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Statistical-mechanical analysis of adaptive filter with clipping saturation-type nonlinearity

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Abstract—Adaptive signal processing is used in broad areas. In most practical adaptive systems, there exists substantial nonlinearity that cannot be neglected. In this paper, we analyze the behaviors of an adaptive system in which the output of the adaptive filter has the clipping saturation-type nonlinearity by a statistical-mechanical method. To represent the macroscopic state of the system, we introduce two macroscopic variables. By considering the limit in which the number of taps of the unknown system and adaptive filter is large, we derive the simultaneous differential equations that describe the system behaviors in the deterministic and closed form. Although the derived simultaneous differential equations cannot be analytically solved, we discuss the dynamical behaviors and steady state of the adaptive system by asymptotic analysis, steady-state analysis, and numerical calculation. As a result, it becomes clear that the saturation value S has the critical value S_C at which the mean-square stability of the adaptive system is lost. That is, when $S > S_C$, both the mean-square error (MSE) and mean-square deviation (MSD) converge, i.e., the adaptive system is mean-square stable. On the other hand, when $S < S_C$, the MSD diverges although the MSE converges, i.e., the adaptive system is not mean-square stable. In the latter case, the converged value of the MSE is a quadratic function of S and does not depend on the step size. Finally, S_C is exactly derived by asymptotic analysis.

Index Terms—adaptive filter, adaptive signal processing, system identification, LMS algorithm, clipping saturation-type nonlinearity, statistical-mechanical analysis, long-filter assumption

I. INTRODUCTION

ADAPTIVE signal processing is used in wide areas such as communication systems and acoustic systems [1], [2]. Active noise control (ANC) [3]–[6], active vibration control (AVC) [7], acoustic echo cancellation [8], and system identification [9] are examples of specific applications of adaptive signal processing. Adaptive signal processing using linear filters has been variously and theoretically analyzed [1], [2]. Statistical-mechanical method [10] is also used in analyses of active noise control [11], [12].

In most practical adaptive systems, for example, power amplifiers and transducers such as loudspeakers and microphones, there exist substantial nonlinearities that cannot be neglected [1], [2]. Such nonlinearities are inevitable and it is extremely important to investigate in detail their effects on the total performance of adaptive systems.

There are many studies on adaptive signal processing systems including nonlinear components [13–35]. In some of

these studies, nonlinearities where input signal and/or error signal are/is expressed by their signs (± 1) or three values ($-1, 0, +1$) have been investigated [13]–[23]. Bershad [24] analyzed the case in which the update by the least-mean-square (LMS) algorithm [36] has $(1 - e^{-ax})$ saturation-type nonlinearity on the basis of the small-step-size assumption. Costa *et al.* [25] analyzed the case in which the output of the adaptive filter has an error function (erf) saturation-type nonlinearity on the basis of the small-step-size assumption. Costa *et al.* [26], [27] analyzed the active noise control in which the secondary path has an erf saturation-type nonlinearity. Snyder and Tanaka [28] proposed to deal with the primary path nonlinearity in ANC/AVC by replacing the finite-duration impulse response (FIR) filter with a neural network. Costa [29] analyzed a hearing aid feedback canceller with an erf saturation-type nonlinearity. Costa *et al.* [30] analyzed the model in which the output of the adaptive filter has a dead-zone nonlinearity caused by a class B amplifier or a nonlinear actuator on the basis of the small-step-size assumption. Tobias and Seara [31] analyzed the behaviors of the modified LMS algorithm derived from the improved cost function in the case where the output of the adaptive filter has an erf saturation-type nonlinearity. Bershad [32] analyzed the case where the update by the LMS algorithm has an erf saturation-type nonlinearity and extended the analysis to the case of tracking a Markov channel in the context of system identification. As described above, there are many studies on an erf saturation-type nonlinearity of adaptive systems. On the other hand, Hamidi *et al.* [33] reported their analysis, computer simulation, and experimental results of the ANC model in which the output of the adaptive filter has the clipping saturation-type nonlinearity. They proposed to modify the cost function aiming to avoid using a nonlinear region to improve the adaptive algorithm. Stenger and Kellermann [34] proposed the use of clipping-type preprocessing in adaptive echo cancellation to cancel the effect of nonlinear echo paths. Motonaka *et al.* [35] analyzed the model in which both the unknown system and adaptive filter have the Volterra-type nonlinearity [37].

