H\| + 1)^2. \end{aligned} \quad (167)$$

The matrix H is banded, i.e., $H_{i,j} = 0$ for $i-j < 0$ or $i-j > k$. Moreover, $|H_{i,j}| = |h_{i,i-j}| \leq |c_{i-j}| + r_{i-j}$ for $0 \leq i-j \leq k$. It follows that

$$\|H\|_r \leq c_s + r_s, \quad \|H^T\|_r \leq c_s + r_s. \quad (168)$$

Using (168) together with (138) in Lemma 5, we get

$$\|H\| \leq (\|H\|_r\|H^T\|_r)^{1/2} \leq c_s + r_s. \quad (169)$$

By (167) and (169),

$$\text{tr}(\Psi^2) \leq (m+n)((c_s+r_s)^2P + (c_s+r_s)\sqrt{P} + 1)^2. \quad (170)$$

APPENDIX G; PROOF OF LEMMA 3

Fix $H \in \mathcal{H}$. In this appendix we obtain three lower bounds on $\frac{\det(\Omega_H)}{\det(\Omega_c)}$. In general, none of these bounds is tighter than the other two. We prove (95) first. Recall the matrix E_c in (78). Writing $H = E_c + H_c$,

$$\begin{aligned} \det(\Omega_H) &= \det(I_m + PHH^T) \\ &= \det(I_m + PH_cH_c^T + PE_cE_c^T + PE_cH_c^T + PH_cE_c^T) \\ &= \det(\Omega_c) \det(I_m + \Delta), \end{aligned} \quad (171)$$

where

$$\Delta := \Omega_c^{-1}\Omega_H - I_m = P\Omega_c^{-1}E_cE_c^T + P\Omega_c^{-1}E_cH_c^T + P\Omega_c^{-1}H_cE_c^T. \quad (172)$$

We make two observations about the matrix Δ :

- (i) The matrix $\Omega_c^{-1}\Omega_H$ is a product of two positive-definite matrices and hence, its eigenvalues are all real (and positive). Therefore, eigenvalues of Δ are all real numbers as well.
- (ii) The spectral norm $\|\Delta\|$ is bounded as

$$\begin{aligned} \|\Delta\| &\leq P\|\Omega_c^{-1}\| \|E_c\| \|E_c^T\| + P\|\Omega_c^{-1}\| \|E_c\| \|H_c^T\| + P\|\Omega_c^{-1}H_c\| \|E_c^T\| \\ &\stackrel{(a)}{\leq} P\|E_c\|^2 + P\|E_c\| \|H_c\| + P\|\Omega_c^{-1}H_c\| \|E_c\| \\ &\stackrel{(b)}{\leq} Pr_s^2 + Pr_s\|H_c\| + Pr_s\|\Omega_c^{-1}H_c\| \\ &\stackrel{(c)}{\leq} Pr_s^2 + Pr_s\beta + Pr_s\|\Omega_c^{-1}H_c\| \\ &\stackrel{(d)}{\leq} Pr_s^2 + Pr_s\beta + Pr_s\varrho \\ &= r_s(r_s + \beta + \varrho)P, \end{aligned} \quad (173)$$

where (a) is due to $\|M^T\| = \|M\|$ for every matrix M and $\|\Omega_c^{-1}\| \leq 1$, (b) is due to $\|E_c\| \leq r_s$ which itself is a consequence of $\|E_c\|_r \leq r_s$, $\|E_c^T\| \leq r_s$ and (138) in Lemma 5, (c) is due to $\|H_c\| \leq \beta$ which follows from (152) and (d) is due to (153), (154) and the definition of ϱ in (20).

