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Searching for a new probability distribution for modeling non-scale-free heavy-tailed real-world networks

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Abstract

Perhaps the most controversial topic in network science research is whether real-world complex networks are scale-free or not. Recently, Broido and Clauset [A.D. Broido, A. Clauset, Nature Communication, 10, 1017 (2019)] claimed that the degree distributions of real-world networks are rarely power-law under statistical tests. Such complex networks include social, biological, information, temporal, and brain networks are often heavy-tailed where the assumption on scale-free nature of real-world heavy-tailed networks becomes insignificant so as the complex system evolves over time. The failure of power-law distribution in fitting the degree distribution data is mainly due to the presence of an identifiable non-linearity in the entire degree distribution in a log-log scale of a heavy-tailed complex network. Here, we attempt to address this issue by proposing a new class of heavy-tailed probability distributions, for modeling the entire degree distributions of complex networks and capturing the non-linearity of these heavy-tailed networks. The generalized Lomax Model is introduced to fit the degree distributions of non-scale-free real-world networks and applied to model a wide variety of large-scale real-world complex networks. Several statistical properties of the proposed model, such as extreme value and inferential statistical properties are derived into this context. Rigorous experimental analysis showcases the excellent performance of the proposed family of distributions while fitting the heavy-tailed real-world complex networks over fifty real-world data sets in comparison with the state-of-the-art.

Introduction

The structure of a complex system can be well studied by representing the system as a network. As it is known, most of the practical networks such as communication, traffic, spatial, social, temporal, and biological networks are regarded as complex systems with the same properties as heavy-tailed and large-scale [1–3]. Recent research across the field of networks focuses on justifying the common claim about the scale-free property of real-world complex networks [4–6]. We call a network distribution **scale-free** when its degree distribution follows power-law. One of the most important structural

Fig 1. Plots of degree distributions of Twitter and Livejournal network data to show the non-scale-free nature of these heavy-tailed networks.

characteristics of large-scale real-world networks is degree distribution which characterize a network system [7–9]. Empirical studies in the last two decades on real-world complex networks such collaboration, communication, social, biological and temporal networks are assumed to follow a power-law distribution [8, 10–13].

Mathematically, a random variable X is said to follow the power-law model if its probability distribution is of the form $P(x) \propto x^{-\alpha}$, where α is a positive constant and is known as the tail-index or shape parameter of the distribution. Therefore, it is common to claim the scale-free property of real-world networks that the degree distributions of such networks follow a single power-law.

In Figure 1, a closer look at the plots of entire degree distributions for Twitter and Livejournal networks (in log-log scale) offers clear evidence of the presence of an identifiable non-linearity (bend) into it. This fact suggests that a single power-law is inappropriate to fit the entire degree distribution data. Consequently, the current researchers also reported that a baseline power-law is insufficient to properly fit the empirical data in its whole range unless some of the lower degree nodes are left out while model fitting [11, 14–17]. Recently, Broido & Clauset (2019) [18] mentioned that (strict) scale-free networks are rare, which relate to the claims of Stumpf et al. (2005) [19], that all the derived sub-networks of a scale-free network fails to meet the scale-free property. Apart from power-law, researchers have also proposed various heavy-tailed distributions such as lognormal, Pareto lognormal, and double Pareto lognormal for modeling the degree distribution of real-world complex networks [20, 21]. Despite the alternatives, the long-tailed and heavy-tailed behavior of the entire degree distribution of complex real-world networks remains a challenging task in network science, even as it slowly moves toward the fields of modern statistics and data science [15, 22, 23].

Motivation

Over the last two decades, a remarkable amount of research focuses on studying heavy-tailed complex networks in physical, biological, social, and network sciences. A scale-free network is a network whose degree distribution follows a power-law, at least asymptotically. While studying internet networks, Barabási & Albert (1999) [9] observed that some nodes, called hubs, had many more connections than others and coined the term scale-free network to describe that class of networks having a power-law degree distribution. Clauset, Cosma & Newman (2007) [11] studied real networks and found that some of the complex networks exhibit the same property. Nevertheless, recently, Broido & Clauset (2019) [18] claimed that (strict) scale-free network are

actually rare. More specifically, Kaviani et al. (2021) [24] surveyed various notions of weak or strong scale-free networks. Voitalov et al. (2018) [15] showed that the scale-free property is closely related to the Pareto type-II or Lomax distribution. Broido & Clauset (2019) [18] also observed that the recent data concentration on complex network data sets shows that they no longer follow the power-law distribution (scale-free nature), which relates to the earlier claim of Stumpf, Wiuf & May (2005) [19].

The log-log scale plot of the degree distribution of the Livejournal and Twitter networks in Figure 1 shows the unique degree values (x) presented in the horizontal x axis and the corresponding frequency presented in the vertical y axis. A node corresponding to these networks represent a single user and follower of that user is represented by an edge in the network. Figure 1 shows that the straight-line representation using a single power-law fails to fit the degree distribution data in the log-log scale. Generally, a single power-law distribution is applied to fit the degree distribution data only when the values of degree are considered higher than x_{min} (minimum degree). Subsequently, the power-law exponent α is estimated using maximum likelihood estimation (MLE) from the data with the help of x_{min} . This intern suggests that the power-law distribution tends to provide a better fit only when some lower-degree nodes are left out. The insufficiency of such fitting using a single power-law is due to the presence of a non-linearity in the log-log scale of the degree distribution of a complex network [18, 19] as depicted in Figure 1. Such a drawback inspires many researchers to use other heavy-tailed probability models with various exponents for better fitting the entire degree distribution of real-world complex networks. This article proposes a new class of generalized Lomax models for modeling these heavy-tailed real-world complex networks. The non-negative shape parameter of these new family of distributions can be expressed as a nonlinear function of the network data sets.

Contribution

Motivated by the above discussion, this article introduces a “generalized Lomax models” (GLM), a new class of heavy-tailed probability distributions, to describe real networks than the strict scale-free property. In generalized Lomax models, the non-negative shape parameter (tail-index) is supposed to be defined as a nonlinear function of the degree distribution data. An apparent reason for introducing the GLM family compared to other popularly used heavy-tailed distributions is for better fitting and gaining greater flexibility to the entire node degree distribution of complex networks. Alternatively, the proposed generalized Lomax model can be effectively used to model the whole degree distribution without discarding any lower degree nodes of a complex network coming from different disciplines.

The proposed GLM family is closely related to various life distributions such as Exponential, Weibull, and Rayleigh distributions. We study some specific models and few extreme value properties of this new family of distributions. We have theoretically shown that the models of the GLM family belong to the basin of attractions of the Frechet family of distributions, possess heavy-tails and long-tails, and tail-equivalent to Pareto distribution. We also discuss parameter estimation procedures and goodness of fit for the newly introduced GLM family of distributions. Some nonlinear variants of the proposed GLM family provide better fits than the other popular heavy-tailed models studied over fifty real-world complex network data sets. The new family of proposed GLM distributions is compared with several other popularly used distributions, such as Pareto, Exponential, log-normal, and power-law with an exponential cutoff model [25, 26]. The standard statistical measures, viz. Kullback-Leibler divergence (KLD), mean absolute error (MAE), and root mean square error (RMSE) has been used to evaluate the goodness-of-fit test. Empirical results confirm the effectiveness of the new GLM family of distributions compared to other

similar types of distributions in terms of the goodness-of-fit from an application point of view. This newly introduced ‘shape-parameter-based generalization approach’ for the Lomax distribution can be seen as one such class of distributions that can better fit the entire degree distributions of heavy-tailed networks in general.

The organization of the rest of the paper is as follows. The basic notion of the Lomax distributions is discussed in Section . We have introduced the new family of GLM distributions in Section . Section discusses some statistical properties of the new class of the GLM family. Experimental analysis of over fifty real-world complex networks is provided in Section . Finally Section concludes the paper.

Preliminaries

K.S. Lomax in 1954 first proposed the Lomax distribution for modeling business failures [27]. The Lomax distribution is one of the popularly and widely used heavy-tailed distribution, which has applications in the field of economics, actuarial science, life testing, reliability modeling, operation research, and network analysis [27–30]. Lomax distribution closely resembles a Pareto type II distribution with support beginning at zero [27], and it can be motivated in several ways from the theoretical and application point of view. For example, the Lomax distribution can be derived as a special case of a particular compound gamma distribution [31]. Again the Lomax distribution is also represented as limiting distribution of the residual lifetime at a great age. The record values of the Lomax distribution, as well as the amalgamation of the Lomax distribution with Poisson distribution, were well studied in [32,33]. Researchers have also studied the possible extensions and various modifications of the Lomax distribution to model the real-life problems [34]. Recent studies [35] suggest that the Lomax distribution is used for the analysis of heavy-tailed data and can make use of an alternative to the gamma, Exponential, and Weibull distributions. The corresponding probability density function (PDF) and cumulative distribution function (CDF) of the Lomax model are defined as follows:

Definition 1. A random variable X follows Lomax distribution with parameters α, γ , which is denoted by $LM(\alpha, \gamma)$ if the CDF is of the following form:

$$F(x; \alpha, \gamma) = 1 - \left(1 + \frac{x}{\gamma}\right)^{-\alpha}; \quad x > 0,$$

where $\alpha (> 0)$ is a shape parameter (real) and $\gamma (> 0)$ is a scale parameter (real).