As described in this section, although there are many studies on adaptive signal processing with nonlinear components, there are only a few studies on clipping saturation-type nonlinearity, which seems to be the most direct expression of saturation phenomenon; in particular, there have been no analytical studies as far as we searched the literature. Therefore, in this paper, we analyze the behaviors of a system with an adaptive filter whose output has clipping saturation-type nonlinearity by the statistical-mechanical method. The main assumption in the analysis is that an unknown system and an adaptive filter

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have a long impulse response. The small-step-size condition is not assumed. Note that the long-impulse-response assumption or long-filter assumption used in our analysis is reasonable, especially considering actual acoustic systems.

The rest of this paper is organized as follows. In Sec. II, we define the model analyzed in this study. In Sec. III, we describe the statistical-mechanical analysis in detail. In Sec. IV, we demonstrate the validity of the obtained theory by comparing theoretical results with simulation results. In that section, we also clarify that there exists the critical value S_C for the saturation value S at which the properties of the adaptive system markedly change. In addition, some results obtained by the steady-state, asymptotic, and numerical analyses are shown. Furthermore, we obtain the exact expression of S_C by asymptotic analysis. In Sec. V we conclude our study in this paper.

II. MODEL

Figure 1 shows a block diagram of the adaptive system analyzed in this paper. The unknown system G is modeled by an N -tap FIR filter. Its coefficient vector is

$$\mathbf{g} = [g_1, g_2, \dots, g_N]^\top, \quad (1)$$

where $^\top$ denotes the transpose of a vector. Each coefficient g_i is assumed to be independently generated from a distribution with

$$\langle g_i \rangle = 0, \quad \langle g_i^2 \rangle = \sigma_g^2, \quad (2)$$

and is time-invariant. Here, $\langle \cdot \rangle$ denotes expectation. The adap-

Fig. 1: Block diagram of the adaptive system.

ative filter W is also an N -tap FIR filter. Its coefficient vector is

$$\mathbf{w}(n) = [w_1(n), w_2(n), \dots, w_N(n)]^\top, \quad (3)$$

where n denotes the time step. The initial value $w_i(0)$ of each coefficient is assumed to be zero. The input signal $u(n)$ is assumed to be independently drawn from a distribution with

$$\langle u(n) \rangle = 0, \quad \langle u(n)^2 \rangle = \sigma^2. \quad (4)$$

That is, the input signal is white. The tap input vector $\mathbf{u}(n)$ in time step n is

$$\mathbf{u}(n) = [u(n), u(n-1), \dots, u(n-N+1)]^\top. \quad (5)$$

Note that only the mean and variance of the distribution are defined in (4). No specific distributions, for example, the

Gaussian distribution, are assumed. The same statements are true for the coefficients of the unknown system G .

Since both G and W are FIR filters, their outputs are convolutions of their own coefficients and a sequence of input signals. That is, the outputs $d(n)$ of the unknown system G and $y(n)$ of the adaptive filter W are

$$d(n) = \mathbf{g}^\top \mathbf{u}(n) = \sum_{i=1}^N g_i u(n-i+1), \quad (6)$$

$$y(n) = \mathbf{w}(n)^\top \mathbf{u}(n) = \sum_{i=1}^N w_i(n) u(n-i+1), \quad (7)$$

respectively.

The nonlinearity of the adaptive filter W is modeled by the function f set behind W . The function f represents the clipping saturation-type nonlinearity and is expressed as

$$f(x) = \begin{cases} S & (x > S) \\ -S & (x < -S) \\ x & (\text{otherwise}), \end{cases} \quad (8)$$

where S is a nonnegative real number. Figure 2 shows the function f .

Fig. 2: Clipping-type saturation.

The error signal $e(n)$ is generated by adding an independent background noise $\xi(n)$ to the difference between $d(n)$ and $y(n)$. That is,

$$e(n) = d(n) - f(y(n)) + \xi(n). \quad (9)$$

Here, the mean and variance of $\xi(n)$ are zero and σ_ξ^2 , respectively.