It results from (i) and (ii) that no eigenvalue of Δ is larger than $r_s(r_s + \beta + \varrho)P$ in absolute value. Therefore, $\det(I_m + \Delta) \geq (1 - r_s(r_s + \beta + \varrho)P)^m$ where we have assumed r_s is small enough such that $r_s(r_s + \beta + \varrho)P < 1$. Using this in (171),

$$\det(\Omega_H) \geq \det(\Omega_c)(1 - r_s(r_s + \beta + \varrho)P)^m. \quad (174)$$

Next, we develop a second lower bound on $\det(\Omega_H)$. For a real symmetric $n \times n$ matrix X , denote its eigenvalues in increasing order by $\lambda_i(X)$ for $1 \leq i \leq n$, i.e., $\lambda_1(X) \leq \lambda_2(X) \leq \dots \leq \lambda_n(X)$. We need the following lemma which is a direct consequence of Theorem 4.3.1 in [18] due to Weyl:

Lemma 8. *Let A and B be $n \times n$ real symmetric matrices. Then*

$$|\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|, \quad (175)$$

for every $1 \leq i \leq n$.

Denote the eigenvalues of $H^T H$ and $H_c^T H_c$ in increasing order by μ_i and λ_i , respectively. Then

$$\frac{\det(\Omega_H)}{\det(\Omega_c)} = \frac{\det(I_m + P H H^T)}{\det(I_m + P H_c H_c^T)} = \frac{\det(I_n + P H^T H)}{\det(I_n + P H_c^T H_c)} = \prod_{i=1}^n \frac{1 + P \mu_i}{1 + P \lambda_i}. \quad (176)$$

We can write

$$\frac{1 + P \mu_i}{1 + P \lambda_i} = 1 + \frac{P(\mu_i - \lambda_i)}{1 + P \lambda_i} \geq 1 - \frac{P|\mu_i - \lambda_i|}{1 + P \min_i \lambda_i}, \quad (177)$$

for every $1 \leq i \leq n$. By (152), $\min_i \lambda_i \geq \alpha^2$ where α is given in (19). Moreover, $|\mu_i - \lambda_i| \leq \|H^T H - H_c^T H_c\|$ due to Lemma 8. Therefore,

$$\frac{1 + P \mu_i}{1 + P \lambda_i} \geq 1 - \frac{P\|H^T H - H_c^T H_c\|}{1 + \alpha^2 P}. \quad (178)$$

Writing $H = E_c + H_c$, we get

$$\|H^T H - H_c^T H_c\| = \|E_c^T E_c + E_c^T H_c + H_c^T E_c\| \leq \|E_c\|^2 + 2\|E_c\|\|H_c\| \leq r_s^2 + 2r_s\beta, \quad (179)$$

where the last step is due to $\|H_c\| \leq \beta$ and $\|E\| \leq r_s$. By (176), (177), (178) and (179)

$$\det(\Omega_H) \geq \det(\Omega_c) \left(1 - \frac{r_s(r_s + 2\beta)P}{1 + \alpha^2 P}\right)^n, \quad (180)$$

where we have assumed r_s is small enough such that $\frac{r_s(r_s+2\beta)P}{1+\alpha^2P} < 1$.

Finally, we prove (94). Define

$$\Omega(t) = I_m + P(H_c + tE_c)(H_c + tE_c)^T \quad (181)$$

and

$$F(t) = \ln \det(\Omega(t)), \quad (182)$$

where $0 \leq t \leq 1$. The function F is differentiable on $[0, 1]$ and we can apply the Mean Value Theorem to write

$$F(1) - F(0) = F'(u), \quad (183)$$

for some $u \in (0, 1)$. Note that $\Omega(0) = \Omega_c$ and $\Omega(1) = \Omega_H$. Therefore, one has the bound

$$|\ln \det(\Omega_H) - \ln \det(\Omega_c)| \leq \max_{0 \leq t \leq 1} |F'(t)|. \quad (184)$$

Using the formula [16]

$$d(\ln \det(A)) = \text{tr}(A^{-1}dA) \quad (185)$$

for any differentiable positive-definite matrix A ,

$$F'(t) = \text{tr}(\Omega^{-1}(t)\Omega'(t)), \quad (186)$$

where

$$\Omega'(t) = \frac{d}{dt}\Omega(t) = PH_cE_c^T + PE_cH_c^T + 2tPE_cE_c^T. \quad (187)$$