The PDF for the Lomax distribution is given by

$$f(x; \alpha, \gamma) = \frac{\alpha}{\gamma} \left(1 + \frac{x}{\gamma}\right)^{-\alpha-1}; \quad x > 0$$

with mean = $\frac{\gamma}{\alpha-1}$, $\alpha > 1$ and variance = $\frac{\alpha\gamma^2}{(\alpha-1)^2(\alpha-2)}$, $\alpha > 2$.

The quantile function is given by

$$Q(u; \alpha, \gamma) = \gamma \left[(1-u)^{-1/\alpha} - 1 \right],$$

and the expression for the r^{th} order moment of the Lomax distribution is as follows [36]:

$$E[X^r] = \frac{\alpha\gamma^r \Gamma(r+1) \Gamma(\alpha-r)}{\Gamma(\alpha+1)}; \quad \alpha > r, \quad r = 1, 2, 3, \dots$$

The survival function is given by

$$S(x; \alpha, \gamma) = \left(1 + \frac{x}{\gamma}\right)^{-\alpha}; \quad x > 0.$$

The hazard function is given by

$$h(x; \alpha, \gamma) = \frac{\alpha}{\gamma} \left(1 + \frac{x}{\gamma}\right)^{-1}; \quad x > 0$$

which is a decreasing function of x .

It is interesting to see that the Exponential distribution arises as a limit of the Lomax distribution when the shape parameter or tail-index (α) increases. This can be seen through a re-parameterization of

$$\gamma = \frac{\alpha(1 + o_\alpha(1))}{\lambda},$$

where $o_\alpha(1) \rightarrow 0$ as $\alpha \rightarrow \infty$ and $\lambda > 0$. Thus, we have

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} P(X > x) &= \lim_{\alpha \rightarrow \infty} \left(1 + \frac{x}{\gamma}\right)^{-\alpha} \\ &= \lim_{\alpha \rightarrow \infty} \left(1 + \frac{\lambda x}{\alpha(1 + o_\alpha(1))}\right)^{-\alpha} \\ &= e^{-\lambda x}; \quad \lambda > 0. \end{aligned}$$

We can conclude that the limiting distribution of the Lomax distribution for $\alpha \rightarrow \infty$ is the Exponential distribution [32]. However, the Lomax distribution fails to provide greater flexibility while modeling heavy-tailed data sets in its whole range. In this paper, we introduce a new method of generalizing Lomax distributions by modeling the tail index of the Lomax model for fitting the entire degree distributions of real-world heavy-tailed networks.

Proposed Model

An essential structural characteristic in the study of heavy-tailed real-world complex networks is their degree distributions. Empirical observations in the analysis of the pattern of real-world complex networks have led to the claim that their degree distributions follow, in general, a single power-law. However, while fitting, a closer observation suggests that the single power-law distribution is inappropriate to model the network data in its whole range. We first introduce a new class of generalized Lomax models (GLM) whose members belong to a maximum domain of attraction of the Frechet distribution and are right tail equivalent to Pareto distribution. These newly introduced heavy-tailed families will suffice the need for new probability distributions for modeling network data.

Family of Generalized Lomax Models (GLM)

We consider a real, continuous, and positive function $g : (0, \infty) \rightarrow \mathcal{R}^+$ which is differentiable on $(0, \infty)$. Also, we assume that g satisfies conditions stated below.

1. $\lim_{z \rightarrow 0^+} (1 + z)^{g(z)} = 1;$
2. $\lim_{z \rightarrow \infty} (1 + z)^{g(z)} = \infty;$

3. $\frac{g'(z)}{g(z)} \geq \frac{-1}{(1+z)\log(1+z)}, z > 0,$ 146
 where $g'(z)$ is the first-order derivative of $g(z)$. 147

It is noted that the condition 3 is equivalent to

$$\frac{d}{dz} \log(g(z)) \geq -\frac{d}{dz} \log[\log(1+z)], z > 0.$$

Then, for any function g with a strictly positive and finite limit at infinity 148

$$\lim_{z \rightarrow \infty} g(z) = \alpha > 0, \quad (1)$$

and satisfying the above three conditions, we define the new class of GLM distributions 149
 as follows: 150

$$F(x) = \begin{cases} 1 - \left(1 + \frac{x}{\gamma}\right)^{-g(x/\gamma)}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0, \end{cases} \quad (2)$$

where, $\gamma (> 0)$ is a scale parameter. 151

It is very easy to verify that: 152

1. $F(x)$ is non-decreasing since

$$g\left(\frac{x}{\gamma}\right) + \left(1 + \frac{x}{\gamma}\right) \log\left(1 + \frac{x}{\gamma}\right) g'\left(\frac{x}{\gamma}\right) > 0 \text{ for } x > 0.$$

2. $\lim_{x \rightarrow -\infty} F(x) = F(-\infty) = 0$ and $F(\infty) = 1$. 153

3. $F(x)$ is right continuous. 154

Thus, $F(x)$ is a standard CDF and any continuous random variable X satisfying the above-mentioned conditions are called class of GLM distributions. The CDF of standard GLM distributions can also be expressed as follows:

$$F(x) = 1 - \exp\left[-g\left(\frac{x}{\gamma}\right) \log\left(1 + \frac{x}{\gamma}\right)\right], x > 0.$$

The survival function is given by 155

$$S(x) = \begin{cases} \left(1 + \frac{x}{\gamma}\right)^{-g(x/\gamma)}, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0, \end{cases} \quad (3)$$

For GLM family of distributions, the survival distribution function decrease as a power function. The probability density function (pdf) of GLM is

$$f(x) = \begin{cases} \frac{S(x)}{\gamma} \left[\frac{g(x/\gamma)}{(1+x/\gamma)} + g'\left(\frac{x}{\gamma}\right) \log\left(1 + \frac{x}{\gamma}\right) \right], & x > 0 \\ 0, & x \leq 0. \end{cases}$$

The hazard function is given by

$$h(x) = \begin{cases} \frac{1}{\gamma} \left[\frac{g(x/\gamma)}{(1+x/\gamma)} + g'\left(\frac{x}{\gamma}\right) \log\left(1 + \frac{x}{\gamma}\right) \right], & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

Table 1. Some examples of GLM family of distributions (with $\lim_{z \rightarrow \infty} g(z) = \alpha > 0$).

$g(z), \forall z > 0$	β	Distribution	$f(x), \forall x > 0$
α		Lomax	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma}\right]^{-(\alpha+1)}$
$\alpha \left[1 + \frac{\beta}{1 + \log(1+z)}\right]$	$\beta \geq -1$	GLM Type-I	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma}\right]^{-(\alpha+1)} \left[1 + \frac{\beta}{(1 + \log(1+x/\gamma))^2}\right] \exp \left[-\alpha \beta \frac{\log(1+x/\gamma)}{1 + \log(1+x/\gamma)}\right]$
$\alpha \left[1 + \frac{\beta z}{(1+z) \log(1+z)}\right]$	$\beta \geq -1$	GLM Type-II	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma}\right]^{-(\alpha+1)} \left[1 + \frac{\beta}{(1+x/\gamma)}\right] \exp \left[-\alpha \beta \left(\frac{x/\gamma}{1+x/\gamma}\right)\right]$
$\alpha \left[\frac{z}{1+z}\right]^\beta$	$\beta > -1$	GLM Type-III	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma}\right]^{-1} \left[1 + \frac{\gamma \beta \log(1+x/\gamma)}{x}\right] \left[\frac{x/\gamma}{1+x/\gamma}\right]^\beta \exp \left[-\alpha \log \left(1 + \frac{x}{\gamma}\right) \left(\frac{x/\gamma}{1+x/\gamma}\right)^\beta\right]$
$\alpha \left[\frac{\log(1+z)}{1 + \log(1+z)}\right]^\beta$	$\beta > -1$	GLM Type-IV (MLM [17])	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma}\right]^{-1} \left[1 + \frac{\beta}{(1 + \log(1+x/\gamma))}\right] \left[\frac{\log(1+x/\gamma)}{1 + \log(1+x/\gamma)}\right]^\beta \exp \left[-\alpha \frac{\log^{\beta+1}(1+x/\gamma)}{[1 + \log(1+x/\gamma)]^\beta}\right]$

Table 1 shows some examples of GLM family of distributions satisfying limiting condition as in (1), i.e., $\lim_{z \rightarrow \infty} g(z) = \alpha > 0$. It is observable that the Lomax distribution belongs to this family and corresponds to the simplest choice $g(z) = \alpha$. Some of the models are novel among Lomax, GLM Type-I, GLM Type-II, GLM Type-III, and GLM Type-IV distributions presented in Table 1. Out of which GLM Type-IV is very similar to Modified lomax model (MLM) proposed in [16,17]. Though both the models (GLM Type-IV and MLM distributions) are derived from completely different phenomena but achieve the same goal of building a heavy-tailed lomax model for explaining nonlinearity in the degree distributions of the real-world complex network data sets. Other GLM type models are completely new whereas GLM family of distribution covers as particular cases several very popular probability distributions as indicated in Table 2. These models treat the tail index of the Lomax distribution as a nonlinear function of the data with two parameters α and β . To select a model, one can choose the nonlinear exponent $g(z)$ that meets the empirical characteristics of the given data sets. Some graphics of the new models derived from this newly introduced GLM family of distributions are illustrated in Figures 2, 3, 4, and 5. This example list of GLM models can easily be expanded to more models by introducing some other forms of $g(z)$ satisfying the above-mentioned conditions and $\lim_{z \rightarrow \infty} g(z) = \alpha > 0$ or by increasing the number of parameters in the function g . The members of this GLM family belong to a maximum domain of attraction of Frechet distribution and are heavy-tailed, as shown in Section .

