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Performance Analysis for Discrete-Time Linear Systems with Saturated Linear Feedback: a Nonlinear Saturation-Dependent Gain-Scheduling Approach

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Abstract

In this paper, we propose a new technique for the performance analysis of discrete-time linear systems controlled by a saturated linear control law. Two performance indices, the computation of invariant sets and the $\mathcal{L}_2$ performance, are considered. The main contributions of the paper are the following: i) a new linear parameter varying system framework is presented to model the saturated system, ii) a nonlinear saturation-dependent auxiliary feedback matrix is considered, iii) new sufficient conditions for the performance analysis are proposed. It is shown that the conditions can be expressed as a set of linear matrix inequalities. Furthermore, it is shown that the conditions are guaranteed to be less conservative than existing solutions in the literature. Three numerical examples are presented to illustrate the effectiveness of the proposed method.

Key words: Discrete-Time Linear System, Actuator Saturation, LPV Modeling, Invariant Set, $\mathcal{L}_2$ Gain Analysis, Linear Matrix Inequalities (LMI)

1 Introduction

Saturation is probably the most commonly encountered non-linearity in control engineering. Analyzing the system performance that can be achieved un-

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der the input saturation is of great importance, and has received the attention of many researchers, for example [10], [15], [22], [26], and references therein.

With the absolute stability analysis tools, such as the circle and Popov criteria, several methods have been proposed for the performance analysis of saturated systems such as the estimation of the domain of attraction [9], [16], $\mathcal{L}_2$ and/or $\mathcal{L}_\infty$ disturbance rejection [21]. In [14], a piecewise quadratic estimate of the domain of attraction for a continuous-time saturated system is considered. The main idea is to use an initial ellipsoidal estimation obtained by means of the circle criterion.

One of the most relevant methods to the performance analysis of saturated systems is based on a linear difference inclusion (LDI) framework. The idea is to express the saturated linear feedback law as a convex hull of a group of 2^m linear feedback laws, where m is the control input dimension. Using this framework, in [11], [4], [2], an estimation of the domain of attraction is obtained, and in [6], [24], the problem of $\mathcal{L}_2$ gain analysis is addressed. In conjunction with the LDI framework, various Lyapunov functions were developed, for example, quadratic Lyapunov functions [11], [12], saturation-dependent Lyapunov function [4], [24], composite Lyapunov function and max quadratic Lyapunov function [13]. However all the existing results were obtained by using a linear saturation-independent auxiliary feedback law.

In this paper we present an approach to the performance analysis of discrete-time saturated linear systems using the LDI framework. Given a system with m saturated control inputs, we show how to select an LDI in such a way the performance is optimized. Our idea is to use a nonlinear saturation-dependent auxiliary feedback law, whereby the real-time information on the severity of saturation is fully exploited. The linear parameter varying modeling framework is used to model the resulting system. The obtained conditions are converted into linear matrix inequalities (LMI) constraints.

The conference contribution [20] touches on the contents of this paper.

The paper is structured as follows. Section 2 describes the problem formulation and some preliminaries. Section 3 is dedicated to the main results of the paper. Three numerical examples with comparison to earlier solutions are evaluated in Section 4, before drawing the conclusions in Section 5.

Notation: A positive-definite (semi-definite) matrix P is denoted by $P \succ 0$ ($P \succeq 0$). $\mathbf{0}$, $\mathbf{I}$, $\mathbf{1}$ are, respectively, the zero matrix, the identity matrix, and the all-ones vector of appropriate dimensions. For a given $P \succeq 0$, $\mathcal{E}(P)$ represents the following ellipsoid

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^T P^{-1} x \leq 1\}$$

For a given matrix H of appropriate dimension, $\mathcal{L}(H)$ is used to denote the following symmetric polyhedron

$$\mathcal{L}(H) = \{x \in \mathbb{R}^n : -\mathbf{1} \leq Hx \leq \mathbf{1}\}$$

The inequalities are to be interpreted element-wise.

For symmetric matrices, the symbol $(*)$ denotes each of its symmetric block.

2 Problem Formulation and Preliminaries

2.1 Problem Formulation

Consider the following discrete-time linear system

$$\begin{cases} x(k+1) = Ax(k) + B\text{sat}(u(k)) + Ew(k) \\ z(k) = Cx(k) + Dw(k) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the disturbance, $z \in \mathbb{R}^p$ is the performance output, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $E \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times q}$.

The saturation function $\text{sat}(u) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as

$$\begin{aligned} \text{sat}(u) &= [\text{sat}(u_1) \text{sat}(u_2) \dots \text{sat}(u_m)]^T, \\ \text{sat}(u_j) &= \begin{cases} -1, & \text{if } u_j \leq -1 \\ u_j, & \text{if } -1 \leq u_j \leq 1, \forall j = \overline{1, m} \\ 1, & \text{if } u_j \geq 1 \end{cases} \end{aligned} \quad (2)$$

The objective of this paper is to carry out systematically an analysis of system (1), under a given linear state feedback law

$$u(k) = Kx(k) \quad (3)$$

The following two problems will be considered

- (1) In the absence of w , we would like to compute an invariant set Ω as large as possible so that if $x(k) \in \Omega$, we have $x(k+1) \in \Omega$, $\forall k \geq 0$.
- (2) With a given bound on the $\mathcal{L}_2$ norm of w , i.e., $\|w\|_2^2 \leq \beta$, we would like to determine a number $\alpha > 0$ as small as possible, so that under the condition $x(0) = 0$, we have $\|z\|_2 \leq \alpha\|w\|_2$. Performing this analysis for each $\beta \in (0, \infty)$, we obtain an estimate of the nonlinear $\mathcal{L}_2$ gain.

It is assumed that all eigenvalues of $A + BK$ are in the interior of the unit circle.

2.2 Previous Works: LDI Modeling Framework

In the following we recall the linear differential inclusion (LDI) modeling framework, which was proposed in [1], [11], [12]. This framework can be considered as a generalization of the circle criterion [16] for the saturation non-linearity.

