Quantum Optimal Control for Bose-Einstein-Condensates at Complex Space

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Abstract. Quantum control of Bose-Einstein-Condensates is interesting topic in the areas of control and physics. In this work, Gross-Pitaevskii equation expressed Bose-Einstein-Condensates is considered as control target. Full theoretical proof for the existence of quantum optimal control is provided for cubical Schrödinger equation in complex Hilbert spaces.

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1. Introduction

Bose-Einstein-Condensates (BEC) theory is proposed by Bose and Einstein at 1922, one dimension bosonic atoms can be collapsed to condensate for 10^3 particles in lab at JILA 1995 of last century; Nobel Prize 2001. After that, experimental and theoretical investigation sufficiently reported in literatures for 2D and 3D dimension (cf. [2], [5]). Atomic condensates ^{87}Rb , ^7Li , ^{23}Na , ^{52}Cr , ^{39}K had been made in laboratory. BEC topic had been spread to a various fields: mathematics, chemistry, and so forth. At the viewpoint of control, BEC is seldom regarded as control purpose. Physically, control of a condensate could be occurred at experiment methodology, experimentation result is already appeared [2].

Theoretic control theory is delay for Bose-Einstein-Condensate. For nonlinear Gross-Pitaevskii (G-P) equation, control can be forced at condensate creation consisting of two external factors. One is the electromagnetic field, usual in the quadratic form of spatial variable, another is optical potential, trapped laser for ultracold particles, is in the form of time independent shaped pulse, wave function. To existing physical BEC experiments, a few question is arising in here: it is necessary to apply control theory to a condensates; it is useful for creating a condensate; control can make sense in practice. Absolutely. Qualitatively experimental result for BEC status of a particle could not as access for others. Theoretical survey of quantitatively controlling is indicating adjustment of magnetic trap minimum, and optical confinement. Quantum control for Bose-Einstein-Condensate can be found for control of external force at Hamiltonian feedback controller, numerical control of magnetic field.

In this paper, quantum optimal control is interesting in the theoretical aspect for Bose-Einstein-Condensates, a description of Gross-Pitaevskii equation, in complex space systematically. Taking G-P equation in the framework of variational method in the complex Hilbert space; applying control theory to quantum system of G-P equation to BEC; supplying full proof for the existence quantum optimal control. Finally, theoretical support to find substantial control of bosons from ground state to condensate status.

Divided this paper into a couple of sections. Section 2 is physical background of Bose-Einstein-Condensates. Section 3 is to do mathematical setting for BEC in complex Hilbert spaces. Section 4 is quantum control theory for BEC using the framework of variational method. Section 5 concludes current work.

2. Physical background

In physical field, as an ultra-cold vapor of bosonic atoms are trapped in electro-magnetic well, one can create pure condensates as they are cooling to the temperature than a BEC criticalness. As that taking place, a BEC is located into a optical lattices, a molasses surrounding the particles. Quantum system for ultracold atoms is unique well-posed for experimental controlling. Suppose $\mathbf{n}$ number particles are in a condensates at optical lattice, each particle as P_n for $n = 1, 2, \dots, \mathbf{n}$. Set $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbf{R}^3$.

Bose-Einstein-Condensates can be usually taken as Gross-Pitaevskii (G-P) equation, a cubically nonlinear Schrödinger equation (cf. [4])

$$\mathbf{i}\hbar\frac{\partial\psi_n}{\partial t} = -\frac{\hbar^2}{2M}\Delta\psi_n + u_n(\mathbf{x})\psi_n + \mathbf{n}\nu|\psi_n|^2\psi_n, \quad (1)$$

for $n = 1, 2, \dots, \mathbf{n}$. In here, $\hbar$ is Planck constant, M is atomic mass, $\psi_n(t, \mathbf{x})$ is condensate wave function of n -th particle P_n at BEC, $\mathbf{n}$ is total number of atoms in the condensate, and $\nu = 4\pi\hbar^2 a/M$ is interacting constant, $a \in \mathbf{R}$ is characteristic scatter length. For control purpose, neglect parameters and formulation in magnetic field and optical regime, total external force $V(u_n)$ is unified set $u_n(\mathbf{x}) = v_n^1(\mathbf{x}) + v_n^2(\mathbf{x})$, and external potential $v_n^1(\mathbf{x})$ is electro-magnetic field for experiment BEC of a particle P_n , in general,

$$v_n^1(\mathbf{x}) = M\sigma_0^2 \frac{|\mathbf{x}|^2}{2} \text{ for } \sigma_0 \in \mathbf{R}.$$