Closeness of GLM and Life distributions

We provide some examples of an extended family of GLM distributions with $\lim_{z \rightarrow \infty} g(z) = \infty$ in Table 2 which are closely related with the life distributions. As discussed in Section , the Exponential distribution arises as a limiting distribution for the Lomax distribution when the shape parameter α approaches infinity. However, these models in Table 2 do not satisfy the restriction on g to be strictly positive with a finite limit at infinity as in condition (1). However, these are examples of the choices of $g(z)$ satisfying $\lim_{z \rightarrow \infty} g(z) = \infty$ which corresponds to some popular distributions as the particular cases. Table 2 displays the well-known Exponential, Rayleigh, Weibull, Gompertz, and Benini distribution as the member of an extended GLM family of distributions satisfying $\lim_{z \rightarrow \infty} g(z) = \infty$. In contrast to the GLM family of distributions, the distributions in the extended GLM family are light-tail distributions that belong to the basin of attraction of the Gumbel family of extreme value distributions, for example, Exponential, Rayleigh, Weibull, etc. For this extended GLM family of distributions, the survival distribution function decreases at an exponential rate.

Fig 2. Plots of PDFs of the GLM Type-I distribution for different values of α , β and γ .

Fig 3. Plots of PDFs of the GLM Type-II distribution for different values of α , β and γ .

Fig 4. Plots of PDFs of the GLM Type-III distribution for different values of α , β and γ .

Fig 5. Plots of PDFs of the GLM Type-IV distribution for different values of α , β and γ .

Table 2. Other extended family of GLM distributions with $\lim_{z \rightarrow \infty} g(z) = \infty$.

$g(z), \forall z > 0$	β	Distribution	$f(x), \forall x > 0$
$\frac{z}{\log(1+z)}$		Exponential	$\frac{1}{\gamma} \exp \left[-\frac{x}{\gamma} \right]$
$\frac{z^2}{2\log(1+z)}$		Rayleigh	$\frac{x}{\gamma^2} \exp \left[-\frac{x^2}{(2\gamma^2)} \right]$
$\frac{z^\beta}{\log(1+z)}$	$\beta > 0$	Weibull	$\frac{\beta}{\gamma} \left(\frac{x}{\gamma} \right)^{\beta-1} \exp \left[-\left(\frac{x}{\gamma} \right)^\beta \right]$
$\frac{\beta(e^z - 1)}{\log(1+z)}$	$\beta > 0$	Gompertz	$\frac{\beta}{\gamma} e^{\frac{x}{\gamma}} \exp \left[-\beta \left(e^{\frac{x}{\gamma}} - 1 \right) \right]$
$\alpha[1 + \beta \log(1+z)]$	$\beta \geq 0$	Benini	$\frac{\alpha}{\gamma} \left[1 + \frac{x}{\gamma} \right]^{-(\alpha+1)} \left[1 + 2\beta \log \left(1 + \frac{x}{\gamma} \right) \right] \exp \left[-\alpha\beta \log^2 \left(1 + \frac{x}{\gamma} \right) \right]$

Statistical properties of the GLM family

191

We study several extreme value properties of the new family of standard GLM distributions from the perspective of extreme value and risk theory [37]. Also, we discuss the parameter estimation and goodness of fit for the proposed generalized Lomax models.

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Extreme Value Properties

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Theorem 1. *The general CDF (2) of the GLM family of distributions satisfying (1)*

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(a) *is heavy-tailed.*

198

(b) *is right-tail equivalent to Pareto distribution.*

199

(c) *belongs to dominated-variation distributions.*

200

(d) *belongs to the family of long-tailed distributions.*

201

(e) *belongs to the maximum domain of attraction (MDA) of the Frechet distribution.*

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Proof. (a) Any probability distribution function F is called heavy-tail if it satisfies the following:

$$\lim_{x \rightarrow \infty} \exp\{\lambda x\}(1 - F(x)) = \infty, \text{ for any } \lambda > 0.$$

Thus, for the GLM family of distributions, we observe that

$$\lim_{x \rightarrow \infty} \exp\{\lambda x\}(1 - F(x)) = \lim_{x \rightarrow \infty} \exp \left[\lambda x - g \left(\frac{x}{\gamma} \right) \log \left(1 + \frac{x}{\gamma} \right) \right] = \infty,$$

where $\lambda, \gamma > 0$.

203

(b) To show the tail-equivalent property, we compute

$$\lim_{x \rightarrow \infty} \frac{1 - F(x)}{1 - G(x)} = \lim_{x \rightarrow \infty} \frac{\exp \left[-g \left(\frac{x}{\gamma} \right) \log \left(1 + \frac{x}{\gamma} \right) \right]}{\exp \left[-\alpha \log \left(1 + \frac{x}{\gamma} \right) \right]} = 1, \forall \gamma > 0,$$

where $G(x)$ is the CDF of the Pareto Type-II distribution.

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(c) Any probability distribution function F is said to belong to the class dominated-variation distributions if

$$\limsup_{x \rightarrow \infty} \frac{1 - F(x)}{1 - F(2x)} < \infty.$$

$$\begin{aligned}
\text{Thus, for the GLM family of distributions, } \lim_{x \rightarrow \infty} \frac{1 - F(x)}{1 - F(2x)} &= \lim_{x \rightarrow \infty} \exp \left[g \left(\frac{x}{\gamma} \right) \log \left(\frac{1 + \frac{2x}{\gamma}}{1 + \frac{x}{\gamma}} \right) \right] \\
&= \exp (\alpha \log 2) \\
&= 2^\alpha < \infty,
\end{aligned}$$

where $\alpha > 0$.

(d) Any probability distribution function F is called long-tailed distributions if F has (right) unbounded support and for any fixed $k > 0$

$$\lim_{x \rightarrow \infty} \frac{1 - F(x + k)}{1 - F(x)} = 1.$$

$$\text{Thus, } \lim_{x \rightarrow \infty} \frac{1 - F(x + k)}{1 - F(x)} = \lim_{x \rightarrow \infty} \exp \left[-g \left(\frac{x}{\gamma} \right) \log \left(1 + \frac{(k/\gamma)}{(1 + \frac{x}{\gamma})} \right) \right] = 1,$$

where $\gamma > 0$.

(e) It should be noted that any function g as defined in Section satisfying $\lim_{z \rightarrow \infty} g(z) = \alpha > 0$, is slowly varying at infinity:

$$\lim_{z \rightarrow \infty} \frac{g(tz)}{g(z)} = 1, \forall t > 0.$$

$$\begin{aligned}
\text{Now, } \lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} &= \lim_{x \rightarrow \infty} \frac{(1 + \frac{tx}{\gamma})^{-g(\frac{tx}{\gamma})}}{(1 + \frac{x}{\gamma})^{-g(\frac{x}{\gamma})}} \\
&= t^{-\alpha}, \forall t > 0 \text{ and } \gamma > 0.
\end{aligned}$$

Thus, GLM family of distributions comprise of continuously changing distributions at infinity and belong to the MDA of the Frechet distribution.

□

Parameter estimation

Let $x_1, x_2, \dots, x_n$ be a sample of size n from standard GLM(α, β, γ) distributions (GLM Type-I, Type-II, Type-III and Type-IV) as reported in Table 1. We present the procedure for parameter estimation including the log-likelihood functions and corresponding normal equations only for a particular GLM model, namely GLM Type-IV distribution. Similar derivations can be obtained for the other three GLM Type distributions.