By Cauchy-Schwarz inequality,

$$|F'(t)| = |\text{tr}(\Omega^{-1}(t)\Omega'(t))| \leq \|\Omega^{-1}(t)\|_2 \|\Omega'(t)\|_2. \quad (188)$$

We have

$$\|\Omega^{-1}(t)\|_2^2 = \text{tr}(\Omega^{-2}(t)) \leq m\lambda_{\max}(\Omega^{-2}(t)) \leq m, \quad (189)$$

where the last step is due to the fact that no eigenvalue of $\Omega^{-1}(t)$ is larger than one. Also,

$$\begin{aligned}
\|\Omega'(t)\|_2^2 &= \text{tr}((\Omega'(t))^2) \\
&\leq m\lambda_{\max}((\Omega'(t))^2) \\
&= m\lambda_{\max}^2(\Omega'(t)) \\
&\leq m\|\Omega'(t)\|^2 \\
&\leq m(2r_s(r_s + \beta)P)^2,
\end{aligned} \tag{190}$$

where once again we have used $\|H_c\| \leq \beta$ and $\|E_c\| \leq r_s$. By (184), (188), (189) and (190),

$$|\log \det(\Omega_H) - \log \det(\Omega_c)| \leq \frac{2m}{\ln 2} r_s(r_s + \beta)P. \tag{191}$$

APPENDIX H; PROOF OF LEMMA 4

We write

$$\inf_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} \underline{y}^T \Omega_H^{-1} \underline{y} = \inf_{m(1-\eta') < t < m(1+\eta')} \min_{\underline{y} \in \mathbb{R}^m: \underline{y}^T \Omega_c^{-1} \underline{y} = t} \underline{y}^T \Omega_H^{-1} \underline{y}. \tag{192}$$

To achieve $\min_{\underline{y} \in \mathbb{R}^m} \underline{y}^T \Omega_H^{-1} \underline{y}$, the gradient of $\underline{y}^T \Omega_H^{-1} \underline{y}$ must be parallel to the gradient of $\underline{y}^T \Omega_c^{-1} \underline{y}$, i.e.,

$$\Omega_H^{-1} \underline{y} = \lambda \Omega_c^{-1} \underline{y}, \tag{193}$$

where λ is the Lagrange multiplier. This says that the ‘‘optimum’’ $\underline{y}$ must be an eigenvector for the matrix $\Omega_c \Omega_H^{-1}$ with corresponding eigenvalue λ . Multiplying both sides of (193) by $\underline{y}^T$ from left, we conclude that

$$\inf_{\underline{y} \in \mathbb{R}^m: \underline{y}^T \Omega_c^{-1} \underline{y} = t} \underline{y}^T \Omega_H^{-1} \underline{y} = t \lambda_{\min}(\Omega_c \Omega_H^{-1}), \tag{194}$$

where $\lambda_{\min}(\Omega_c \Omega_H^{-1})$ is the minimum eigenvalue of $\Omega_c \Omega_H^{-1}$. Putting (192) and (194) together,

$$\inf_{\underline{y} \in \mathcal{T}_{\eta'}^{(m)}(\Omega_c)} \underline{y}^T \Omega_H^{-1} \underline{y} = m(1 - \eta') \lambda_{\min}(\Omega_c \Omega_H^{-1}). \tag{195}$$

Moreover,

$$\begin{aligned}
\lambda_{\min}(\Omega_c \Omega_H^{-1}) &= \frac{1}{\lambda_{\max}(\Omega_c^{-1} \Omega_H)} \\
&\stackrel{(a)}{=} \frac{1}{\lambda_{\max}(I_m + \Delta)} \\
&\stackrel{(b)}{\geq} \frac{1}{1 + \|\Delta\|} \\
&\stackrel{(c)}{\geq} \phi_5,
\end{aligned} \tag{196}$$

where (a) is due to the definition of Δ in (172), (b) is due to $\|\Delta\|$ being an upper bound on $\lambda_{\max}(\Delta)$ and (c) is due to (173) and the definition of ϕ_5 .

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