Optical potential for particle P_n in physical experiments (cf. [4]) as

$$v_n^2(\mathbf{x}) = \mu \sum_{t=1}^3 \frac{\hbar^2 \mathbf{x}^2}{M} \sin^2(x_t x_t), x_t \in \mathbf{R},$$

and $\mu > 0$ is a dimensionless parameter for expressing the depth of optical lattice. System (1) is representing an interacting n particles BEC condensate.

In practice, although Bose-Einstein-Condensates is taking as Gross-Pitaevskii equation in the form of cubically complex Schrödinger equation, the task in this work is to discuss control theory in complex functional analysis space to ensure control solution is making sense in real space, and meaningful in realistic world. That is, complex G-P equation (1) is formula reflection of a real BEC condensate.

3. Variational method framework

Despite it is real particles of bosonic atoms in a condensate, to test and check control theory is worked for BEC condensate. Consider Bose-Einstein-Condensate in the form of Gross-Pitaevskii equation (1), to do mathematically setting in complex Hilbert spaces. Suppose Ω is an open bounded set of $\mathbf{R}^3$ and $Q = (0, T) \times \Omega$ for $T > 0$, $(t, \mathbf{x}) \in Q$. In G-P system (1), $u_n(\mathbf{x})$ is control input of particle P_n . Control $u_n(\mathbf{x})$ is depended on spatial variable $\mathbf{x}$, and independent of time t . For complex-valued function ψ_n , define complex solution space $\mathbb{L}^2(\Omega)$ and $\mathbb{H}_0^1(\Omega)$ corresponding to $L^2(\Omega)$ and $H_0^1(\Omega)$, use the notations $\mathbb{H} = \mathbb{L}^2(\Omega)$, $\mathbb{V} = \mathbb{H}_0^1(\Omega)$ for two complex Hilbert spaces. Further, $\mathbb{V}'$ is complex conjugate space of $\mathbb{V}$. Actually, $(\mathbb{V}, \mathbb{H})$ is a complex Gelfand triple spaces $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}'$, two embeddings are continuous, dense and compact. For $\psi_n = \psi_n^1 + \mathbf{i}\psi_n^2 \in \mathbb{L}^2(\Omega)$, $\psi_n^1, \psi_n^2 \in L^2(\Omega)$, define norm of ψ_n in complex space $\mathbb{L}^2(\Omega)$ as

$$\|\psi_n\|_{\mathbb{L}^2(\Omega)} = (\|\psi_n^1\|_{L^2(\Omega)}^2 + \|\psi_n^2\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

If $\boldsymbol{\psi}^n = \boldsymbol{\psi}_n^1 + \mathbf{i}\boldsymbol{\psi}_n^2 \in \mathbb{L}^2(\Omega)$, then inner product of ψ_n and $\boldsymbol{\psi}_n$ in complex space $\mathbb{L}^2(\Omega)$ can be defined by

$$(\psi_n, \boldsymbol{\psi}_n)_{\mathbb{L}^2(\Omega)} = ((\psi_n^1, \boldsymbol{\psi}_n^1)_{L^2(\Omega)} + (\psi_n^2, \boldsymbol{\psi}_n^2)_{L^2(\Omega)}) + \mathbf{i}((\psi_n^2, \boldsymbol{\psi}_n^1)_{L^2(\Omega)} - (\psi_n^1, \boldsymbol{\psi}_n^2)_{L^2(\Omega)}).$$