The log-likelihood function $\ell \equiv \ell(x; \alpha, \beta, \gamma)$ for the vector of parameters $\Theta = (\alpha, \beta, \gamma)^T$ corresponding to GLM Type-IV distributions is given by

$$\begin{aligned}
\ell &= n \log(\alpha) - \sum_{i=1}^n \log(\gamma + x_i) + \sum_{i=1}^n \log[\beta + 1 + w_i] \\
&\quad + \beta \sum_{i=1}^n \log[w_i] - (\beta + 1) \sum_{i=1}^n \log[1 + w_i] - \alpha \sum_{i=1}^n \frac{[w_i]^{\beta+1}}{[1 + w_i]^\beta};
\end{aligned} \tag{4}$$

where n is the sample size and $w_i = \log \left(1 + \frac{x_i}{\gamma} \right)$. The maximum likelihood estimates of the unknown parameter vector (α, β, γ) are those that maximizes the log-likelihood function ℓ in Eqn. (4).

The normal equations can be obtained by taking the partial derivatives of Eqn. (4) w.r.t. α , β , γ and equating them to zero as follows.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \frac{w_i^{\beta+1}}{(1+w_i)^\beta} \quad (5)$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^n \frac{1}{(1+\beta+w_i)} + \sum_{i=1}^n \log \left(\frac{w_i}{1+w_i} \right) \times \left[1 - \frac{\alpha w_i^{\beta+1}}{(1+w_i)^\beta} \right] \quad (6)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \gamma} = & - \sum_{i=1}^n \frac{1}{(\gamma+x_i)} + \sum_{i=1}^n \frac{x_i}{\gamma(\gamma+x_i)} \times \left\{ \frac{\beta+1}{(1+w_i)} - \frac{\beta}{w_i} - \frac{1}{(1+\beta+w_i)} \right\} \\ & + \alpha \sum_{i=1}^n \frac{x_i}{\gamma(\gamma+x_i)} \left[\frac{(1+\beta+w_i)w_i^\beta}{(1+w_i)^{\beta+1}} \right] \end{aligned} \quad (7)$$

The MLEs of the three parameters for the GLM Type-IV distributions with α , β , and γ can be obtained by setting the above partial derivatives to zero and then solving them simultaneously. Eqns. (5), (6) and (7) does not have closed-form solutions, thus, an iterative algorithm can be adapted to solve these equations numerically. For practical implementation of the model, we fit the standard GLM models in the whole range of the data sets with quasi-Newton BFGS numerical algorithm with initial values to be chosen as $(\hat{\alpha}_0, \hat{\beta}_0, \hat{\gamma}_0) = (1, 0, 1)$ to find the MLE estimates of the parameters.

Goodness of fit

How well the large-scale real-world complex network data sets correspond to the fitted (hypothesized) standard GLM family of distributions can be determined by the goodness-of-fit test. To this end, we have derived the p -value corresponding to the Chi-square statistic test, which will evaluate the goodness-of-fit for the GLM Type-I, Type-II, Type-III, and Type-IV distributions. We obtain the p -values using bootstrap resampling computational technique as follows: First, we decide the best fit of the proposed GLM family of distributions corresponding to the data after estimating the parameters using MLE and then evaluate the p -values through Chi-square statistic test for the goodness-of-fit of the best-fitted GLM model for the data. Then we generate 50000 synthetic network data sets from the concerned GLM distribution and calculate the Chi-square statistic (p -values) for each of the generated synthetic data sets. Finally, we obtain the p -value for the generated synthetic data sets as the fraction of GLM synthetic data sets with a Chi-square value greater than the empirical one. Higher p -values signify that the proposed model is ‘most’ suitable and effective for the data set. Furthermore, we compute few other standard statistical measures viz. KLD, RMSE, and MAE for quantifying and comparing the goodness-of-fit for the proposed standard GLM family of distributions to the other commonly used heavy-tailed distributions for modeling various heavy-tailed real-world complex networks.

Real-world Applications

Description of data sets

We consider large scale network data sets from different disciplines, namely social networks, collaboration networks, web graphs, citation networks, biological networks, product co-purchasing networks, temporal networks, communication networks, ground-truth networks, and brain networks. We study several individual data sets from each discipline. These data sets are publicly available at

Table 3. Characteristics and statistical measures for fifty real-world complex network data sets

Data sets		No. of Nodes	No. of Edges	Statistical Measures		
				s.d.	mean	CV
Social Networks	ego-Twitter(In)	81,306	1,768,149	57.965	21.747	2.6654
	ego-Gplus(In)	107,614	13,673,453	1404.8	283.42	4.9568
	soc-Slashdot	70,068	358,647	35.069	10.237	3.4260
	soc-Delicious(In)	536,108	1,365,961	39.826	10.673	3.7312
	soc-Digg(In)	770,799	5,907,132	166.61	46.584	3.5765
	soc-Academia	200,169	1,398,063	48.297	14.259	3.3871
	LiveJournal(In)	4,847,571	68,993,773	44.969	15.368	2.9261
	Dogster-Friendship	426,821	8,546,581	284.06	40.033	7.0950
	Higgs-Twitter(In)	456,626	14,855,842	350.91	54.786	6.4051
	Artist-Facebook	50,615	819,307	63.427	32.366	1.9596
	Athletes-Facebook	13,866	86,859	17.978	12.438	1.4453
Citation Networks	cit-HepTh(In)	27,770	352,807	43.139	15.220	2.8342
	cit-HepPh(In)	34,546	421,578	27.286	14.933	1.8271
	cit-Patents(In)	3,774,768	16,518,948	6.9125	5.0687	1.3637
	cit-Citeseer(In)	227,320	814,134	9.8260	5.4322	1.8088
Collaboration Networks	ca-CondMat	23,133	93,497	10.671	8.0189	1.3308
	ca-AstroPh	18,772	198,110	30.568	21.103	1.4484
	ca-GrQc	5,242	14,496	7.9186	5.5284	1.4322
	ca-HepPh	12,008	118,521	46.654	19.696	2.3687
	ca-HepTh	9,877	25,998	6.1867	5.2618	1.1757
Web Graphs	Google(In)	875,713	5,105,039	43.320	7.1444	6.0634
	BerkStan(In)	685,230	7,600,595	300.08	12.316	24.364
	Wikipedia2009(In)	1,864,433	4,507,315	12.846	4.8903	2.6268
	WikipediaLinkFr(In)	4,906,478	113,122,279	1864.4	48.608	38.356
	Hudong(In)	1,984,484	14,869,483	199.28	16.467	12.101
Biological Networks	Yeast-PPIN	2,361	7,182	8.0800	6.0838	1.3281
	Diseasome	3,926	7,823	9.1009	5.5334	1.6447
	Bio-Mouse-Gene	45,101	14,506,199	856.67	643.27	1.3317
	Bio-Dmela	7,393	25569	10.782	6.9170	1.5587
	Bio-WormNet-v3	16,347	762,822	138.17	93.328	1.4805
Product co-purchasing networks	amazon0601(In)	403,394	3,387,388	15.279	8.3989	1.8191
	amazon0505(In)	410,236	3,356,828	15.313	8.1826	1.8714
	amazon0312(In)	400,727	3,200,444	15.073	7.9865	1.8873
Temporal Networks	sx-mathoverflow(In)	24,818	506,550	31.476	10.424	3.0195
	sx-stackoverflow(In)	2,601,977	63,497,050	186.00	27.647	6.7278
	sx-superuser(In)	194,085	1,443,339	23.782	5.8239	4.0836
	sx-askubuntu (In)	159,316	964,437	18.404	4.3856	4.1966
Communication Networks	Email-Enron	36,692	183,831	36.100	10.021	3.6027
	Wiki-Talk(In)	2,394,385	5,021,410	12.259	2.1195	5.7844
	Rec-Libimseti(In)	220,970	17,359,346	413.71	102.85	4.0227
Ground-truth Networks	Wiki-Topcats	1,791,489	28,511,807	283.78	15.915	17.831
	com-Friendster	65,608,366	1,806,067,135	137.81	55.056	2.5031
	com-LiveJournal	3,997,962	34,681,189	42.957	17.349	2.4759
	com-Orkut	3,072,441	117,185,083	154.78	76.281	2.0291
	com-Youtube	1,134,890	2,987,624	50.754	5.2650	9.6398
Brain Networks	Human25890-session1	177,584	15,669,036	319.01	176.47	1.8078
	Human25890-session2	723,881	158,147,409	667.91	436.94	1.5286
	Human25864-session2	692,957	133,727,516	554.48	385.96	1.4366
	Human25913-session2	726,197	183,978,766	446.92	258.99	1.7256
	Human25886-session1	780,185	158,184,747	558.41	405.50	1.3771

<http://snap.stanford.edu/data/index.html>. These are the most standard network data sets that have heavy-tailed behaviors and are used for modeling in the statistical paradigm [10, 25, 26]. Previous studies focused on using standard single statistical distributions, namely power-law, Lomax (Pareto Type-II), Exponential, Log-normal for modeling this wide variety of network data sets [1, 11]. But these models fail in capturing the lower-degree nodes while modeling the degree distributions. To overcome the drawback, we consider a new family of the proposed GLM family of distributions. Note that the proposed family of heavy-tailed GLM distributions can model these large-scale network data sets in the whole range. An overview of these publicly available network data sets is presented in Table 3. Some statistical measures, for example, mean, standard deviation (s.d.), and calculated CV corresponding to the degree distributions of each network data set, are also given in Table 3. It is important to note that the empirical CV for all the data sets is greater than one, as reported in Table 3.