The LMS algorithm [36] is used to update the adaptive filter. That is,

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu e(n) \mathbf{u}(n), \quad (10)$$

where μ is a positive real number and is called the step size.

III. ANALYSIS

In this section, we theoretically analyze the behaviors of the adaptive system with clipping saturation-type nonlinearity by the statistical-mechanical method. From (9), the MSE is expressed as

$$\langle e^2 \rangle = \langle (d - f(y) + \xi)^2 \rangle \quad (11)$$

$$= \langle d^2 \rangle + \langle f(y)^2 \rangle - 2 \langle df(y) \rangle + \sigma_\xi^2. \quad (12)$$

In this section, we omit the time step n unless otherwise stated to avoid a rather cumbersome notation. We assume $N \rightarrow \infty$ ¹ while keeping

$$\rho^2 \triangleq N\sigma^2 \quad (13)$$

constant in accordance with the statistical-mechanical method. Note that when we set $\rho^2 = 1$, it corresponds to the normalized LMS (NLMS) algorithm. Then, from the central limit theorem, both d and y are stochastic variables that obey the Gaussian distribution. Their means are zeros, and the variance-covariance matrix is

$$\Sigma = \rho^2 \begin{pmatrix} \sigma_g^2 & r \\ r & Q \end{pmatrix}. \quad (14)$$

Here, Q and r are macroscopic variables that are defined as

$$Q \triangleq \frac{1}{N} \mathbf{w}^\top \mathbf{w}, \quad (15)$$

$$r \triangleq \frac{1}{N} \mathbf{g}^\top \mathbf{w}, \quad (16)$$

respectively. The derivation of the means and variance-covariance matrix is given in detail in Appendix A.

We obtain three sample means in (12) as follows by carrying out the Gaussian integration for d and y :

$$\langle d^2 \rangle = \rho^2 \sigma_g^2, \quad (17)$$

$$\begin{aligned} \langle f(y)^2 \rangle &= S^2 + (\rho^2 Q - S^2) \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right) \\ &\quad - S \sqrt{\frac{2\rho^2 Q}{\pi}} \exp \left(-\frac{S^2}{2\rho^2 Q} \right), \end{aligned} \quad (18)$$

$$\langle df(y) \rangle = \rho^2 r \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right), \quad (19)$$

where $\operatorname{erf}(\cdot)$ is an error function defined as

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt. \quad (20)$$

Equation (17) is easily derived from (14). Equations (18) and (19) are derived in detail in Appendices B and C, respectively.

From (12), (17), (18), and (19), we obtain the MSE as

$$\begin{aligned} \langle e^2 \rangle &= \rho^2 \sigma_g^2 + S^2 + (\rho^2 Q - 2\rho^2 r - S^2) \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right) \\ &\quad - S \sqrt{\frac{2\rho^2 Q}{\pi}} \exp \left(-\frac{S^2}{2\rho^2 Q} \right) + \sigma_\xi^2. \end{aligned} \quad (21)$$

This formula shows that the MSE is a function of the macroscopic variables Q and r . Therefore, we derive differential equations that describe the dynamical behaviors of these variables in the following. Multiplying both sides of (10) on the left by $\mathbf{g}^\top$ and using (16), we obtain

$$Nr(n+1) = Nr(n) + \mu e(n)d(n). \quad (22)$$

We introduce time t defined by

$$t \triangleq n/N, \quad (23)$$

and use it to represent the adaptive process. Then, t becomes a continuous variable since the limit $N \rightarrow \infty$ is considered. This is a standard method in the statistical-mechanical analysis of on-line learning [38].