For $\psi_n \in \mathbb{H}_0^1(\Omega)$, the norm of ψ_n in complex space $\mathbb{H}_0^1(\Omega)$ is

$$\|\psi_n\|_{\mathbb{H}_0^1(\Omega)} = (\|\psi_n^1\|_{H_0^1(\Omega)}^2 + \|\psi_n^2\|_{H_0^1(\Omega)}^2)^{\frac{1}{2}}.$$

If $\psi_n \in \mathbb{H}_0^1(\Omega)$, then inner product of ψ_n and ψ_n in complex space $\mathbb{H}_0^1(\Omega)$ is defined as

$$(\psi_n, \psi_n)_{\mathbb{H}_0^1(\Omega)} = (\psi_n^1, \psi_n^1)_{H_0^1(\Omega)} + (\psi_n^2, \psi_n^2)_{H_0^1(\Omega)}.$$

Definition 1 Complex space $W(0, T)$ is said as solution space of ψ_n

$$W(0, T) = \{\psi_n | \psi_n \in L^2(0, T; \mathbb{V}), \psi_n' \in L^2(0, T; \mathbb{V}')\}.$$

The norm of ψ_n in complex space $W(0, T)$ can be defined as

$$\|\psi_n\|_{W(0, T)} = \left(\|\psi_n\|_{L^2(0, T; \mathbb{V})}^2 + \|\psi_n'\|_{L^2(0, T; \mathbb{V}')}^2 \right)^{\frac{1}{2}}.$$

If $\psi_n, \psi_n \in W(0, T)$, then definition of inner product is

$$(\psi_n, \psi_n) = (\psi_n, \psi_n)_{\mathbb{V}} + (\psi_n', \psi_n')_{\mathbb{V}'}.$$

Thus, $W(0, T)$ is a complex Hilbert space equipped by above norm and inner product.

Definition 2 A function ψ_n is said as weak solution of BEC system in the form of G-P equation (1) for n -th particle P_n , $n = 1, 2, \dots, \mathbf{n}$, if $\psi_n \in W(0, T)$ and satisfy

$$\begin{aligned} & \int_0^T \int_{\Omega} \mathbf{i} \hbar \frac{\partial \psi_n}{\partial t} \mathbf{n}^n d\mathbf{x} dt \\ &= \frac{\hbar^2}{2M} \int_0^T \int_{\Omega} \psi_{n\mathbf{x}} \mathbf{n}_{\mathbf{x}}^n d\mathbf{x} dt + \int_0^T \int_{\Omega} u_n(\mathbf{x}) \psi_n \mathbf{n}^n d\mathbf{x} dt + \mathbf{n} \nu \int_0^T \int_{\Omega} |\psi_n|^2 \psi_n \mathbf{n}^n d\mathbf{x} dt, \end{aligned} \quad (2)$$

where $\mathbf{n}^n$ is a arbitrary test function by the means of distribution in space $\mathcal{D}'(0, T)$, $\mathbf{n}^n \in C^1(0, T; \mathbb{V})$ and $\mathbf{n}^n(T) = 0$.

Citing [3, 9], from weak form (2) to get existence theorem of weak solution.

Theorem 1 For $\psi_{n0} \in \mathbb{V}$ of particle P_n , $n = 1, 2, \dots, \mathbf{n}$, then there exists a unique solution $\psi_n \in W(0, T)$ for G-P equation (1).

Proof. As in [3], using Faedo-Galerkin method to construct an approximate solution for BEC system (1). From Gelfand triple spaces, $\mathbb{V} \hookrightarrow \mathbb{H}$ is compact, then there exists an orthogonal basis of $\mathbb{H}$, $\{w^j\}_{j=1}^{\infty}$ consisting of eigenfunctions of $A = \Delta$, such that $Aw^j = \lambda^j w^j$ for all j , $0 < \lambda^1 \leq \lambda^2 \leq \dots, \lambda^j \rightarrow \infty$ as $j \rightarrow \infty$. G^i is the orthogonal projection of $\mathbb{H}$ (or $\mathbb{V}$) onto the space spanned by $\{w^1, \dots, w^i\}$. For each