Analysis of results

This section shows the effectiveness of the proposed GLM family of distributions by comparing it with six other established, well-known models, such as Lomax, power-law, power-law with cutoff, Log-normal, and Exponential distributions. The performances of these commonly used heavy-tailed and life distribution models are reported in Table 6. Finally, we test the adequacy of the GLM family of distributions compared to these heavy-tailed and light-tailed distributions over fifty large-scale real-world network data sets in the whole range. To do the numerical computations for fitting the new family of GLM distributions in the whole range for large-scale real-world network data sets, we use the R statistical software function ‘*optim*’. The ‘*optim*’ function makes use of the limited memory quasi-Newton L-BFGS-B optimization algorithm in the background by taking the initial parameter values as $(\hat{\alpha}_0, \hat{\beta}_0, \hat{\gamma}_0) = (1, 0, 1)$.

We fitted the GLM Type-I, GLM Type-II, GLM Type-III, and GLM Type-IV models over the entire degree distribution of the complex network data sets by applying MLE to estimate the parameters, as discussed in Section and . We measure the goodness-of-fit of the proposed GLM distributions through the bootstrap resampling Chi-square test. Table 4 represents the estimated values of the parameters for four newly introduced GLM models along with the p -values. It is observed that the obtained p -values > 0.4 for all the cases. Also, it is clear that the MLE estimated parameter values for all the network data sets satisfy the condition $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. The empirical analysis shows that the proposed standard GLM distributions fit well in the majority of the network data sets under consideration. We provide enough statistical evidence of fitting the proposed GLM models in Table 5. We leverage some popular statistical measures, viz., MAE, RMSE, and KLD, to evaluate and compare the performance of the proposed standard GLM distributions with others as shown in Table 5.

The overall performance of the proposed GLM models are better in terms of RMSE, MAE, and KLD values for most of the networks compared to other competing distributions, which suggests that the proposed GLM family plausibly fits the observed node distribution. Empirical data analysis suggests that the parameter estimation yield the values for specifying the best fit of the proposed models. However, the estimated values alone do not give any effective information regarding the validity or the goodness-of-fit of the underlying models. From Table 5 and 6, it can be concluded that the GLM Type-I model outperforms all the competitive models for four data sets. In contrast, the GLM Type-III model outperforms others for eight network data sets from various domains. The GLM Type-II model performs superior to others for thirteen data sets. The power-law model with exponential cutoff [11] performed better than the GLM models for four out of fifty real-world complex heavy-tailed network data sets. Overall, the performance of the GLM Type-IV model is very consistent across all the data sets,

Table 4. Parameter estimates for the GLM models over 50 different real-world network data

Data	GLM Type-I				GLM Type-II				GLM Type-III				GLM Type-IV			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	p	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	p	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	p	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	p
ego-Twitter(In)	1.4425	0.4908	16.404	0.9600	1.4267	-0.0018	11.098	0.9789	1.9244	0.4791	35.361	0.8746	1.9922	-0.3591	30.543	0.9920
ego-Gplus(In)	0.6257	0.4903	15.236	0.9540	0.6283	-0.0015	10.577	0.9217	0.6666	-0.5953	8.8974	0.8792	0.7108	-0.4983	23.077	0.9963
soc-Slashdot	0.8463	0.1220	0.8679	0.9515	0.8478	-0.0047	0.7968	0.9746	0.8684	-0.5504	0.5696	0.9825	0.8663	-0.6228	1.0461	0.9955
soc-Delicious(In)	1.0804	0.5820	3.4346	0.9782	1.0853	-0.0001	2.2622	0.9638	1.0904	-0.3791	1.7701	0.9600	1.3630	-0.6819	5.3709	0.9960
soc-Digg(In)	0.7204	0.4587	4.3544	0.9810	0.7235	-0.0025	3.1117	0.9573	0.7544	-0.5379	2.4772	0.9271	0.7931	-0.6928	5.5163	0.9890
soc-Academia	1.7382	0.4769	15.286	0.5872	1.7182	-0.0229	10.202	0.6480	2.6546	0.2978	38.846	0.5901	2.7429	-0.3737	36.644	0.6087
LiveJournal(In)	1.1060	0.5473	6.4094	0.8713	1.1016	-0.0010	4.2266	0.8010	1.0846	-0.3344	3.3178	0.8245	2.6892	-0.7272	51.933	0.8983
Dogster-Friendship	1.5772	-0.6461	11.177	0.9178	1.5188	-0.7341	8.0293	0.9200	1.5493	-0.6744	9.5169	0.9246	1.5634	0.3108	14.057	0.9500
Higgs-Twitter(In)	1.6379	0.7828	8.8420	0.9457	1.6437	-0.0010	33.501	0.9652	1.0009	3.5570	11.578	0.9700	1.6797	-0.0347	36.204	0.9870
Artist-Facebook	1.7616	0.4736	38.243	0.9871	1.7268	0.0024	25.915	0.9889	1.0003	2.2910	24.774	0.9677	2.0117	-0.1445	39.337	0.9812
Athletes-Facebook	3.0346	-0.4698	15.827	0.9023	2.9388	-0.5970	11.569	0.9230	3.1509	-0.0843	21.497	0.9450	3.1229	0.1406	21.180	0.9640