Define $\mathcal{M} = \{1, 2, \dots, m\}$, and $\mathcal{V}$ as the set of all subsets of $\mathcal{M}$, i.e., $\mathcal{V} = \{S : S \subseteq \mathcal{M}\}$. Note that the empty set belongs to $\mathcal{V}$. Define also S^c as the complementary of S in $\mathcal{M}$, i.e., $S^c = \{i \in \mathcal{M} : i \notin S\}$. For example, if $m = 2$, then $\mathcal{M} = \{1, 2\}$ and $\mathcal{V} = \{S_1, S_2, S_3, S_4\}$ with

$$\begin{aligned} S_1 &= \emptyset, S_2 = \{1\}, S_3 = \{2\}, S_4 = \{1, 2\}, \\ S_1^c &= \{1, 2\}, S_2^c = \{2\}, S_3^c = \{1\}, S_4^c = \emptyset \end{aligned}$$

Denote e_j as the j th standard basis of $\mathbb{R}^m$, i.e.,

$$e_j = [0 \ \dots \ 0 \ \underbrace{1}_{j\text{th element}} \ 0 \ \dots \ 0]^T$$

Associated to S_l , $\forall l = \overline{1, 2^m}$, consider the following scalars $v_{l(j)}$, $\forall j = \overline{1, m}$,

$$\begin{cases} -1 \leq v_{l(j)} \leq 1, & \text{if } j \in S_l \\ v_{l(j)} = 0, & \text{if } j \notin S_l \end{cases} \quad (4)$$

The following lemma is taken from [22]. It has been proposed originally in [1]. It will be used to model the saturation non-linearity (2).

Lemma 1: [22] With $v_{s(j)}$ defined as in (4), the following equation holds

$$\text{sat}(Kx) = \sum_{l=1}^{2^m} \lambda_l \left(\sum_{j \in S_l^c} e_j K_j x + \sum_{j \in S_l} e_j v_{l(j)} \right) \quad (5)$$

where K_j denotes the j th row of K , $j = \overline{1, m}$.

For example, if $m = 1$, we have

$$\text{sat}(Kx) = \lambda_1 Kx + \lambda_2 v_{2(1)}$$

with $\lambda_1 + \lambda_2 = 1, \lambda_l \geq 0, l = 1, 2$. If $m = 2$, we have

$$\begin{bmatrix} \text{sat}(K_1 x) \\ \text{sat}(K_2 x) \end{bmatrix} = \lambda_1 \begin{bmatrix} K_1 x \\ K_2 x \end{bmatrix} + \lambda_2 \begin{bmatrix} v_{2(1)} \\ K_2 x \end{bmatrix} + \lambda_3 \begin{bmatrix} K_1 x \\ v_{3(2)} \end{bmatrix} + \lambda_4 \begin{bmatrix} v_{4(1)} \\ v_{4(2)} \end{bmatrix}$$

with $\sum_{l=1}^4 \lambda_l = 1, \lambda_l \geq 0, l = \overline{1, 4}$.

Define $\mathcal{D}$ as the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. For example, if $m = 2$ then

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

There are 2^m elements in $\mathcal{D}$. Define $E_l, l = \overline{1, 2^m}$ as an element in $\mathcal{D}$. Then $\mathcal{V} = \{E_l, l = \overline{1, 2^m}\}$. Define also $E_l^- = I_m - E_l$. Clearly, E_l^- is also an element of $\mathcal{D}$.

In Lemma 1, if we select $v_{l(j)} = v_j = H_j x, l = \overline{1, 2^m}, j = \overline{1, m}$, then we obtain the following result, which is proposed in [11]

Lemma 2: [11] Let $K, H \in \mathbb{R}^{m \times n}$. For all $x \in \mathcal{L}(H)$, one has

$$\text{sat}(Kx) = \sum_{l=1}^{2^m} \lambda_l (E_l K + E_l^- H) x \quad (6)$$

where $\sum_{l=1}^{2^m} \lambda_l = 1, \lambda_l \geq 0$

For the single input case, Lemma 1 and Lemma 2 are the same if a linear auxiliary feedback gain is chosen for $v_{2(1)}$. For the multi-input case, Lemma 1 provides an extra degree of freedom. Hence for the performance analysis, conditions based on Lemma 1 are less conservative than that are based on Lemma 2. However, this comes with a cost of a higher computational complexity.

Substituting (6) in (1), one obtains

$$x(k+1) = \left(A + \sum_{l=1}^{2^m} \lambda_l (E_l K + E_l^- H) \right) x(k) + Ew(k) \quad (7)$$

Hence (1) can be modeled as an uncertain time-varying system $\forall x : -\mathbf{1} \leq Hx \leq \mathbf{1}$, whereby the parameters $\lambda_l, l = \overline{1, 2^m}$ are unknown and time-varying. It was shown in [11] that for the estimation of the domain of attraction, conditions using (7) are less conservative than that are based on the circle criterion or the vertex analysis. Furthermore, as it is proved in [11], Lemma 2

provides a necessary and sufficient conditions for an ellipsoid to be invariant for the single input case, i.e., $m = 1$.

In [4], it was noticed that the parameters $\lambda_l, l = \overline{1, 2^m}$ reflects the severity of the saturation function. Consequently, $\lambda_l, l = \overline{1, 2^m}$ are functions of x . To see this, consider the case $m = 1$, and assume $H = 0$. In this case, $\forall x, -1 \leq Hx = 0 \leq 1$. Using (6), one obtains

$$sat(Kx) = \lambda_1 Kx$$

If $-1 \leq Kx \leq 1$, then the saturation function becomes

$$sat(Kx) = Kx$$

therefore $\lambda_1 = 1$ and $\lambda_2 = 0$. If $Kx = 3$, then

$$sat(Kx) = 1 = \frac{1}{3}Kx$$

therefore $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$.

Similarly, if we assume $H = \frac{1}{2}K$, then using (6), one gets,

$$sat(Kx) = (\lambda_1 + \frac{1}{2}\lambda_2)Kx$$

for all x such that $-1 \leq \frac{1}{2}Kx \leq 1$. If $-1 \leq Kx \leq 1$, we have $\lambda_1 = 1, \lambda_2 = 0$. If $Kx = \frac{3}{2}$, then $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$.

Using the available information of $\lambda_l, l = \overline{1, 2^m}$, a saturation-dependent Lyapunov function was proposed in [4] to estimate the domain of attraction, and in [24] to estimate the $\mathcal{L}_2$ gain. The conditions are proved to be less conservative than that are based on quadratic Lyapunov function.