$i \in N$, an approximate solution is defined for G-P equation (1) by $\psi_n^i(t) = \sum_{j=1}^i a_n^{ij}(t) w_j$,

$n = 1, 2, \dots, \mathbf{n}$, and $a_n^{ij}(t)$ is real-valued coefficient function. Taking $\mathbf{n}^n = w^j$ at weak form (2), then approximate solution $\psi_n^i(t)$ satisfy ordinary differential equation ($1 \leq i \leq N$) as

$$\begin{cases} \int_0^T \int_{\Omega} \mathbf{i} \hbar \frac{\partial \psi_n^i}{\partial t} w^j d\mathbf{x} dt \\ = \frac{\hbar^2}{2M} \int_0^T \int_{\Omega} \nabla \psi_n^i \nabla w^j d\mathbf{x} dt + \int_0^T \int_{\Omega} u_n(\mathbf{x}) \psi_n^i w^j d\mathbf{x} dt + \mathbf{n} \nu \int_0^T \int_{\Omega} |\psi_n^i|^2 \psi_n^i w^j d\mathbf{x} dt. \\ \psi_n^i(0) = \psi_{n0}^i, \end{cases} \quad (3)$$

Substitute ψ_n^i as approximate form in (3), from standard theory of ODE, ensure a unique local solution $\{\psi_n^i\}$, $i = 1, 2, \dots, N$ for approximate system (3). For ψ_{n0}^i , there exists a $\psi_{n0} \in \mathbb{V}$ such that

$$\psi_{n0}^i \rightarrow \psi_{n0} \text{ strongly in } \mathbb{H}_0^1(\Omega).$$

Take analogous argument for $\{\psi_n^i\}$, there is a function $\psi_n \in \mathbb{V}$, that $\|\psi_n^i - \psi_n\|_{\mathbb{V}} \rightarrow 0$. Approximate solution ψ_n^i is bounded in $L^\infty(0, T; \mathbb{V})$. Suppose that $\{\psi_n^i\}$ and $\{\psi_n^k\}$ are two solutions of G-P equation (1) for ψ_{n0}^i and ψ_{n0}^k . From weak form (2) to get

$$\begin{aligned} & \frac{i\hbar}{2} \frac{d}{dt} I^{ik}(t) + \frac{\hbar^2}{2M} \|\psi_n^i - \psi_n^k\|_{\mathbb{V}}^2 \\ & \leq C(t) \left(\|\psi_n^i - \psi_n^k\|_{\mathbb{H}}^2 + \mathbf{n}\nu \left\| |\psi_n^i|^2 \psi_n^i - |\psi_n^k|^2 \psi_n^k \right\|_{\mathbb{H}}^2 \right). \end{aligned}$$

In here, $C(t) = \max\{u_n(\mathbf{x}), 2\mathbf{n}\nu(|\psi_n^i|^2 + |\psi_n^k|^2)\}$ and $I^{ik}(t) = \|\psi_n^i(t) - \psi_n^k(t)\|_{\mathbb{H}}^2 + \|\psi_n^i(t) - \psi_n^k(t)\|_{\mathbb{V}}^2$. Set $\lambda = \max\{\frac{\hbar}{2}, \frac{\hbar^2}{2M}\}$ and $C'(t) = C(t)/\lambda$ that

$$\frac{d}{dt} I^{ik}(t) \leq C'(t) I^{ik}(t)$$

Set $I^{ik}(0) = \|\psi_{n0}^i - \psi_{n0}^k\|_{\mathbb{H}}^2 + \|\psi_{n0}^i - \psi_{n0}^k\|_{\mathbb{V}}^2$, from Gronwall inequality that

$$I^{ik}(t) \leq \exp\left(\int_0^t C'(s) ds\right) I^{ik}(0).$$

It implied $\psi_n^i \rightarrow \psi_n^k$ as $i, k \rightarrow \infty$. Taking $\psi_n^k = \psi_n$ to find that $\psi_n^i \rightarrow \psi_n$ in $C(0, T; \mathbb{H})$. Thus, $C(0, T; \mathbb{H}) \subset W(0, T)$ to sure Theorem 1.