cit-HepTh(In)	1.5176	0.5016	12.196	0.8020	1.5010	0.0016	8.2280	0.8622	1.7932	0.4842	19.284	0.8238	1.8410	-0.3093	16.416	0.8730
cit-HepPh(In)	1.6478	0.4899	15.227	0.9800	1.6216	8.8e-04	10.267	0.9823	2.4380	0.3225	35.806	0.9730	2.5553	-0.3622	34.349	0.9900
cit-Patents(In)	3.0822	0.4155	12.407	0.7600	2.9957	0.0013	8.6685	0.7829	4.3878	0.1154	20.912	0.7760	4.4822	-0.2534	21.689	0.8080
cit-Citeseer(In)	1.9770	0.5650	6.8768	0.5620	1.9529	-0.0012	4.4284	0.5930	2.2391	0.3025	8.3053	0.6210	2.2630	-0.2788	7.4150	0.6350
ca-CondMat	3.0282	-0.7145	7.8751	0.9790	2.8862	-0.7832	5.5961	0.9821	3.1195	-0.2201	10.704	0.9328	3.1068	0.3615	10.535	0.9896
ca-AstroPh	1.9271	0.4732	30.925	0.9537	1.8673	-0.0010	20.590	0.9480	1.0040	28.012	6.4950	0.9729	16.434	17.276	9.0101	0.9990
ca-GrQc	1.6553	-0.9564	3.5532	0.6982	1.4603	-0.8822	4.4838	0.7290	1.5173	-0.4341	1.6545	0.7628	2.2624	3.5861	0.6765	0.7849
ca-HepPh	0.8466	-0.8552	2.9314	0.8092	0.7828	-0.7766	2.7426	0.8009	0.6181	1.1938	3.6423	0.7985	0.9798	2.8780	0.6791	0.8163
ca-HepTh	2.0930	-0.2740	2.0444	0.8710	1.9578	-0.9675	5.4783	0.8892	2.0088	-0.9969	2.6700	0.9023	2.9417	5.2791	0.4825	0.9332
Google(In)	1.0913	0.6327	1.9727	0.8938	1.0992	0.0007	1.2666	0.9012	1.1111	-0.4390	0.9454	0.9520	1.1999	-0.6399	2.0429	0.9780
BerkStan(In)	1.4163	-0.4362	5.1998	0.5010	1.1456	-0.7842	8.0937	0.5541	1.1368	-0.0213	3.1245	0.6020	1.4129	1.8449	0.7592	0.6250
Wikipedia2009(In)	1.3230	0.5888	2.0563	0.8970	1.3377	-0.0038	1.3722	0.9261	1.3513	-0.3821	1.0539	0.9789	1.3988	-0.6291	1.9658	0.9891
WikipediaLinkP(In)	0.7804	0.5529	4.3367	0.8800	0.7822	0.0021	2.8959	0.8128	0.8015	-0.5066	2.2526	0.8735	1.0988	-0.7123	9.8888	0.9152
Hudong(In)	0.9796	-0.2066	15.002	0.9200	0.7881	0.0085	0.3133	0.9374	0.7990	-0.7314	0.1727	0.9720	1.1567	10.921	0.0013	0.9883
Yeast-PPIN	1.4329	0.5402	5.2496	0.9680	1.4013	0.0015	3.4116	0.9718	1.2137	2.6648	7.8313	0.9870	10.535	-0.4527	175.29	0.9930
Diseasome	1.1069	0.6460	2.2879	0.7028	1.1030	0.0031	1.4262	0.7620	1.1031	-0.3999	1.0749	0.7988	10.969	-0.9493	134.52	0.8090
Bio-Mouse-Gene	0.0579	-0.7322	3.9518	0.9380	0.0512	-0.5732	3.5602	0.9678	0.1190	-0.9903	4.7784	0.9890	6.3e-08	-1.2e-02	2.1e+00	0.9898
Bio-Dmela	1.5268	0.5971	6.1136	0.8972	1.5076	0.0003	3.8602	0.9342	3.5180	0.3183	39.215	0.9679	14.979	-0.5053	498.27	0.9806
Bio-WormNet-v3	0.4182	0.5801	12.836	0.9780	0.4034	-0.0002	7.9435	0.9890	0.5142	-0.6171	9.3771	0.9829	5.6496	-0.9801	704.71	0.9938
amazon0601(In)	3.4448	-0.6406	10.273	0.5026	3.2935	-0.7309	7.3059	0.5567	3.5283	-0.1446	14.262	0.5830	3.8261	-0.7137	19.522	0.6010
amazon0505(In)	3.4863	-0.4779	11.883	0.5100	3.3632	-0.6095	8.5613	0.5671	3.6248	-0.0677	16.618	0.5892	3.8367	-0.8006	19.984	0.6880
amazon0312(In)	3.4280	-0.5110	10.948	0.4997	3.3018	-0.6336	7.8797	0.5102	3.5387	-0.0850	15.171	0.5287	3.7631	-0.8179	18.747	0.5890
sx-mathoverflow(In)	1.3002	-0.9831	4.2711	0.8751	1.0922	-0.9254	5.2890	0.9127	1.1282	-0.6887	6.4165	0.9457	1.4452	2.4236	0.8241	0.9846
sx-stackoverflow(In)	1.0282	-0.2710	3.8714	0.7982	1.0402	-0.0038	4.8809	0.8086	1.0604	-0.6329	3.1170	0.9780	1.0218	-0.8224	4.4865	0.9490
sx-superuser(In)	1.6803	-0.8466	5.1872	0.9421	1.3392	-0.6496	2.1189	0.9537	1.3425	-0.4221	2.1865	0.9788	1.7401	2.1405	0.7284	0.9780
sx-askubuntu(In)	2.0897	-0.1210	6.1834	0.7002	1.6205	-0.6537	7.0789	0.7546	1.5954	-0.9139	4.1416	0.8249	2.1923	2.2069	0.7665	0.9300
Email-Enron	1.2232	-0.0338	2.5047	0.9045	1.2149	-0.2599	1.9463	0.9246	1.0797	1.7058	3.4351	0.9550	1.2417	-0.1275	2.9045	0.9641
Wiki-Talk(In)	0.4360	18.634	2.2229	0.9452	1.5175	0.6688	3.2e-05	0.8990	0.9895	59.520	0.0548	0.9782	1.5167	-0.2846	0.0016	0.9900
Rec-Libimseti(In)	0.5100	0.5417	5.2411	0.9247	0.5055	0.0004	3.3914	0.9354	0.5536	-0.5814	3.2391	0.9556	2.5008	-0.8496	331.18	0.9670
Wiki-Topcats	1.1907	-0.3884	2.3364	0.7400	1.1697	-0.4969	1.8217	0.7893	1.1968	-0.7570	1.6464	0.8420	1.1811	0.1998	2.6412	0.8310
com-Friendster	0.4495	0.5330	2.1401	0.8080	0.4481	-0.0014	1.4159	0.8123	0.4878	-0.5807	1.2679	0.8450	4.5863	-0.9188	590.01	0.9000
com-LiveJournal	1.2144	0.5169	9.3591	0.6798	1.2065	0.0010	6.2874	0.7246	1.0222	30.033	26.596	0.7800	2.8206	-0.6020	65.638	0.7980
com-Orkut	3.2813	-0.6192	94.848	0.9120	3.1372	-0.7163	67.315	0.9570	0.9977	183.89	46.878	0.9544	3.7049	0.1292	167.93	0.9890
com-Youtube	1.0835	0.4666	0.5561	0.7840	1.0994	-0.0046	0.4157	0.8701	1.1142	-0.5816	0.2706	0.8701	1.6113	8.3355	0.0094	0.8410
Human25890-session1	1.2418	0.5115	123.15	0.8701	1.2183	0.0025	81.378	0.8018	1.1098	1.8328	136.28	0.7812	1.6098	-0.2076	168.75	0.8710
Human25890-session2	1.3134	0.4457	353.84	0.8990	1.2927	0.0019	246.67	0.9102	2.8936	0.2284	1711.6	0.9920	14.423	-0.3466	18886.3	0.9980
Human25864-session2	0.2129	-0.2582	5.2852	0.9781	1.2291	6.9e-04	207.25	0.9631	0.2186	-0.9777	4.7560	0.9800	16.250	-0.3379	19217.8	0.9640
Human25913-session2	0.6684	0.5383	58.072	0.8403	0.6570	-0.0006	37.681	0.8476	0.7088	-0.5805	33.302	0.9257	7.1013	-0.4681	5779.8	0.9290
Human25886-session1	1.5185	0.4334	430.83	0.8902	1.4983	8.4e-04	303.87	0.8935	1.1350	2.4212	525.20	0.9456	21.591	-0.3119	26975.9	0.9768