The auxiliary feedback matrix H in [4], [11], [24] is a decision variable. It can be optimized to provide a less conservative estimation of the domain of attraction and/or of the $\mathcal{L}_2$ gain. However, in [11], [4], [24], H is a constant matrix, and does not depend on $\lambda_l, l = \overline{1, 2^m}$. Hence, the real-time information on the severity of saturation is not exploited. The main objective of this paper is to show that by selecting a nonlinear saturation-dependent auxiliary feedback matrix H , a significant improvement in the performance analysis can be obtained. For this purpose, some preliminary results are recalled in the next section.

2.3 Preliminaries

The following double sum positivity problem of the form

$$x^T \left(\sum_{l_1=1}^{2^m} \sum_{l_2=1}^{2^m} \lambda_{l_1} \lambda_{l_2} \Gamma_{l_1 l_2} \right) x \geq 0 \quad (8)$$

will be dealt several times in this paper, where the coefficients λ_l satisfy

$$\sum_{l=1}^{2^m} \lambda_l = 1, \lambda_l \geq 0, \forall l = \overline{1, 2^m}$$

Lemma 3: The double sum (8) is positive, if, see [25],

$$\begin{cases} \Gamma_{ll} \succeq 0, l = \overline{1, 2^m} \\ \Gamma_{l_1 l_2} + \Gamma_{l_2 l_1} \succeq 0, \forall l_1, l_2 = \overline{1, 2^m}, l_1 < l_2 \end{cases} \quad (9)$$

Lemma 4: The double sum (8) is positive, if, see [23],

$$\begin{cases} \Gamma_{ll} \succeq 0, l = \overline{1, 2^m} \\ \frac{2}{2^m-1} \Gamma_{l_1 l_1} + \Gamma_{l_1 l_2} + \Gamma_{l_2 l_1} \succeq 0, \forall l_1, l_2 = \overline{1, 2^m}, l_1 \neq l_2 \end{cases} \quad (10)$$

Remark 1: It is well known [23] that Lemma 4 is less conservative Lemma 3, i.e., if there exist matrices $\Gamma_{l_1 l_2}$ satisfying (10), they also satisfy (9). The main advantage of (9) with respect to (10) is that (9) has a fewer number of LMI constraints than (10). Hence the computational complexity is reduced.

Lemma 5: Given matrices P, G of appropriate dimension with $P \succ 0$. Then, see [5]

$$(G - P)^T P^{-1} (G - P) \succeq 0 \Leftrightarrow G^T P^{-1} G \succeq G^T + G - P \quad (11)$$

Lemma 6: For given matrices $F \in \mathbb{R}^{n_f \times n}$, and $P \succeq 0$, $\mathcal{E}(P, 1) \subseteq \mathcal{L}(F)$ if and only if, see [10]

$$1 - e_j F P F^T e_j^T \succeq 0, j = \overline{1, n_f} \quad (12)$$

where e_j is the j th standard basic of $\mathbb{R}^{n_f}$.

Concerning the LMIs, we will make use of the following results.

Property 1 (Congruence): Let P and Q are matrices of appropriate dimension, where $P = P^T$, and Q is a full rank matrix. It holds that

$$P \succeq 0 \Leftrightarrow Q^T P Q \succeq 0 \quad (13)$$

Property 2: (Schur complements): Consider a matrix M , with

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

and M_{11}, M_{22} being square matrices. Then, see [3]

$$\begin{aligned} M \succeq 0 &\Leftrightarrow \begin{cases} M_{11} \succeq 0, \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} \succeq 0 \end{cases} \\ &\Leftrightarrow \begin{cases} M_{22} \succeq 0, \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T \succeq 0 \end{cases} \end{aligned} \quad (14)$$

3 Performance Analysis

3.1 LPV Modeling

In this section, we present an LPV framework to model the saturated system (1). As will be shown, our LPV model allows to consider a nonlinear saturation-dependent auxiliary feedback matrix in contrast to the LPV model (7) in [4], [24].

Substituting (3) to (1), one obtains

$$x(k+1) = Ax(k) + B \text{sat}(Kx(k)) + Ew(k) \quad (15)$$

Using Lemma 1, there exist $\lambda_l(k)$ and $v_{l(j)}(k)$, $l = \overline{1, 2^m}$, $j = \overline{1, m}$ such that

$$\begin{cases} \sum_{l=1}^{2^m} \lambda_l(k) = 1, \lambda_l(k) \geq 0, l = \overline{1, 2^m} \\ \begin{cases} -1 \leq v_{l(j)}(k) \leq 1, & \text{if } j \in S_l \\ v_{l(j)}(k) = 0, & \text{if } j \notin S_l \end{cases} \end{cases}$$

and

$$x(k+1) = Ax(k) + \sum_{l=1}^{2^m} \lambda_l(k) \left(\sum_{j \in S_l^c} B_j K_j x(k) + \sum_{j \in S_l} B_j v_{l(j)}(k) \right) + Ew(k)$$

where B_j is the j th column of B , $j = \overline{1, m}$. Thus, with $A = \sum_{l=1}^{2^m} \lambda_l A$

$$x(k+1) = \sum_{l=1}^{2^m} \lambda_l(k) \left\{ (A + \sum_{j \in S_l^c} B_j K_j) x(k) + \sum_{j \in S_l} B_j v_{l(j)}(k) \right\} + Ew(k) \quad (16)$$

Define

$$\begin{cases} \mathcal{A}_l = A + \sum_{j \in S_l^c} B_j K_j, \mathcal{B}_l = \begin{bmatrix} \mathbf{0} & B_j & \mathbf{0} \end{bmatrix}_{j \in S_l}, \\ v = [v_{2(1)} \ v_{3(2)} \ \dots \ v_{2^m(m)}]^T \end{cases} \quad (17)$$

Note that $v \in \mathbb{R}^{m2^{m-1} \times 1}$. For example, if $m = 1$, then $v = v_{2(1)}$ and

$$\mathcal{A}_1 = A + BK, \mathcal{A}_2 = A, \mathcal{B}_1 = \mathbf{0}_{n \times 1}, \mathcal{B}_2 = B$$