If the adaptive filter is updated Ndt times in an infinitely small time dt , we can obtain Ndt equations as

$$Nr(n+1) = Nr(n) + \mu e(n)d(n), \quad (24)$$

$$Nr(n+2) = Nr(n+1) + \mu e(n+1)d(n+1), \quad (25)$$

$$\vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} Nr(n+Ndt) &= Nr(n+Ndt-1) \\ &\quad + \mu e(n+Ndt-1)d(n+Ndt-1). \end{aligned} \quad (26)$$

Summing all these equations, we obtain

$$N(r+dr) = Nr + Ndt\mu \langle ed \rangle. \quad (27)$$

Here, from the law of large numbers, we have represented the effect of the probabilistic variables by their means since the updates are executed Ndt times, that is, many times, to change r by dr . This property is called *self-averaging* in statistical mechanics [10]. From (9) and (27), we obtain a differential equation that describes the dynamical behavior of r in a deterministic form as follows:

$$\frac{dr}{dt} = \mu \left(\langle d^2 \rangle - \langle df(y) \rangle \right). \quad (28)$$

Next, squaring both sides of (10) and proceeding in the same manner as for the derivation of the above differential equation for r , we can derive a differential equation for Q , which is given by

$$\begin{aligned} \frac{dQ}{dt} &= \mu^2 \left(\langle d^2 \rangle - 2 \langle df(y) \rangle + \langle f(y)^2 \rangle + \sigma_\xi^2 \right) \\ &\quad + 2\mu \left(\langle dy \rangle - \langle yf(y) \rangle \right). \end{aligned} \quad (29)$$

Equations (28) and (29) include five sample means. However, three of the five means were given in (17), (18), and (19). We obtain the remaining two means as follows by carrying out the Gaussian integration for d and y :

$$\langle dy \rangle = \rho^2 r, \quad (30)$$

$$\langle yf(y) \rangle = \rho^2 Q \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right). \quad (31)$$

Equation (30) is easily derived from (14). Equation (31) is derived in detail in Appendix D.

Substituting (17)–(19), (30), and (31) into (28) and (29), we obtain the concrete formula of the simultaneous differential equations as follows:

$$\frac{dr}{dt} = \mu \rho^2 \left(\sigma_g^2 - r \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right) \right), \quad (32)$$

$$\begin{aligned} \frac{dQ}{dt} &= \mu \rho^2 \left(\mu (\rho^2 Q - 2\rho^2 r - S^2) - 2Q \right) \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right) \\ &\quad - \mu^2 \rho^2 S \sqrt{\frac{2\rho^2 Q}{\pi}} \exp \left(-\frac{S^2}{2\rho^2 Q} \right) \\ &\quad + \mu \rho^2 (\mu (\rho^2 \sigma_g^2 + S^2 + \sigma_\xi^2) + 2r). \end{aligned} \quad (33)$$

¹This is called *the thermodynamic limit* in statistical mechanics.

We numerically solve the derived simultaneous differential equations, since they cannot be analytically solved. Substituting the obtained numerical solution into (21), we obtain the MSE learning curves.

From (15) and (16), we can also obtain the mean-square deviation (MSD) as a function of the macroscopic variables Q and r as follows:

$$\text{MSD} = \|\mathbf{g} - \mathbf{w}\|_2^2 \quad (34)$$

$$= \|\mathbf{g}\|_2^2 - 2\mathbf{g}^\top \mathbf{w} + \|\mathbf{w}\|_2^2 \quad (35)$$

$$= N(\sigma_g^2 - 2r + Q). \quad (36)$$

Equation (36) shows that the MSD is proportional to the tap length N in the model setting in this paper; therefore, we normalize the MSD by N and call it the normalized MSD.

IV. RESULTS AND DISCUSSION

A. Learning Curves

We first investigate the validity of the theory by comparing with simulation results with regard to the dynamical behaviors of the MSE and normalized MSD, that is, the learning curves. Figures 3 and 4 show the learning curves obtained using the theory derived in the previous section, along with the corresponding simulation results. In these figures, “ μ ” denotes the step size μ . In these figures, the curves represent theoretical results and the polygonal lines represent simulation results. In both the theoretical calculation and simulation, $\rho^2 = \sigma_g^2 = 1$. In the theoretical calculation, the results are obtained by substituting Q and r , which are respectively obtained by solving (32) and (33), into (21). Here, (32) and (33) are numerically solved by the Runge–Kutta method. In the computer simulations, the number of taps of the unknown system $\mathbf{G}$ and the adaptive filter $\mathbf{W}$ is $N = 200$. For all the computer simulations, ensemble means for 500 trials are plotted. Each coefficient g_i of the unknown system is independently generated from the Gaussian distribution with a mean of zero and a variance of $\sigma_g^2 = 1$.

