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On the node reliability of Harary graphs: an analysis through real-world topologies

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Abstract—The literature on network reliability shows that Harary networks are designed so that the link reliability is maximum in many cases. The following question: “what are the best topology networks in maximizing node reliability?” is still open. In this paper, we performed computational experiments with eleven real-world networks and their corresponding Harary graphs. The node reliability of both sets of networks was computed. Computational results point out that the Harary network has a topology with high node reliability if compared to the real-world networks studied.

Index Terms—Harary graph, Network, node reliability.

I. INTRODUCTION

The reliability of a network is defined as the probability that the network remains connected even after removing a subset of links or nodes. A network is more reliable if it has less probability of becoming disconnected after some random failures. The problem of determining the most reliable networks often appears in the literature according to the following four types: (A) the nodes are reliable and only links may fail; (B) the links are reliable, and only the nodes may fail; (C) links and nodes may fail independently; (D) an optimization problem is described where an objective function is optimized under some constraints involving a reliability measure. Approaches of types (A) and (B) are related to a reliability measure that depends on the link and node connectivity parameters and the cardinality of the link cut set or node cut set. Approaches of type (C) are more related to Monte Carlo simulation models, and in (D), dynamic and heuristic methods are amongst the most used to address the problem. Some references about that subjects are [1], [4], [6], [7], [8], [14], [20], [21], [22], [24], [25]. In designing a communication system, it is desirable that it still works in case of disruption or failure of some of its nodes [18]. Here, we address the problem of type (B). In this context, since Harary graphs [11] are very well-known in the literature as the ones with the largest link and node connectivity under certain conditions to the order and number of links, such class of graphs plays an important role in our

study. It turns out that some Harary topologies are also the most reliable network topologies in the context of the problem of type (A), [1]. However, to the best of our knowledge, there is no paper showing how good is the node reliability of a Harary network topology compared to any network of the same order and size. In this paper, computational experiments are performed over a dataset of eleven real-world optical backbone networks. We computed the node reliability of each real-world network topology and compared it to the node reliability of the corresponding Harary network. Besides, we propose a simplified formula to compute a network’s node reliability, which shows to be useful for networks with a high number of nodes and links.

II. NETWORK RELIABILITY

A physical network topology can be modelled as a graph G with node set $V(G)$ of cardinality $|V| = N$ and link set $E(G)$ of cardinality $|E| = L$. A set of links $S \subseteq E(G)$ is a link cut set if $G \setminus S$ is disconnected, i.e., if the removal of S from G disconnects the graph G . The link (resp. node) connectivity of a graph G , $k'(G)$ (resp. $k(G)$), is the size of the smallest link (resp. node) cutset of G . For given N and k , Harary [11] shows a procedure to generate a graph H on N nodes and minimum number of links L that maximizes the link and node connectivity. Those graphs are called Harary graphs, [11], and such graphs satisfy the following equality:

$$k(H) = k'(H) = \left\lfloor \frac{2L}{N} \right\rfloor.$$

In [17], the authors present a variation of Hakimi’s algorithm, [10], that builds Harary graphs from N and L given as inputs. The goal of the Hakimi’s algorithm is to share the links between N nodes in order to have a graph as well-balanced as possible. It uses the classic Euclidean division procedure to distribute links between nodes by $L = qN + r$, $0 \leq q < N$ and $q = \lfloor \frac{L}{N} \rfloor$. Also, the algorithm considers the distances of nodes labeled as i and j as $|j - i|_N = \min(j - i \bmod n, (i - j) \bmod n)$ according to Algorithm 1. For instance, we consider the real-world optical backbone CESNET network with $N = 12$ nodes and $L = 19$ links, which is presented in Figure 1 and its corresponding Harary graph obtained by Algorithm 1 in Figure 2. It is proved, [6], that the Harary graph of Figure 2 has higher link reliability than the one of Figure 1. We are interested to observe what happens to the node reliability of real networks compared to the Harary topology. Assuming that links are perfect and nodes may fail randomly and independently at probability $1 - p$

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Algorithm 1: Modified Hakimi's algorithm, [17]

Data: L, N given as input
Result: Harary graph, $H = (V, E)$
 $V \leftarrow \{1, \dots, N\}$
 $E \leftarrow \emptyset$
 $q \leftarrow \lfloor \frac{L}{N} \rfloor$
 $r \leftarrow L - qN$
for $p = 1, \dots, q$ **do**
 if $|(j - i)|_N = p$ **then**
 $E \leftarrow E \cup (i, j)$
 end
end
 $s \leftarrow \min \{r, \frac{N+1}{2}\}$
for $i = 1, \dots, s$ **do**
 $j \leftarrow i + \lfloor \frac{N}{2} \rfloor$
 $E \leftarrow E \cup (i, j)$
end
if $r > s$ **then**
 place $r - s$ links between two non-adjacent
 vertices according to the lexicographic ordering.
end