If $m = 2$, then $v = [v_{2(1)} \ v_{3(2)} \ v_{4(1)} \ v_{4(2)}]^T$ and

$$\begin{cases} \mathcal{A}_1 = A + BK, \mathcal{A}_2 = A + B_2 K_2, \mathcal{A}_3 = A + B_1 K_1, \mathcal{A}_4 = A \\ \mathcal{B}_1 = \mathbf{0}_{n \times 4}, \mathcal{B}_2 = \begin{bmatrix} B_1 & \mathbf{0}_{n \times 3} \end{bmatrix}, \mathcal{B}_3 = \begin{bmatrix} \mathbf{0}_{n \times 1} & B_2 & \mathbf{0}_{n \times 2} \end{bmatrix}, \mathcal{B}_4 = \begin{bmatrix} \mathbf{0}_{n \times 2} & B \end{bmatrix} \end{cases}$$

Using (17), (16) can be rewritten as

$$x(k+1) = \left(\sum_{l=1}^{2^m} \lambda_l(k) \mathcal{A}_l \right) x(k) + \left(\sum_{l=1}^{2^m} \lambda_l(k) \mathcal{B}_l \right) v(k) + Ew(k) \quad (18)$$

The auxiliary variable $v(k)$ can be considered as a control input for the system (18). Hence the problem of carrying out the performance analysis for the system (15) becomes the problem of selecting the optimal input $v(k)$ to obtain the best performance for (18).

Remark 2: If Lemma 2 is used to model the saturation non-linearity instead of Lemma 1, then (18) can still be used to model (15) but with the following matrices $\mathcal{A}_l$, $\mathcal{B}_l$, and v

$$\begin{cases} \mathcal{A}_l = A + BE_l K, \mathcal{B}_l = BE_l^-, l = \overline{1, 2^m} \\ v = [v_1 \ v_2 \ \dots \ v_m]^T \end{cases} \quad (19)$$

Note that the dimension of v in (19) is m . For example if $m = 2$ then $v = [v_1 \ v_2]^T$ and

$$\begin{cases} \mathcal{A}_1 = A + BK, \mathcal{A}_2 = A + B_2 K_2, \mathcal{A}_3 = A + B_1 K_1, \mathcal{A}_4 = A \\ \mathcal{B}_1 = \mathbf{0}_{n \times 2}, \mathcal{B}_2 = \begin{bmatrix} B_1 & \mathbf{0}_{n \times 1} \end{bmatrix}, \mathcal{B}_3 = \begin{bmatrix} \mathbf{0}_{n \times 1} & B_2 \end{bmatrix}, \mathcal{B}_4 = B \end{cases}$$

Consider the following auxiliary feedback law

$$v(k) = F(\lambda(k))G(\lambda(k))^{-1}x(k) \quad (20)$$

with

$$F(\lambda(k)) = \sum_{l=1}^{2^m} \lambda_l(k) F_l, G(\lambda(k)) = \sum_{l=1}^{2^m} \lambda_l(k) G_l \quad (21)$$

where F_l, G_l are unknown matrices that will be treated as decision variables.

Remark 3: In the literature [11], [24], [18], only a *linear saturation-independent* control law $v(k) = Hx(k)$ was considered with $H = FG^{-1}$. Clearly, this is a particular case of (20) with $F_l = F$ and $G_l = G, \forall l = \overline{1, 2^m}$. The *nonlinear saturation dependent* control law (20) takes the real time information of the saturation into account. As will be shown in the examples, a less conservative estimate of the performance is obtained.

Substituting (20) into (16), one obtains the following closed-loop system

$$x(k+1) = \mathcal{A}_c(\lambda(k))x(k) + Ew(k) \quad (22)$$

where

$$\mathcal{A}_c(\lambda(k)) = \mathcal{A}(\lambda(k)) + \mathcal{B}(\lambda(k))F(\lambda(k))G(\lambda(k))^{-1}$$

It should be stressed that system (15) can be modeled as (22) only for x such that

$$-\mathbf{1} \leq F(\lambda(k))G(\lambda(k))^{-1}x(k) \leq \mathbf{1} \quad (23)$$

Define the following saturation-dependent Lyapunov function

$$V(k, x(k)) = x(k)^T (P(\lambda(k)))^{-1} x(k) \quad (24)$$

with

$$P(\lambda(k)) = \sum_{l=1}^{2^m} \lambda_l(k) P_l \quad (25)$$

where $P_l \succeq 0, l = \overline{1, 2^m}$ are unknown matrices that will be treated as decision variables.

With slight abuse of notation, $\mathcal{A}_c(k), \mathcal{A}(k), \mathcal{B}(k), G(k), P(k)$ are used to denote $\mathcal{A}_c(\lambda(k)), \mathcal{A}(\lambda(k)), \mathcal{B}(\lambda(k)), G(\lambda(k)), P(\lambda(k))$.

3.2 Invariant Set Computation

In this section, we are interested in computing an invariant set as large as possible *in the absence of* w . For this purpose, the following definitions are recalled [19].

Definition 1 (Domain of Attraction): A set Ω is said to be inside the domain of attraction for system (15), if for any initial condition $x(0) \in \Omega$, one has $\lim_{k \rightarrow \infty} x(k) = 0$.

Definition 2 (Invariant Set): A set Ω is said to be invariant for system (15), if $\forall x(k) \in \Omega$, one has $x(k+1) \in \Omega$, $\forall k \geq 0$, and $\lim_{k \rightarrow \infty} x(k) = 0$.

Clearly from Definition 2, if a set is invariant, then it contains the origin in its interior. It is also clear that if a set is invariant then it is inside the domain of attraction, but not the other way around, i.e., if a set is inside the domain of attraction, then it is not necessarily invariant.

The following theorem provides the theoretical support of the algorithm proposed to calculate an invariant set for the system (15).

Theorem 1: Consider the system (15). Assume that there exist matrices $P_l \succeq 0$, F_l, G_l , $l = \overline{1, 2^m}$ satisfying the following matrix inequalities

$$\begin{bmatrix} 1 & e_j F(k) \\ F(k)^T e_j^T & G(k) + G(k)^T - P(k) \end{bmatrix} \succeq 0, \quad j = \overline{1, m2^{m-1}} \quad (26)$$

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k) + \mathcal{B}(k)F(k) \\ (*) & G(k) + G(k)^T - P(k) \end{bmatrix} \succeq 0 \quad (27)$$

then, the set $V(k, x(k)) \leq 1$ is invariant.