4. Applying quantum control theory

In complex Hilbert space, consider theoretical existence of quantum optimal control for Bose-Einstein-Condensates. Suppose $\mathcal{U} = L^2(\Omega)$ is the space of controls u_n , and $\mathcal{U}_{ad}$ is a admissible set of $\mathcal{U}$. Mathematically set control $\mathbf{u} = (v^1, \dots, u^n, \dots, v^{\mathbf{n}*})$, and corresponding optimal control $\mathbf{u}^* = (v^{1*}, \dots, v^{n*}, \dots, v^{\mathbf{n}*})$. Set ground state of n -th particle P_n as $\psi_n(0, u_n) = \psi_{n0}$. Cost function of n -th particle for G-P system (1) is in the form of

$$J(u_n) = \epsilon^1 \|\psi_n^f(u_n) - \psi_n^{\text{target}}(u_n)\|_{\mathbb{H}}^2 + \epsilon^2 \|u_n\|_{\mathcal{U}}^2. \quad (4)$$

In criteria function (4), $u_n \in \mathcal{U}_{ad}$, $\psi_n^{\text{target}}(u_n)$ is target state, $\psi_n^f(u_n)$ is observed final state of particle P_n at final time t_f . $\epsilon^i, i = 1, 2$ are weighted coefficients for balancing the values of inhere cost and running cost. Total cost function $J(\mathbf{u}) = \sum_{n=1}^{\mathbf{n}} J(u_n)$ for a condensate of n particles.

Two fundamental problems of quantum optimal control for Bose-Einstein-Condensates:

- i). find quantum optimal control u_n^* , $n = 1, 2, \dots, \mathbf{n}$ for G-P system (1).
- ii). find Euler-Lagrange system for u_n^* .

For n -th particle P_n , u_n^* is said as quantum optimal control for G-P system (1) to cost function (4).

Theorem 2 For $\psi_{n0} \in \mathbb{V}$ of n -th particle P_n , $n = 1, 2, \dots, \mathbf{n}$, if $\mathcal{U}_{ad}$ is closed convex (bounded) admissible subset of $\mathcal{U}$, then there exists at least one quantum optimal control u_n^* of Bose-Einstein-Condensates (1) subject to cost function (4).

Proof. Set $J = \inf_{u_n \in \mathcal{U}_{ad}} J(u_n)$, since $\mathcal{U}_{ad}$ is non-empty, there is a sequence $\{u_n^k\}$ in $\mathcal{U}_{ad}$ such that $\inf_{u_n \in \mathcal{U}_{ad}} J(u_n) = \lim_{k \rightarrow \infty} J(u_n^k) = J$. Since $\{J(u_n)\}$ is bounded in $\mathbf{R}^+$, and $\mathcal{U}_{ad}$ is closed and convex (bounded) subset of $\mathcal{U}$, there exist a subsequences $\{u_n^{k'}\}$ of $\{u_n^k\}$ can be extracted, and a $u_n^* \in \mathcal{U}_{ad}$, such that

$$u_n^{k'} \rightarrow u_n^* \text{ weakly in } \mathcal{U} \text{ as } k' \rightarrow \infty. \quad (5)$$

From the existence theorem 1 of weak solution to get estimate $\|\psi_n\|_{\mathbb{H}}^2 + \|\psi_n\|_{\mathbb{V}}^2$ is bounded for ψ_n of n -th particle P_n . For control $u_n^{k'}$, from boundedness of $\mathcal{U}_{ad}$ that

$$\psi_n(u_n^{k'}) \text{ is bounded in } L^2(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}).$$