Table 5. Performance measures of the proposed generalized Lomax models over 50 real world network data

Data sets		GLM Type-I			GLM Type-II			GLM Type-III			GLM Type-IV (MLM [17])		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	27.161	0.01264	2.3209	29.325	0.01354	2.4709	17.465	0.00828	1.3814	16.800	0.00819	1.3498
	ego-Plus(In)	10.028	0.06357	0.3927	10.579	0.06444	0.4034	7.6188	0.06088	0.2591	1.6115	0.05001	0.1825
	soc-Slashdot	30.607	0.01406	2.3965	30.581	0.01407	2.3944	29.255	0.01355	2.3074	31.527	0.01365	2.3951
	soc-Delicious(In)	88.643	0.01255	4.6171	91.403	0.01326	4.7889	87.087	0.01293	4.6383	79.809	0.00839	3.7730
	soc-Digg(In)	23.306	0.02362	1.0020	24.568	0.02391	1.0214	21.642	0.02287	0.9678	13.634	0.02182	0.8440
	soc-Academia	46.35	0.00932	1.5352	49.037	0.01020	1.6258	16.758	0.00356	0.5958	16.323	0.00351	0.5705
	LiveJournal(In)	1710.8	0.01968	52.426	1760.4	0.02112	54.325	1647.4	0.02078	52.644	243.99	6.13e-04	5.4026
	Dogster-Friendship	28.237	0.01245	0.7818	27.864	0.01215	0.7650	34.434	0.01353	0.8853	32.203	0.01328	0.8502
	Higgs-Twitter(In)	20.611	0.00787	0.4575	20.613	0.00793	0.4622	20.2112	0.00787	0.4639	19.821	0.00785	0.4710
	Artist-Facebook	19.710	0.01147	2.3187	11.861	0.01193	2.4949	8.4631	0.01143	2.0212	11.708	0.01079	2.1381
	Athletes-Facebook	5.4833	0.00899	1.4784	5.6072	0.00901	1.4950	4.2557	0.00876	1.2988	4.4304	0.00879	1.3252
Citation Networks	cit-HepTh(In)	7.0784	0.01531	0.7060	7.8479	0.01584	0.7536	3.3792	0.01358	0.5150	3.2640	0.01354	0.5071
	cit-HepPh(In)	18.816	0.01235	2.8438	20.303	0.01311	3.0223	10.051	0.00834	1.9528	9.6810	0.00821	1.9016
	cit-Patents(In)	2400.3	0.00165	213.79	2580.7	0.00192	230.33	321.64	1.26e-04	36.668	445.80	1.61e-04	47.603
	cit-Citeseer(In)	24.451	0.00256	3.0602	27.533	0.00278	3.3824	41.668	0.00231	3.8355	40.728	0.00228	3.3778
Collaboration Networks	ca-CondMat	13.146	0.00471	3.8859	12.914	0.00471	3.8557	15.609	0.00483	4.2729	14.830	0.00479	4.1570
	ca-AstroPh	31.827	0.03335	7.1574	31.943	0.03425	7.2577	31.189	0.03423	6.9837	23.890	0.02756	5.9796
	ca-GrQc	10.147	0.02796	5.3808	9.9131	0.02801	5.2724	10.069	0.02800	5.3306	15.850	0.03055	7.2247
	ca-HepTh	13.630	0.06938	4.1517	13.468	0.06945	4.1297	13.197	0.06814	4.0911	13.944	0.06959	4.1919
Web Graphs	Google(In)	337.11	0.01504	14.019	333.95	0.01546	14.129	319.73	0.01505	13.993	360.62	0.01368	13.845
	BerkStan(In)	42.907	0.02851	0.7056	34.360	0.02711	0.5673	35.906	0.02704	0.6067	71.819	0.03116	0.9478
	Wikipedia2009(In)	92.743	0.00189	7.8955	100.61	0.00197	8.3468	95.333	0.00192	8.0388	86.510	0.00169	7.7408
	WikipediaLinkFr(In)	141.99	0.02955	0.4321	144.48	0.03088	0.4451	136.35	0.02841	0.4202	124.98	0.01776	0.3174
Biological Networks	Hadong(In)	11.465	0.3076	0.3076	25.955	0.00525	0.4596	23.271	0.00512	0.4378	8.0517	0.00433	0.2502
	Yeast-PPIN	7.8751	0.01962	4.0614	8.0721	0.02006	4.1484	4.3043	0.01797	2.6537	4.3766	0.01487	2.6321
	Diseaseome	11.411	0.09954	3.3500	11.371	0.10023	3.3509	11.172	0.09984	3.3264	8.8683	0.08090	2.7445
	Bio-Mouse-Gene	15.990	0.20137	2.4457	14.861	0.19761	2.4118	17.602	0.20148	2.4549	7.6919	0.18941	2.2557
Product co-purchasing networks	Bio-Dmela	8.7148	0.01674	3.3483	8.7368	0.01715	3.3947	17.519	0.01442	4.1165	16.173	0.01305	4.0968
	Bio-WormNet-v3	12.897	0.09075	3.7004	12.848	0.09247	3.7318	11.202	0.08337	3.4857	14.054	0.04648	2.0666
	amazon0601(In)	82.087	0.00335	7.1527	81.665	0.00325	7.0597	97.130	0.00391	8.4329	94.347	0.00374	8.1863
	amazon0505(In)	104.28	0.00366	8.5019	104.27	0.00358	8.4494	113.45	0.00422	9.2333	109.95	0.00412	9.1836
Temporal Networks	amazon0312(In)	91.763	0.00335	7.5804	91.416	0.00327	7.5191	102.88	0.00388	8.4072	100.89	0.00430	8.5465
	sx-mathoverflow(In)	13.268	0.01537	1.7301	13.092	0.01508	1.6809	13.769	0.015238	1.7288	19.706	0.01879	2.4647
	sx-stackoverflow(In)	43.321	0.00341	0.9205	41.452	0.00345	0.9064	50.608	0.00337	0.9064	39.654	0.00336	0.8694
	sx-superuser(In)	33.298	0.00409	2.3678	26.814	0.00341	1.7797	31.548	0.00342	1.8848	79.777	0.00654	4.5409
Communication Networks	sx-askubuntu(In)	46.943	0.00669	3.3637	36.935	0.00479	2.3870	41.091	0.00460	2.5579	106.04	0.01100	6.2022
	Email-Enron	76.151	0.03529	5.2324	76.417	0.03526	5.2335	75.369	0.03499	5.1948	74.667	0.03523	5.2075
	Wiki-Talk(In)	325.74	0.00149	14.119	671.976	0.00357	25.903	654.64	0.00255	23.816	670.47	0.00356	25.871
	Rec-Libimseti(In)	66.955	0.09759	1.7695	66.498	0.09977	1.7924	65.039	0.08939	1.6652	23.341	0.02163	0.4953
Ground-truth Networks	Wiki-Topcats	10.906	0.00190	0.0987	10.206	0.00189	0.0951	13.254	0.00192	0.1098	11.375	0.00190	0.1011
	com-Friendster	43.97K	0.06351	3427.1	42.64K	0.06400	3413.5	43.76K	0.05975	3326.1	8266.8	0.00126	411.15
	com-LiveJournal	1685.8	0.02289	48.512	1738.8	0.02462	50.298	447.06	0.00584	18.469	165.79	0.00084	6.9832
	com-Orkut	165.35	0.00548	4.7694	164.41	0.00538	4.6646	315.83	0.03085	16.651	197.89	0.00793	7.0113
Brain Networks	com-YouTube	83.496	0.00177	1.1093	82.441	0.00175	1.0957	78.454	0.00167	1.0403	53.288	0.00122	0.6984
	Human25890-session1	17.029	0.02172	3.6229	17.843	0.02264	3.7572	16.150	0.02068	3.4537	17.309	0.01920	3.7439
	Human25890-session2	94.272	0.04548	14.885	96.959	0.04769	15.296	61.328	0.01488	6.4757	46.276	0.01024	5.8781
	Human25864-session2	147.06	0.14826	33.184	108.04	0.05325	20.661	147.63	0.14423	32.581	64.037	0.01321	9.8670
	Human25913-session2	116.61	0.04524	20.418	119.75	0.04718	20.982	85.326	0.04042	18.015	112.661	0.01347	11.574
	Human25886-session1	113.21	0.0525	22.854	116.77	0.05476	23.393	64.951	0.03826	15.540	65.181	0.01471	12.051

and it gives the ‘best’ fit for 22 out of 50 data sets considered in this study.

Other than the statistical comparison, visual inspections of the fits of different models are used as an essential tool for model validation and comparison. This motivates us to draw the plots of the original frequency distribution and the estimated frequencies (in log-log scale) by GLM family of distributions and the frequency estimated by other competing distributions used in this paper viz. Lomax, log-normal, power-law, power-law cutoff, and Exponential distributions over all the networks under consideration. Figures 6 and 7 provide the plots of different models in the log-log scale for twelve network data sets from other domains. It is visually clear from the Figures 6 and 7 that the proposed GLM models provide a better fit in comparison to the other single heavy-tailed and life distributions under considerations for most of the network data sets. Pictorial representation from the log-log plots shows that the proposed model always passes through the middle of the scatter plot of the observed distribution compared to the competing heavy-tailed distributions in very few cases. From the practitioner’s point of view, the proposed fitted distribution can easily capture the spatial structure of the network by simulating the model parameters. The new concept of the GLM family of distributions provides a better and efficient fit for the entire degree distribution of real-world heavy-tailed complex networks than the other competitive heavy-tailed distributions.

Discussion and Conclusion

Learning from complex networks (such as social, information, biological, temporal, and brain networks) is a difficult proposition. Many studies on complex networks focus on network topology and network statistical mechanics from a scale-free perspective.

Table 6. Performance evaluation of commonly used heavy-tailed models over 50 real-world network data