Proof: It was showing that for all x satisfying (23), (22) can be used to model (15). Using Lemma 6, condition (23) is equivalent

$$1 - e_j F(k) G(k)^{-1} P(k) (G(k)^{-1})^T F(k)^T e_j^T \geq 0, \quad \forall j = \overline{1, m2^{m-1}}$$

Recall that e_j is the j th standard basic of $\mathbb{R}^{m2^{m-1}}$. With Schur complement, one gets, $\forall j = \overline{1, m2^{m-1}}$

$$\begin{bmatrix} 1 & e_j F(k) \\ F(k)^T e_j^T & G(k)^T P(k)^{-1} G(k) \end{bmatrix} \geq 0 \quad (28)$$

Using Lemma 5, condition (28) is satisfied if

$$\begin{bmatrix} 1 & e_j F(k) \\ F(k)^T e_j^T & G(k) + G(k)^T - P(k) \end{bmatrix} \geq 0, \quad \forall j = \overline{1, m2^{m-1}} \quad (29)$$

Note that (29) is (26). It remains to show that $V(k, x(k+1)) \leq 1$ is invariant.

It is required that

$$x(k+1)^T P(k+1)^{-1} x(k+1) - x(k)^T P(k)^{-1} x(k) \leq 0$$

Using (22), one obtains

$$\mathcal{A}_c(k)^T P(k+1)^{-1} \mathcal{A}_c(k) - P(k)^{-1} \preceq 0$$

or equivalently

$$(\mathcal{A}(k)G(k) + B(k)F(k))^T P(k+1)^{-1} (\mathcal{A}(k)G(k) + B(k)F(k)) - G(k)^T P(k)^{-1} G(k) \preceq 0$$

Using Schur complement, one gets

$$\begin{bmatrix} P(k+1) & (\mathcal{A}(k)G(k) + B(k)F(k)) \\ (\mathcal{A}(k)G(k) + B(k)F(k))^T & G(k)^T P(k)^{-1} G(k) \end{bmatrix} \succeq 0 \quad (30)$$

Thus, using Lemma 5, (30) is satisfied if

$$\begin{bmatrix} P(k+1) & (\mathcal{A}(k)G(k) + B(k)F(k)) \\ (\mathcal{A}(k)G(k) + B(k)F(k))^T & G(k)^T + G(k) - P(k) \end{bmatrix} \succeq 0 \quad (31)$$

The proof is complete. $\square$

In Theorem 1, if Lemma 2 is used to model the saturation non-linearity, and by setting $F_l = F, G_l = G, \forall l = \overline{1, 2^m}$, then Theorem 1 in [4] is recovered. This implies that Theorem 1 in [4] is a particular case of ours.

Theorem 1 gives a condition for the set

$$V(k, x(k)) = \{x(k) : x(k)^T P(k)^{-1} x(k) \leq 1\}$$

to be invariant. Note that this set might be non-convex. If we are interested in a convex characterization of the invariant set, then the intersection of the ellipsoids $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ can be used. However in this case, using Theorem 1, we can only guarantee that $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is inside the domain of attraction. In the following we prove that $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is also invariant.

Theorem 2: If there exist $P_l \succeq 0, F_l, G_l$ satisfying (26), (27), then the set $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is invariant.

Proof: Note that $\forall x(0) \in \bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$, one has

$$\lim_{k \rightarrow \infty} x(k) = 0$$

since $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is contained in the domain of attraction. It remains to prove that $\forall x(k) \in \bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$, one has $x(k+1) \in \bigcap_{l=1}^{2^m} \mathcal{E}(P_l), \forall k \geq 0$.

At time k , without loss of generality, assume that $\lambda_{l_0}(k) = 1$, then $\lambda_l(k) = 0, \forall l = \overline{1, 2^m}, l \neq l_0$. Since $x(k) \in \bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ one has $x(k) \in \mathcal{E}(P_{l_0})$. Following Theorem 1, one has

$$x(k+1)^T P(k+1)^{-1} x(k+1) \leq x(k)^T P_{l_0}^{-1} x(k) \leq 1 \quad (32)$$

where $x(k+1) = \mathcal{A}_c(k)x(k)$ with $\mathcal{A}_c(k) = \mathcal{A}_{l_0} + \mathcal{B}_{l_0} F_{l_0} G_{l_0}^{-1}$.

In Theorem 1, no assumption about $\lambda_l(k+1)$ were made, except that

$$\sum_{l=1}^{2^m} \lambda_l(k+1) = 1, \lambda_l(k+1) \geq 0$$

Hence $P(k+1)$ can be any matrices $P_l, l = \overline{1, 2^m}$ and their convex combination. Using (32), it follows that $x(k+1) \in \mathcal{E}(P_l), \forall l = \overline{1, 2^m}$. In other words,

$$x(k+1) \in \bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$$

or the set $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is invariant. $\square$

Using (21), (25), condition (26) holds if and only if

$$\begin{bmatrix} 1 & F_l e_j \\ F_l^T e_j & G_l + G_l^T - P_l \end{bmatrix} \succeq 0, \forall l = \overline{1, 2^m}, \forall j = \overline{1, m2^{m-1}} \quad (33)$$

Rewrite (27) as

$$\sum_{l_1=1}^{2^m} \sum_{l_2=1}^{2^m} \lambda_{l_1} \lambda_{l_2} \Gamma_{l_1 l_2}^r \succeq 0 \quad (34)$$

with $r = \overline{1, 2^m}$, and

$$\Gamma_{l_1 l_2}^r = \begin{bmatrix} P_r & (A_{l_1} G_{l_2} + B_{l_1} F_{l_2}) \\ (A_{l_1} G_{l_2} + B_{l_1} F_{l_2})^T & (G_{l_2} + G_{l_2}^T - P_{l_2}) \end{bmatrix} \quad (35)$$

Combining (33), (34), relaxed LMI conditions can be formulated using Lemmas 3, or 4 as follows

Corollary 1: If there exist matrices $P_l \succeq 0$, G_l, F_l , $l = \overline{1, 2^m}$ such that (33) hold and

$$\begin{cases} \Gamma_{ll}^r \succeq 0, \forall r, \forall l = \overline{1, 2^m} \\ \Gamma_{l_1 l_2}^r + \Gamma_{l_2 l_1}^r \succeq 0, \forall r, l_1, l_2 = \overline{1, 2^m}, l_1 < l_2 \end{cases} \quad (36)$$

or

$$\begin{cases} \Gamma_{ll}^r \succeq 0, \forall r, \forall l = \overline{1, 2^m} \\ 2\Gamma_{l_1 l_1}^r + \Gamma_{l_1 l_2}^r + \Gamma_{l_2 l_1}^r \succeq 0, \forall m, l_1, l_2 = \overline{1, 2^m}, l_1 \neq l_2 \end{cases} \quad (37)$$

then the set $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ is invariant.