Fig. 1: CESNET network G

such that $p \in (0, 1)$, the network reliability with node failures is defined by Equation (1):

$$R(G, p) = \sum_{r=1}^N S'_r(G) p^r (1-p)^{N-r}, \quad (1)$$

where $S'_r(G)$ is the number of connected induced subgraphs of G with r nodes. Notice that in Equation (1), the parameter p is the probability that a node does not fail. Let $\Omega(N, L)$ be the set of all connected networks with N nodes and L links. For a given p , a graph $G \in \Omega(N, L)$ is said to be the most reliable if $R(G, p) \geq R(H, p)$ for any $H \in \Omega(N, L)$. If $G \in \Omega(N, L)$ is the most reliable graph for any p , then G is said to be *uniformly best* [9]. Notice that computing $S'_r(G)$ for all r is a very difficult computational task even for small values of N . Good approaches to $R(G, p)$ can be useful for large networks. Analogously, one can evaluate the graph reliability by considering that links may fail with equal and independent probability p from Equation (2):

$$R_E(G, p) = 1 - \sum_{i=k'(G)}^L S_i(G) p^i (1-p)^{L-i}, \quad (2)$$

Fig. 2: Harary graph H with $N = 12$ nodes and $L = 19$ links, obtained by Algorithm 1.

where $S_i(G)$ is the number of link cut sets of size i . In order to maximize $R_E(G, p)$, G should have maximum link connectivity, $k'(G)$, and minimum link cut set, $S_i(G)$. In [6], under certain conditions, the authors proved that Harary graphs also have maximum reliability when we consider the problem of type (A), i.e., nodes are perfect, and links may fail. However, we are unaware of any paper relating network reliability under node failures to Harary graphs topology. This paper is about the investigation of such relations. Further, we propose simplifying the node reliability function based on our computational results with real-world networks.

III. METHODOLOGY

We selected 11 real-world optical backbone networks from the Topology Zoo [23] corresponding to some of the ones studied by [20]. These are Optical Transport Networks (OTNs), that carry large amounts of data at high rates through the DWDM (Dense Wavelength Division Multiplexing) technique. A single node failure in such a network may disrupt service on many optical links, leading to significant data and revenue losses. For that reason, it is worth investigating the node reliability of real-world optical backbone networks and to identify suitable network topologies regarding node reliability. In Table I each studied network is presented with its number of nodes, links, and average degree. The number of nodes of each network ranges from 12 to 32, while the number of links ranges from 19 to 50. Table I is arranged in non-decreasing order according to the number of nodes. The fourth column is the average degree of each network which is computed by $2L/N$. Our methodology consists of the following steps:

TABLE I: Real-world network topologies

Network Topology	Number of Nodes	Number of Links	Average degree
CESNET	12	19	3.167
ITALY	14	29	4.143
NSFNET	14	21	3.000
GARR-B	16	27	3.375
GERMANY	17	26	3.059
REDIRISNET	17	28	3.294
EON	19	37	3.895
COX	24	40	3.333
CERNET	29	45	3.103
IBN31	31	47	3.032
GEANT2	32	50	3.125

(1) For a fixed $p = 0.05$, we compute $R(G, p)$ for all networks of Table I by computing $S'_1(G), S'_2(G), \dots, S'_N(G)$.

(2) We generate the Harary graph, say H , with the same number of nodes and links of the networks of Table I and compute $R(H, p)$ for $p = 0.05$ by obtaining $S'_1(H), S'_2(H), \dots, S'_N(H)$. All Harary graphs were obtained by applying Algorithm 1.

(3) We compare the results obtained in Steps (1) and (2) to the ones obtained for $R_E(G, p)$ and $R_E(H, p)$ available in [20].