Setting $\psi_n^* = \psi_n(u_n^*)$, there exist a subsequence $\{\psi_n(u_n^{k''})\}$ of $\{\psi_n(u_n^{k'})\}$ and a $\psi_n^* \in W(0, T)$ such that

$$\psi_n(u_n^{k''}) \rightarrow \psi_n^* \text{ weakly in } L^2(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}).$$

as $k'' \rightarrow \infty$. Since the embedding $\mathbb{V} \hookrightarrow \mathbb{H}$ is compact, from Aubin-Lions-Temam theorem (cf. [6]), then there is $\bar{\psi}_n \in \mathbb{H}$ that

$$\psi_n(u_n^{k''}) \rightarrow \bar{\psi}_n \text{ strongly in } L^2(0, T; \mathbb{H}) \quad k'' \rightarrow \infty,$$

and get the convergences for ψ_n as

$$\begin{cases} \frac{\partial \psi_n}{\partial t}(u_n^{k''}) \rightarrow \frac{\partial \bar{\psi}_n}{\partial t} \text{ weakly in } L^2(0, T; \mathbb{V}'), \\ \nabla \psi_n(u_n^{k''}) \rightarrow \nabla \bar{\psi}_n \text{ weakly in } L^2(0, T; \mathbb{H}), \end{cases} \quad (6)$$

as $k'' \rightarrow \infty$. Set $\psi_n^{k''} = \psi_n(u_n^{k''})$, therefore, for n -th particle P_n , by taking test function $w^j = \mathbf{n}^n$ in (3), by the definition of weak form (2) to find

$$\begin{cases} \int_0^T \int_{\Omega} -i\hbar \psi_n^{k''} \frac{\partial \mathbf{n}^n}{\partial t} d\mathbf{x} dt \\ = \int_0^T \int_{\Omega} -\frac{\hbar^2}{2M} \frac{\partial \psi_n^{k''}}{\partial \mathbf{x}} \frac{\partial \mathbf{n}^n}{\partial \mathbf{x}} + u_n(\mathbf{x}) \psi_n^{k''} \mathbf{n}^n j + \mathbf{n} \nu |\psi_n^{k''}|^2 \psi_n^{k''} \mathbf{n}^n d\mathbf{x} dt \end{cases}$$

If using (5), (6), and taking $k'' \rightarrow \infty$ to yield that

$$\begin{cases} \int_0^T \int_{\Omega} -i\hbar \bar{\psi}_n \frac{\partial \mathbf{n}^n}{\partial t} d\mathbf{x} dt \\ = \int_0^T \int_{\Omega} -\frac{\hbar^2}{2M} \frac{\partial \bar{\psi}_n}{\partial \mathbf{x}} \frac{\partial \mathbf{n}^n}{\partial \mathbf{x}} + u_n(\mathbf{x}) \bar{\psi}_n \mathbf{n}^n + \mathbf{n} \nu |\bar{\psi}_n|^2 \bar{\psi}_n \mathbf{n}^n d\mathbf{x} dt \quad \forall \mathbf{n}^n \in C^1(0, T; \mathbb{V}). \end{cases}$$

It inferred that $\bar{\psi}_n$ is a solution of (1) in the sense of distribution $\mathcal{D}'(0, T)$ on $(0, T)$. From uniqueness of weak solution for BEC system to attain $\bar{\psi}_n = \psi_n(u_n^*)$. For ψ_n to get two convergences as

$$\begin{aligned}\psi_n(u_n^{k''}) &\rightarrow \psi_n(u_n^*) \quad \text{strongly in } L^2(0, T; \mathbb{H}), \\ \psi_n^f(u_n^{k''}) &\rightarrow \psi_n^f(u_n^*) \quad \text{strongly in } \mathbb{H}, \quad k'' \rightarrow \infty.\end{aligned}$$

Since the norm $\|\cdot\|_{L^2(\Omega)}$ are lower semi-continuous in weak topology of $L^2(\Omega)$, for n -th particle P_n

$$\liminf_{k'' \rightarrow \infty} \|\psi_n^f(u_n^{k''}) - \psi_n^{\text{target}}\|_{\mathbb{H}}^2 \geq \|\psi_n^f(u_n^*) - \psi_n^{\text{target}}\|_{\mathbb{H}}^2.$$

Vice versa, from weak convergence (5) that $\liminf_{k'' \rightarrow \infty} (u_n^{k''}, u_n^{k''})_{\mathcal{U}} \geq (u_n^*, u_n^*)_{\mathcal{U}}$. For cost function (4), $J = \liminf_{k'' \rightarrow \infty} J(u_n^{k''}) \geq J(u_n^*)$, and $J(u_n^*) = \inf_{u_n \in \mathcal{U}_{ad}} J(u_n)$. That is, u_n^* is quantum optimal control to criteria function (4). That is Theorem 2.