The proof of corollary 1 is straightforward.

In the interest of the size of the domain of attraction, which is proportional $\log \det(P_l)$, the set $\bigcap_{l=1}^{2^m} \mathcal{E}(P_l)$ should be maximized. This can be done by solving the following SDP problem

$$\begin{aligned} \max_{P_l, F_l, G_l} & \left\{ \sum_{l=1}^{2^m} \log \det(P_l) \right\} \\ \text{s.t.} & (33), (36) \end{aligned} \quad (38)$$

or

$$\begin{aligned} \max_{P_l, F_l, G_l} & \left\{ \sum_{l=1}^{2^m} \log \det(P_l) \right\} \\ \text{s.t.} & (33), (37) \end{aligned} \quad (39)$$

Since (38) and/or (39) are a convex SDP problem, they can be solved efficiently using free available LMI parser such as CVX [8], or Yalmip [17]. In the following, we refer to the optimization problems (38), and (39), respectively, as algorithm 1 and algorithm 2.

Remark 4: Note that the number of LMIs in (38) and (39) increases exponentially as the number of the system input m increases.

3.3 $\mathcal{L}_2$ Performance Analysis

In this section, we are interested in estimating the $\mathcal{L}_2$ gain for the system (1), (3). This $\mathcal{L}_2$ gain is defined as follows.

Definition 3 ($\mathcal{L}_2$ gain): For a given $\gamma > 0$, the system (1), (3) is said to be with a $\mathcal{L}_2$ gain less than γ , if for the zero initial condition, one has

$$\sum_{k=0}^{\infty} z(k)^T z(k) - \gamma^2 \sum_{k=0}^{\infty} w(k)^T w(k) \leq 0 \quad (40)$$

for all nonzero $w \in W$.

Recall that the set W is

$$w \in W = \{w \in \mathbb{R}^q : \|w\|_2^2 \leq \beta\}$$

where $\beta > 0$ is a given constant.

The following theorem establishes a sufficient condition to estimate the $\mathcal{L}_2$ gain for the system (1).

Theorem 3: Consider the system (1). For given scalars $\alpha > 0, \beta > 0$, if there exist matrices $P_l \succeq 0, F_l, G_l, l = \overline{1, 2^m}$ and a scalar $\gamma > 0$ such that the following matrix inequalities

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k) + \mathcal{B}(k)F(k) & E & \mathbf{0} \\ (*) & G(k) + G(k)^T - P(k) & \mathbf{0} & G(k)^T C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0 \quad (41)$$

$$\begin{bmatrix} \frac{1}{\alpha + \beta} & e_j F(k) \\ F(k)^T e_j^T & G(k) + G(k)^T - P(k) \end{bmatrix} \succeq 0, \forall j = \overline{1, m2^{m-1}} \quad (42)$$

hold, then, for all $x(0)$ such that $x(0)^T P(0)^{-1} x(0) \leq \alpha$, one has, $\forall k \geq 1$

$$x(k)^T P(k)^{-1} x(k) \leq \alpha + \beta$$

and the following inequality holds

$$\sum_{k=0}^{\infty} z^T(k) z(k) \leq \gamma^2 \sum_{k=0}^{\infty} w^T(k) w(k) + \alpha. \quad (43)$$

Proof: Consider the saturation-dependent Lyapunov function (24). For the $\mathcal{L}_2$ gain, it is required that

$$V(k+1, x(k+1)) - V(k, x(k)) \leq -\frac{1}{\gamma^2} z(k)^T z(k) + w(k)^T w(k) \quad (44)$$

$\forall x(k), \forall x(k+1)$ satisfying (22), and $\forall w(k) \in W$.

If (44) holds, then it follows that

$$V(\infty, x(\infty)) - V(0, x(0)) \leq -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k) \quad (45)$$

Note that (22) is asymptotically stable for states near the origin. It follows that $\lim_{k \rightarrow \infty} x(k) = \mathbf{0}$. Hence $\lim_{k \rightarrow \infty} V(k, x(k)) = 0$. With the zero initial condition, condition (45) becomes

$$0 \leq -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k)$$

or equivalently

$$\sum_{k=0}^{\infty} z(k)^T z(k) \leq \gamma^2 \sum_{k=0}^{\infty} w(k)^T w(k)$$

It is concluded that the system (1) has $\mathcal{L}_2$ gain performance γ .

Rewrite the left hand side of (44) as

$$\begin{aligned} & V(k+1, x(k+1)) - V(k, x(k)) = \\ & = x(k+1)^T P(k+1)^{-1} x(k+1) - x(k)^T P(k)^{-1} x(k) \\ & = \begin{bmatrix} x(k)^T & w(k)^T \end{bmatrix} \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \begin{bmatrix} \mathcal{A}_c(k) & E \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ & \quad - \begin{bmatrix} x(k)^T & w(k)^T \end{bmatrix} \begin{bmatrix} P(k)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \end{aligned} \quad (46)$$

and the right hand side of (44) as

$$\begin{aligned} & -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k) = \\ & = -\frac{1}{\gamma^2} \begin{bmatrix} x(k)^T & w(k)^T \end{bmatrix} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \\ & \quad + \begin{bmatrix} x(k)^T & w(k)^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \end{aligned} \quad (47)$$

Combining (46), (47), one obtains

$$\begin{bmatrix} P(k)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \begin{bmatrix} \mathcal{A}_c(k) & E \end{bmatrix} \succeq \frac{1}{\gamma^2} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} P(k)^{-1} - \frac{1}{\gamma^2} C^T C & -\frac{1}{\gamma^2} C^T D \\ -\frac{1}{\gamma^2} D^T C & -\frac{1}{\gamma^2} D^T D + \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \begin{bmatrix} \mathcal{A}_c(k) & E \end{bmatrix} \succeq 0$$