Data sets		Lomax			power-law			Log-normal			power-law cutoff			Exponential		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	29.366	0.01354	2.4701	204.35	0.1831	10.847	53.863	0.0169	2.9494	68.004	0.0397	4.1974	157.98	0.2733	11.567
	ego-Gplus(In)	10.491	0.06444	0.3033	53.064	0.2299	0.9221	10.155	0.0678	0.2523	30.925	0.1475	0.6821	50.098	1.3131	1.8328
	soc-Slashdot	32.065	0.014102	2.4658	247.87	0.1007	10.074	237.63	0.1058	10.549	19.598	0.0075	1.3599	434.25	0.6381	22.275
	soc-Delicious(In)	91.993	0.01326	4.8060	349.66	0.2021	14.867	281.34	0.0579	10.781	66.896	0.0185	4.2366	535.11	0.4626	25.304
	soc-Digg(In)	24.841	0.02391	1.0269	208.01	0.1601	4.2185	69.438	0.0552	1.9087	65.713	0.0441	1.9042	204.50	0.8907	8.0015
	soc-Academia	48.951	0.01019	1.6178	229.54	0.2027	6.3889	91.003	0.0169	2.0921	62.376	0.0255	1.9845	198.11	0.1924	6.6739
	LiveJournal(In)	1764.9	0.02111	54.400	5025.2	0.1614	127.98	3473.6	0.0355	70.64	808.79	0.0101	24.501	7017.9	0.3440	186.01
	Dogster-Friendship	36.449	0.01700	1.0755	358.27	0.2926	5.6815	42.539	0.0272	1.1459	182.47	0.1862	4.1579	165.19	0.4765	5.4623
	Higgs-Twitter(In)	20.609	0.00793	0.4621	260.32	0.2492	4.8938	41.955	0.0134	0.5995	118.23	0.0914	2.8051	134.68	0.3163	4.4689
	Artist-Facebook	12.923	0.01199	2.6173	100.49	0.1643	14.552	24.071	0.0154	3.026	56.801	0.0458	6.6452	88.351	0.1799	13.623
Citation Networks	cit-HepTh(In)	7.9393	0.01585	0.7585	73.531	0.1741	4.0821	22.59	0.0255	2.331	25.42	0.0464	2.816	58.74	0.2778	4.5286
	cit-HepPh(In)	21.303	0.01317	3.1135	128.55	0.1825	13.234	44.951	0.0189	4.4405	36.887	0.0221	4.5287	107.32	0.1801	14.145
	cit-Patents(In)	2577.5	0.00192	230.35	27.5K	0.2266	2049.5	9612.7	0.0192	725.71	2424.5	0.0061	271.89	13.2K	0.0659	1147.5
	cit-Citeseer(In)	28.032	0.00278	3.3902	889.88	0.3308	49.467	353.26	0.0299	21.921	195.67	0.0131	13.877	629.02	0.1486	44.301
Collaboration Networks	ca-CondMat	36.094	0.00814	7.3904	107.86	0.1025	26.092	42.665	0.0082	6.5746	62.929	0.0287	13.362	64.985	0.0472	16.873
	ca-AstroPh	32.799	0.03457	7.4448	92.255	0.1753	15.158	28.209	0.0312	6.8565	50.604	0.0384	7.4579	68.185	0.1235	14.361
	ca-GrQc	35.935	0.04013	12.286	124.24	0.2554	27.137	30.184	0.0515	12.148	58.305	0.0659	69.169	69.169	0.1418	25.759
	ca-HepPh	19.607	0.07266	4.7763	75.071	0.1769	8.0906	29.993	0.1011	6.8038	50.717	0.1128	8.2477	89.936	0.5187	17.589
	ca-HepTh	61.797	0.01353	20.391	268.91	0.2346	66.108	55.618	0.0178	21.032	89.425	0.0245	27.613	109.96	0.0551	43.882
Web Graphs	Google(In)	337.68	0.01546	14.201	1809.1	0.124	45.023	1514.5	0.0878	40.067	188.01	0.0157	9.6549	2589.4	0.4419	76.441
	BerkStan(In)	105.20	0.0346	1.1962	615.03	0.1863	4.0722	185.04	0.1002	2.0198	322.63	0.1037	2.8203	595.53	0.7438	6.6185
	Wikipedia2009(In)	103.94	0.00197	8.5289	4371.9	0.1352	164.58	2720.9	0.0798	116.43	781.87	0.0082	35.531	4727.1	0.3431	213.66
	WikipediaLinkPr(In)	146.14	0.03082	0.4465	248.09	0.1518	0.7857	240.18	0.0543	0.5234	121.31	0.0622	0.5006	534.61	1.0217	2.0471
Biological Networks	Yeast-PPIN	12.651	0.02389	5.4529	75.325	0.1999	19.013	29.928	0.0496	9.3869	4.9595	0.0175	2.9178	45.462	0.1234	13.786
	Diseaseome	12.451	0.10202	3.4575	26.006	0.2248	5.3567	23.282	0.1552	4.8906	9.3332	0.0822	2.8587	31.709	0.2979	5.7013
	Bio-Mouse-Gene	14.654	0.19473	2.3919	41.371	0.4566	3.9018	17.199	0.1878	2.5372	9.277	0.0943	1.6036	31.649	0.4882	3.6376
Product co-purchasing networks	Bio-Dmela	10.579	0.01759	3.8219	143.71	0.1907	21.415	46.271	0.0426	9.2857	24.091	0.0162	5.0541	86.659	0.1724	18.048
	Bio-WormNet-v3	13.018	0.09249	3.7468	46.259	0.2761	6.8867	17.726	0.0851	3.9352	6.7764	0.0419	2.2424	40.795	0.3082	7.0826
Temporal Networks	amazon0601(In)	147.602	0.00695	10.928	1495.4	0.2708	70.281	286.61	0.0102	16.881	297.39	0.0382	22.199	308.32	0.0574	24.114
	amazon0505(In)	114.048	0.00499	8.7882	1572.9	0.2463	73.003	358.59	0.0125	19.123	260.85	0.0342	20.136	390.13	0.0628	26.178
	amazon0312(In)	92.525	0.00495	8.5742	1562.4	0.2425	71.875	338.03	0.0116	17.742	273.39	0.0352	20.381	383.82	0.0639	26.299
Communication Networks	email-Enron	38.764	0.02621	3.7877	213.91	0.2131	13.600	41.934	0.0634	5.3161	92.603	0.0861	7.9912	129.69	0.4636	15.172
	Wiki-Talk(In)	62.254	0.00345	1.0741	1877.5	0.2016	14.007	341.96	0.0286	4.4267	740.18	0.0685	7.2275	1324.4	0.6829	19.362
	askubuntu(In)	136.85	0.01045	6.8313	900.04	0.1808	33.837	243.42	0.0616	13.199	354.72	0.0570	16.613	609.40	0.3891	34.655
	askubuntu(In)	176.58	0.01707	9.3509	949.66	0.2091	39.419	212.91	0.0649	12.451	389.14	0.0719	20.113	555.44	0.3433	33.693
Ground-truth Networks	Wiki-Popsets	76.155	0.03531	5.2347	246.51	0.1779	14.886	121.47	0.0873	8.445	95.468	0.0689	7.664	230.41	0.5405	18.139
	Wiki-Talk(In)	671.76	0.00357	25.898	9669.4	0.3376	293.63	7978.6	0.1902	246.26	672.32	0.0036	25.905	16.5K	0.4879	542.31
	Rec-Libimseti(In)	66.434	0.09978	1.7923	77.081	0.2198	2.1486	87.472	0.0755	1.4021	28.059	0.0359	0.6971	166.18	0.8547	3.9402
Brain Networks	Human25890-session1	14.955	0.00201	0.1347	565.21	0.1377	2.6145	272.99	0.0464	1.5159	389.86	0.0629	2.2289	832.23	0.0767	5.8936
	Human25890-session2	41.69K	0.06401	3385.2	71.5K	0.1498	4575.6	101K	0.0762	4022.8	17.8K	0.0052	1025.5	193K	0.7216	10.1K
	com-LiveJournal	1741.3	0.02462	50.318	4102.9	0.1823	106.85	2629.9	0.0299	51.656	497.89	0.0104	18.559	5230.3	0.2889	139.74
	com-Orkut	207.43	0.01049	9.9761	2443.6	0.5408	80.712	452.92	0.0459	19.624	261.75	0.0479	16.197	228.83	0.0599	14.496
	com-YouTube	81.409	0.00175	1.0862	1380.5	0.1342	15.690	1422.2	0.1416	17.219	143.79	0.0045	2.2564	2515.4	0.6241	31.101

Fig 6. Plots of degree distributions of (a) ego-Twitter(In), (b) ego-Gplus(In), (c) Artist-Facebook, (d) cit-HepTh(In), (e) ca-CondMat and (f) Google(In) in log-log scale.

Fig 7. Plots of degree distributions of (a) Yeast-PPIN, (b) amazon0601(In), (c) sx-mathoverflow(In), (d) Email-Enron, (e) Wiki-Topcats and (f) Human25890-session1 in log-log scale.

However, the putative scale-free nature of real-world networks has generated a lot of interest in the past two decades. Therefore, most of the recent studies emphasized on the testing of the degree distribution of networks for power-law tails [18, 22, 38–40]. As the question “Are real-world networks scale-free” has important philosophical and conceptual consequences, we have considered this question methodologically. This paper explores a journey towards non-scale-freeness and proposes a new family of generalized Lomax models with nonlinear exponents in the shape parameter for efficient modeling of the heavy-tailed behavior of complex networks. The proposed GLM models provide a better fitting to the whole degree distributions of real-world complex networks than other well-known heavy-tailed distributions which could be thought of as a constructive answer to the overarching question raised in the literature: “Are complex networks scale-free? If not then what?” [18, 22, 40]. With this current paper, we hope to have contributed to this recent methodological progress.

The generalization of Lomax model provides us greater insight into probability distribution theory as well. Interestingly, several well-known probability distributions such as Lomax, Exponential, Rayleigh, Weibull, Gompertz and Benini and GLM Type-IV (MLM Distribution proposed in [16, 17]) arise as particular cases of GLM family of distributions whereas all other GLM type models are completely new. Another important insight from distribution theory point of view is that one may come up with several other choices of $g(z)$ and build new probability distributions from the GLM family of distributions. Theoretical analysis suggests that these newly introduced GLM models belong to the maximum domain of attraction of the Frechet distribution. It has also been observed that the proposed family of distributions (GLM) is right-tail equivalent to the Pareto distribution. We theoretically show that these models are heavy-tailed and long-tailed distributions. These models are highly efficient in modeling the whole degree distribution of real-world complex networks, unlike popularly used power-law and other probabilistic models. From the practitioner’s point of view, the proposed GLM distributions enable us to characterize the patterns of the degree distributions accurately, leading to the effective modeling of the large-scale networks.

The proposed GLM models satisfy several extreme value properties and desired inferential characteristics. The closeness between this heavy-tailed GLM family with life distributions is also discussed. Thus, the current paper may fulfill the search for a new probability distribution for modelling heavy-tailed real-world networks which are rarely scale-free as discussed in [11]. The proposed idea of the ‘shape parameter-based’ generalization for the Lomax distribution presented in this paper can also be applied to generalize other similar income and size distributions, namely Dagum, Burr, Beta-prime, and Log-logistic distributions. It is important to note from the experimental results and evaluations that four GLM Type models mostly outperforms all the well-known degree distribution models for a wide variety of real-world network data sets. Though the choice of $g(z)$ is still very subjective and based on the data set type; one may choose the desired Type from the GLM family of distributions. An immediate extension of this work is to verify these newly introduced GLM family of distributions for modeling other centrality measures (betweenness, closeness, etc.) of heavy-tailed real-world networks.

Data and Codes

For the sake of reproducibility, all codes and data sets are made publicly available at <https://github.com/tanujit123/nonscalefree>.

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