IV. COMPUTATIONAL EXPERIMENTS

All routines were implemented in Python 3.7 using a computer of 8Gb RAM memory. We start presenting the node connectivity of each real-world network and the node connectivity of the corresponding Harary graph with the same number of links and nodes. Intuitively, we can think that the higher the node connectivity, the higher is the node reliability. The results are displayed in Table II. As theoretically expected, the Harary graph version of each network has higher node connectivity, except for the IBN31 network, where the node connectivities are equal. Table III presents the link and node

TABLE II: Networks' node connectivity

Network	$k(G)$	$k(H)$
CESNET	2	3
ITALY	2	4
NSFNET	2	3
GARR-B	2	3
GERMANY	2	3
REDIRISNET	2	3
EON	2	3
COX	1	3
CERNET	1	3
IBN31	3	3
GEANT2	1	3

reliabilities for all selected real-world networks and the node reliability of the corresponding Harary graph. We observed that, in all cases, $R(H, p) \geq R(G, p)$, pointing out that Harary graphs can represent a good topology in the sense of node reliability, and not only to the link reliability. The link reliability results were obtained from [20]. Comparing $R(G, p)$ and $R_E(G, p)$, notice that $R(G, p) \leq R_E(G, p)$ for all networks, and this fact may occur because a node failure affects one or more links although no mathematical proof has been done so far. Computing $S'_r(G, p)$ and $S'_r(H, p)$ for all

TABLE III: Link and node reliabilities results for the selected real-world networks, and node reliability of the corresponding Harary graphs

Network	$R_E(G, p)$	$R(G, p)$	$R(H, p)$
CESNET	0.987786	0.977832	0.998645
ITALY	0.996706	0.986030	0.999776
NSFNET	0.995581	0.993416	0.997940
GARR-B	0.993065	0.974732	0.998294
GERMANY	0.982903	0.965531	0.997271
REDIRISNET	0.981346	0.968150	0.998017
EON	0.992832	0.982579	0.998664
COX	0.987016	0.930437	0.996828
CERNET	0.974957	0.805304	0.994266
IBN31	0.995861	0.992850	0.992850
GEANT2	0.984636	0.835157	0.993563

$r = 1, \dots, N$ is known for its high demanding computational

time. For instance, computing the node reliability of the GEANT2 network spent about 72 hours. Even though we were able to compute it for all studied networks to understand how the number of connected induced subgraphs of G e H with r nodes really influences the node reliability. Figures 3 and 4 together are important to understand such influence. In Figure 3 the number of nodes r versus the number of connected induced subgraphs with r nodes, $S'_r(G)$, is displayed for the CESNET network. Note that in all cases, in particular when r is close to N , $S'_r(H) \geq S'_r(G)$. The same situation occurs in the majority cases for other analysed networks, which is expected since in Table III we obtained $R(H, p) \geq R(G, p)$. Also, to understand the most relevant number of connected

Fig. 3: Number of nodes (r) versus Number of connected induced subgraphs with r nodes ($S'_r(G)$): CESNET network. In gray, $S'_r(G)$, and in black, $S'_r(H)$.

induced subgraphs of G e H with r nodes in terms of influencing the node reliability, we compute the cumulative values of $R(G, p)$ as we obtain the number of node cut sets $S'_r(G)$ for $r = 1, \dots, N$. The results are displayed in Figure 4. It seems that the last terms of the function are most significant to the final computation of $R(G, p)$. Hence, we propose an approximation of the node reliability function considering only the last $N/2$ final terms of Equation (1) as follows by Equation (3):

$$R'(G, p) = \sum_{r=\frac{N}{2}}^N S'_r(G) p^r (1-p)^{N-r}. \quad (3)$$

In Table IV, we compute the percentual difference between the node reliabilities for each network using $p = 0.05$ as follows:

$$gap = 100 \left(\frac{R(G, p) - R'(G, p)}{R(G, p)} \right).$$

The results of Table IV show that the proposed function $R'(G, p)$ is quite good for all eleven networks.

V. CONCLUSION

Designing a network in views of optimizing node reliability is desirable in several practical situations. The comparison of the node reliabilities of each of the eleven real-world networks with the node reliability of its corresponding Harary

TABLE IV: Percentual difference between the reliability and the approximation of the node reliability function

Network	gap (%)
CESNET	2.4e-4
ITALY	7.6e-5
NSFNET	4.1e-5
GARR-B	5.3e-6
GERMANY	2.3e-7
REDIRISNET	5.8e-7
EON	9.7e-8
COX	2.3e-9
CERNET	3.9e-12
IBN31	0.0
GEANT2	4.52e-13

Fig. 4: Accumulated values of $R(G, p)$

graph has shown that, in all cases, the node reliability of a Harary graph is larger. Our results led us to conjecture that the Harary network topology is one of the most reliable topologies in terms of node reliability, besides having maximal link reliability as proved in the literature. Therefore, the Harary topology should be considered when designing a network with optimized node reliability. Further, we proposed a simplified formula that can be useful in networks with many nodes and links.

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