Theorem 3 For $\psi_{n0} \in \mathbb{V}$, and control problem for G - P system (1) to cost function (4), if $\mathcal{U}_{ad}$ is closed convex (bounded) admissible subset of $\mathcal{U}$, then quantum optimal control u_n^* is characterized by Euler-Lagrange system as

$$\begin{cases} i\hbar \frac{\partial \psi_n}{\partial t} = -\frac{\hbar^2}{2M} \Delta \psi_n + u_n^*(\mathbf{x}) \psi_n + \mathbf{n} \nu |\psi_n|^2 \psi_n & \text{in } Q, \\ \psi_n(u_n^*, 0) = \psi_{n0} & \text{on } \Omega, \\ i\hbar \frac{\partial \sigma_n}{\partial t} = -\frac{\hbar^2}{2M} \Delta \sigma_n + 2\mathbf{n} \nu |\psi_n| \psi_n \sigma_n + \mathbf{n} \nu |\psi_n|^2 \sigma_n & \text{in } Q, \\ i\sigma_n(T) = \psi_n^f(u_n^*) - \psi_n^{\text{target}} & \text{on } \Omega, \end{cases}$$

$$\int_0^T \int_{\Omega} \sigma^* \sigma_n(u_n^*)(u - u_n^*) d\mathbf{x} dt + (u_n^*, u_n - u_n^*)_{\mathcal{U}} \geq 0 \quad \forall u_n \in \mathcal{U}_{ad}. \quad (7)$$

In here, $\sigma_n \in W(0, T)$ is solution of adjoint system corresponding to ψ_n of BEC system. (7) is necessary optimality condition for quantum optimal control u_n^* of n -th particle P_n , $n = 1, 2, \dots, \mathbf{n}$. σ^* is conjugate operator of σ in $\sigma(\psi_n) = u_n^*(\mathbf{x}) \psi_n$.

Applying control theory to Bose-Einstein-Condensate is accomplished for the existence of quantum optimal control to Gross-Pitaevskii equation respect to quadratic criteria function. If using practical formulation of electro-magnetic field $v_n^1(\mathbf{x})$ and optical force $v_n^2(\mathbf{x})$, control of a condensate can be performed directly and straightforwardly. According to apparatus, detail $v_n^1(\mathbf{x}), v_n^2(\mathbf{x})$ can be set up in lab as guidance for controlling particle in physical experiments. For configuration (σ_0, μ) of controls fields v_n^1 and v_n^2 of atomic chips, second order necessary and sufficient equality for criteria function $J(u_n)$ or external force potential $V(u_n)$ at $u_n^* = (v_n^{1*}, v_n^{2*})$ is in variational form as $J''(u_n^*) = 0$ or $\frac{\partial V(u_n^*)}{\partial u_n^2} = - \int_{\Omega} u_n^* \text{Re}[\psi_n(u_n^*)] \text{Re}[\sigma_n(u_n^*)] d\mathbf{x}$. Theoretical conclusion would be worked for real-time computer-aided numerical control by adopting of adequate computational algorithm.

In contrast, question from control are left for Bose-Einstein-Condensates. People create condensate of atoms or bosons at low temperature under the magnetic field and

optical trapping lattice, to make all the particles at the same quantum state. Can BEC condensate be for other purpose at other areas, industrial; aerospace; medical therapy; computer; materials. Condensate out of the door of lab, and into an ideal tools, it is not a dream.

5. Concludes work

Quantum control for Bose-Einstein-Condensate is considered for full proof of existence of optimal control in complex Hilbert space. Control theory is applied to BEC in the framework of variational method. It is a confidential theoretical work for complementary of BEC control theory corresponding to computational approach of [7, 8].

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