All initial values of the coefficients $w_i(0)$ are set to zero in the simulation, and the initial condition $Q(0) = r(0) = 0$ is used in the theoretical calculation.

Figures 3 and 4 show that the theory derived in this paper predicts the simulation results well in terms of average values. From Figs. 3a and 3b, it seems that the MSE almost converges at $t = 50$ regardless of the step size μ for both $S = 1$ and 3. However, Fig. 4a shows that the normalized MSD continues to increase for $S = 1$.

Next, we show the MSE at $t = 10, 100$, and 1000 in Fig. 5 to investigate the relationship between the saturation value S and the MSE. In the computer simulations, the medians and standard deviations in 100 trials are plotted using error bars. Figures 5a–5c show that the MSE increases when S is in the range of 1.1–1.3, and this tendency becomes obvious with time.

B. Critical value S_C and steady-state analysis when $S > S_C$

To clarify the phenomenon described in the previous subsection, we investigate the steady-state values of the macroscopic variables Q, r , and MSE. If the steady-state values of Q

Fig. 3: MSE learning curves ($\rho^2 = \sigma_g^2 = 1, \sigma_\xi^2 = 0$)

Fig. 4: Normalized MSD learning curves ($\rho^2 = \sigma_g^2 = 1, \sigma_\xi^2 = 0$)

Fig. 5: MSE ($\rho^2 = \sigma_g^2 = 1, \sigma_\xi^2 = 0$)

and r exist, they can be obtained by numerically solving the simultaneous equations that are obtained by substituting zeros into the right-hand sides of the simultaneous differential equations (32) and (33). Figure 6 shows the results for Q . When the saturation value S is larger than $S_C = 1.25331 \dots$, the solution is found. However, when S is smaller than S_C , no solution is found. In addition, Q diverges in the limit when S approaches $S_C + 0$. Since Q is proportional to the ℓ_2 -norm of $\mathbf{w}$ as seen from (15), the divergence of Q means the divergence of the coefficient vector $\mathbf{w}$ of the adaptive filter $\mathbf{W}$. When $S > S_C$, we can obtain the steady-state MSE by substituting the steady-state values of Q and r into (21). Note that it will become clear that the exact expression of $S_C = 1.25331 \dots$ is $\sqrt{\pi}/2$ as described in Sec. IV-D.

C. Asymptotic analysis when $S < S_C$

As described in the previous subsection, Q diverges in the limit $t \rightarrow \infty$ when $S < S_C$. However, Fig. 5 shows that

Fig. 6: Steady-state Q ($\rho^2 = \sigma_g^2 = 1, \sigma_\xi^2 = 0$)

the MSE seems to converge even when $S < S_C$. Therefore, the asymptotic analysis of the behavior of the system when $S < S_C$ is shown in this subsection. From (15) and (16), we obtain

$$r = \sigma_g \sqrt{Q} \cos \theta. \quad (37)$$

Here, θ is the angle between vectors $\mathbf{g}$ and $\mathbf{w}$. As described in Sec. IV-B, when $S < S_C$, $Q \rightarrow \infty$ in the limit $t \rightarrow \infty$. Therefore, from (32), (33), and (37), we obtain

$$\cos \theta = 1. \quad (38)$$

Equation (38) is derived in detail in Appendix E, where the approximations

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \simeq \frac{2}{\sqrt{\pi}} x \quad (x \ll 1), \quad (39)$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \simeq 1 + x \quad (x \ll 1), \quad (40)$$

are used.

Equation (38) means that the directions of $\mathbf{g}$ and $\mathbf{w}$ coincide even when $S < S_C$, although $\mathbf{w}$ diverges as described in Sec. IV-B. Note that this property does not depend on μ, ρ^2, σ_g^2 , nor σ_ξ^2 .