Thus, with Schur complement, one gets

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E \\ (*) & P(k)^{-1} - \frac{1}{\gamma^2} C^T C & -\frac{1}{\gamma^2} C^T D \\ (*) & (*) & -\frac{1}{\gamma^2} D^T D + \mathbf{I} \end{bmatrix} \succeq 0$$

or equivalently

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E \\ (*) & P(k)^{-1} & \mathbf{0} \\ (*) & (*) & \mathbf{I} \end{bmatrix} - \frac{1}{\gamma^2} \begin{bmatrix} \mathbf{0} \\ C^T \\ D^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & C & D \end{bmatrix} \succeq 0$$

Using Schur complement, one obtains

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E & \mathbf{0} \\ (*) & P(k)^{-1} & \mathbf{0} & C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0 \quad (48)$$

Pre- and post-multiplication of (48) by

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (*) & G(k)^T & \mathbf{0} & \mathbf{0} \\ (*) & (*) & \mathbf{I} & \mathbf{0} \\ (*) & (*) & (*) & \mathbf{I} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (*) & G(k) & \mathbf{0} & \mathbf{0} \\ (*) & (*) & \mathbf{I} & \mathbf{0} \\ (*) & (*) & (*) & \mathbf{I} \end{bmatrix}$$

one gets

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k) + \mathcal{B}(k)F(k) & E & \mathbf{0} \\ (*) & G(k)^T P(k)^{-1} G(k) & \mathbf{0} & G(k)C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0 \quad (49)$$

Using Lemma 5, condition (49) holds if

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k) + \mathcal{B}(k)F(k) & E & \mathbf{0} \\ (*) & G(k) + G(k) - P(k) & \mathbf{0} & G(k)C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0$$

This condition is (41).

Using (44), $\forall x(0)$ such that $x(0)^T P(0)^{-1} x(0) \leq \alpha$, one gets

$$x(k)P(k)^{-1}x(k) \leq x(0)^T P(0)^{-1}x(0) - \frac{1}{\gamma^2} \sum_{t=0}^{k-1} z(t)^T z(t) + \sum_{t=0}^{k-1} w(t)^T w(t)$$

Consequently,

$$x(k)P(k)^{-1}x(k) \leq x(0)^T P(0)^{-1}x(0) + \sum_{t=0}^{k-1} w(t)^T w(t)$$

Hence

$$x(k)P(k)^{-1}x(k) \leq \alpha + \beta \quad (50)$$

Recall that

$$-\mathbf{1} \leq v(k) = F(k)G(k)^{-1}x(k) \leq \mathbf{1}$$

Thus, using Lemma 5, one obtains

$$\frac{1}{\alpha + \beta} - e_j F(k)G(k)^{-1}P(k)(G(k)^{-1})^T F(k)^T e_j^T \succeq 0, \forall j = \overline{1, m2^{m-1}}$$

Using Schur complement, this condition is equivalently rewritten as

$$\begin{bmatrix} \frac{1}{\alpha + \beta} & e_j F(k) \\ F(k)^T e_j^T & G(k)^T P(k)^{-1} G(k) \end{bmatrix} \succeq 0$$

Using Lemma 4, one gets

$$\begin{bmatrix} \frac{1}{\alpha + \beta} & e_j F(k) \\ F(k)^T e_j^T & G(k) + G(k)^T - P(k) \end{bmatrix} \succeq 0$$

The proof is complete. $\square$

By using Lemma 2 to model the saturation non-linearity and by setting $F_l = F, G_l = G$ in the conditions of Theorem 3, one recover Theorem 1 in [24]. Hence the result in [24] is a particular case of ours.

Condition (42) holds if and only if

$$\begin{bmatrix} \frac{1}{\alpha+\beta} & e_j F_l \\ e_j F_l^T & G_l + G_l^T - P_l \end{bmatrix} \succeq 0, \forall l = \overline{1, 2^m}, \forall j = \overline{1, m2^{m-1}} \quad (51)$$

Define $\gamma_s = \gamma^2$. Rewrite (41) as

$$\sum_{l_1=1}^{2^m} \sum_{l_2=1}^{2^m} \lambda_{l_1} \lambda_{l_2} \Theta_{l_1 l_2}^r \succeq 0 \quad (52)$$

with $r = \overline{1, 2^m}$, and

$$\Theta_{l_1 l_2}^r = \begin{bmatrix} P_r & (A_{l_1} G_{l_2} + B_{l_1} F_{l_2}) & E & \mathbf{0} \\ (*) & (G_{l_2} + G_{l_2}^T - P_{l_2}) & 0 & G_{l_2} C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma_s \mathbf{I} \end{bmatrix} \quad (53)$$

Combining (52) and Lemma 3 or Lemma 4, one obtains the following corollary.

Corollary 2: System (1), (3) is with a $\mathcal{L}_2$ gain less than $\gamma = \sqrt{\gamma_s}$, if for given scalars $\alpha > 0, \beta > 0$, there exist $P_l \succeq 0, G_l, F_l, l = \overline{1, 2^m}$, and a scalar $\gamma_s > 0$, such that (51) hold and

$$\begin{cases} \Theta_{ll}^r \succeq 0, \forall r, \forall l = \overline{1, 2^m} \\ \Theta_{l_1 l_2}^r + \Theta_{l_2 l_1}^r \succeq 0, \forall r, l_1, l_2 = \overline{1, 2^m}, l_1 < l_2 \end{cases} \quad (54)$$

or

$$\begin{cases} \Theta_{ll}^r \succeq 0, \forall r, l = \overline{1, 2^m} \\ 2\Theta_{l_1 l_1}^r + \Theta_{l_1 l_2}^r + \Theta_{l_2 l_1}^r \succeq 0, \forall r, l_1, l_2 = \overline{1, 2^m}, l_1 \neq l_2 \end{cases} \quad (55)$$

Using Corollary 2, the problem of optimizing the $\mathcal{L}_2$ gain γ can be formulated as

$$\begin{aligned} & \min_{P_l, G_l, F_l} \gamma \\ & \text{subject to (51), (54)} \end{aligned} \quad (56)$$

or

$$\begin{aligned} & \min_{P_l, G_l, F_l} \gamma \\ & \text{subject to (51), (55)} \end{aligned} \quad (57)$$

For further use, we refer to the optimization problems (56), and (57), respectively, as algorithm 3 and algorithm 4.

Remark 3: In the unsaturated linear system case, it is well known [7] that the parameter β has no impact on the $\mathcal{L}_2$ gain γ . However, in the presence of the saturation non-linearity, this is no longer the case, i.e., γ is a function of β . Using (51), it is clear that this function is non-increasing, i.e., $\gamma_1 \leq \gamma_2$ if $\beta_1 \geq \beta_2$.

4 Examples

Three examples are considered in this section. The CVX toolbox was used to solve SDP optimization problems.

4.1 Example 1

This example is taken from [4]. Consider the following system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \text{sat}(u(k)) \quad (58)$$

The LQ controller with $Q = \mathbf{I}$, and $R = 0.1$ is used, giving the state feedback gain

$$K = [-0.6167 \quad -1.2703]$$

For this example, results are almost identical for Algorithms 1 and 2. Hence only results of Algorithm 2 are presented. By solving the SDP problem (39), one obtains

$$P_1 = \begin{bmatrix} 66.5120 & -26.0475 \\ -26.0475 & 40.4144 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 103.7959 & -29.9170 \\ -29.9170 & 20.4263 \end{bmatrix}$$

Fig. 1 shows the intersection of two ellipsoids $\mathcal{E}(P_1, 1)$ and $\mathcal{E}(P_2, 1)$ (solid cyan and solid violet). For comparison, Fig. 1 also shows the intersection of two ellipsoids (dashed yellow and dashed red) obtained by using Theorem 1 in [4], and the ellipsoid obtained by using [11] (dash-dot green). We can see that the estimate of the invariant set obtained by using the nonlinear saturation-dependent auxiliary feedback gain is larger than that by the linear saturation independent ones.

For different initial conditions, Fig. 2 presents the state trajectories in the phase plane.

Fig. 1. Invariant sets for our approach (solid cyan and solid violet), for [4] (dashed yellow and dashed red), and for [11] (dash-dot green) for example 1.

Fig. 2. State trajectories for different initial conditions for example 1.

4.2 Example 2

Consider system (1), (3) with

$$A = \begin{bmatrix} 0 & 1 \\ -0.6 & -0.62 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ K = \begin{bmatrix} 0 & -0.7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

For this example, the $\mathcal{L}_2$ gain of the closed-loop system in the linear region is 4.7760. For $\alpha = 0$, $\beta = 15$, by using algorithms 3 and 4, one gets, respectively,

$\gamma = 13.4399$, $\gamma = 13.2680$. The matrices obtained using algorithm 4 are

$$\begin{aligned} P_1 &= \begin{bmatrix} 33.1911 & -17.1499 \\ -17.1499 & 17.0760 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 21.0983 & -9.7412 \\ -9.7412 & 19.6851 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} 33.1911 & -17.1499 \\ -17.1499 & 17.0760 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 21.0983 & -9.7412 \\ -9.7412 & 19.6851 \end{bmatrix}, \\ F_1 &= \begin{bmatrix} 0.9861 & -0.0336 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1.1544 & -0.7637 \end{bmatrix} \end{aligned}$$

For different $\beta \in [0.5 \ 20]$, Fig. 3 presents the $\mathcal{L}_2$ gain using algorithm 3 (dashed red), and algorithm 4 (solid blue). It can be observed that algorithm 4 slightly outperforms algorithm 3. For comparison, Fig. 3 also presents the $\mathcal{L}_2$ gain obtained by Theorem 2 in [6] (dashed violet), and by Theorem 1 in [24] (dash-dot yellow). Note that the dashed violet curve diverges to infinity as β approaches to 2.1. This implies that the maximal tolerable disturbance by [6] is 2.1.

Fig. 3. $\mathcal{L}_2$ gain as a function of β for Algorithm 3 (dashed red), for Algorithm 4 (solid blue), for Theorem 2 in [6] (dashed violet), and for Theorem 1 in [24] (dash-dot yellow).

4.3 Example 3

To further illustrate the performance of our approach, we consider the following multi-input system

$$x(k+1) = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.42 & 0.9 \\ 0.38 & 0.67 \end{bmatrix} \text{sat}(u(k)) \quad (59)$$

Design the linear control law by the LQ approach with $Q = \mathbf{I}$, $R = 0.1\mathbf{I}$, giving the matrix gain

$$K = \begin{bmatrix} 0.0843 & -0.6372 \\ -0.8990 & -0.8826 \end{bmatrix}$$

For this example, Algorithm 1 and Algorithm 2 give almost the same results. Hence only the results of Algorithm 2 is shown. By solving (39), one gets

$$P_1 = \begin{bmatrix} 36.3443 & 3.3024 \\ 3.3024 & 21.4723 \end{bmatrix}, P_2 = \begin{bmatrix} 36.3000 & -9.2123 \\ -9.2123 & 14.9668 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 36.2838 & 4.7992 \\ 4.7992 & 18.6164 \end{bmatrix}, P_4 = \begin{bmatrix} 36.6749 & -7.5698 \\ -7.5698 & 12.3944 \end{bmatrix}$$

Fig. 4 presents the intersection of four ellipsoids $\bigcap_{l=1}^4 \mathcal{E}(P_l, 1)$ (solid lines). For comparison, Fig. 4 also shows the intersection of four ellipsoids (dashed lines) obtained by using Theorem 1 in [4], and the ellipsoid with [11] (dash-dot green). Finally, for different initial conditions, Fig. 5 presents the state trajec-

Fig. 4. Invariant sets for our approach (solid lines), for [4] (dashed lines), and for [11] (dash-dot green) for example 3.

jectories in the phase plane.

5 Conclusion

In this paper, a novel approach to the performance analysis of a saturated linear system is proposed. The main contribution of the paper is a new nonlinear saturation-dependent auxiliary feedback law. Using the linear parameter varying system modeling framework, sufficient conditions for the computation of

Fig. 5. State trajectories for different initial conditions for example 3.

invariant set and for the $\mathcal{L}_2$ gain analysis are presented. The obtained conditions are expressed as a set of linear matrix inequalities constraints. It is shown that the proposed conditions are less conservative than the existing results in the literature. Three numerical examples with comparison to earlier solutions in the literature demonstrate the effectiveness of this new method.

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