Fig. 7: Steady-state $\cos \theta = r/(\sigma_g \sqrt{Q})$ ($\rho^2 = \sigma_g^2 = 1, \sigma_\xi^2 = 0$)

As described so far, Q diverges and $\cos \theta = 1$ in the limit $t \rightarrow \infty$ when $S < S_C$. Then, the MSE is

$$\langle e^2 \rangle = S^2 - 2\sigma_g \rho \sqrt{\frac{2}{\pi}} S + \sigma_g^2 \rho^2 + \sigma_\xi^2. \quad (41)$$

Equation (41) is derived in detail in Appendix F. Equation (41) shows that the MSE converges although w diverges when $S < S_C$. The converged value does not depend on the step size μ , and it is a quadratic function of S . The converged value takes a minimum value $\sigma_g^2 \rho^2 (1 - \frac{2}{\pi}) + \sigma_\xi^2$ when $S = \sigma_g \rho \sqrt{\frac{2}{\pi}}$. Figure 7 shows the steady-state values of $\cos \theta = r/(\sigma_g \sqrt{Q})$. In this figure, for $S > S_C$, values calculated using the steady-state values of Q and r that were given in Sec. IV-B are plotted. On the other hand, for $S < S_C$, the value obtained using (38) is plotted.

Figure 8 shows the steady-state values of the MSE. In this figure, for $S > S_C$, results of the steady-state analysis described in Sec. IV-B are plotted. The results show that the steady-state MSE depends on μ . On the other hand, for $S < S_C$, the result obtained using (41) for the asymptotic analysis described in Sec. IV-C is plotted.

Fig. 8: Steady-state MSE ($\rho^2 = \sigma_g^2 = 1$)

Figure 9 shows the steady-state normalized MSD. This figure shows that the normalized MSD diverges because Q diverges in the limit $S \rightarrow S_C + 0$. That is, the condition in which the adaptive system is mean-square stable [2] is $S > S_C$.

D. Exact derivation of the critical value S_C

As described so far, the properties of the adaptive system markedly change at $S = S_C$. That is, when $S > S_C$, the MSE and normalized MSD converge. That is, the adaptive system is mean-square stable. Converged values depend on the step-size μ . On the other hand, when $S < S_C$, the normalized MSD diverges and the MSE converges to the values that do

Fig. 9: Steady-state normalized MSD ($\rho^2 = \sigma_g^2 = 1$)

not depend on μ . In addition, the angle between the coefficient vector $\mathbf{g}$ of the unknown system and the coefficient vector $\mathbf{w}$ of the adaptive filter converges to zero.

In this subsection, we analytically obtain the critical value S_C . As described in Sec. IV-B,

$$\lim_{S \rightarrow S_C + 0} \lim_{t \rightarrow \infty} \frac{dr}{dt} = \lim_{S \rightarrow S_C + 0} \lim_{t \rightarrow \infty} \frac{dQ}{dt} = 0. \quad (42)$$

As described in Secs. IV-B and IV-C,

$$\lim_{S \rightarrow S_C + 0} \lim_{t \rightarrow \infty} Q = \infty, \quad (43)$$

$$\lim_{S \rightarrow S_C + 0} \lim_{t \rightarrow \infty} \cos \theta = 1. \quad (44)$$

Substituting (39), (40), and (42)–(44) into (32) and (33) and solving for S_C , we obtain

$$S_C = \sigma_g \rho \sqrt{\frac{\pi}{2}}. \quad (45)$$

Equation (45) shows that the critical value S_C is proportional to σ_g , which is the standard deviation of elements of the unknown system $\mathbf{G}$ and ρ , which corresponds to the standard deviation of the input signal $u(n)$. In addition, we see that the critical value $S_C = 1.25331 \dots$ numerically obtained in Sec. IV-B is, in fact, $\sqrt{\frac{\pi}{2}}$.

V. CONCLUSIONS

We have analyzed the behaviors of the adaptive system in which the output of the adaptive filter has the clipping saturation-type nonlinearity by the statistical-mechanical method. To represent the macroscopic state of the system,

we have introduced two macroscopic variables Q and r . By considering the limit in which the number N of taps of the unknown system and adaptive filter is large, we have derived the simultaneous differential equations that describe the system behaviors in the deterministic and closed form. Although the derived simultaneous differential equations cannot be analytically solved, we have discussed the dynamical behaviors and steady state of the adaptive system by asymptotic analysis, steady-state analysis, and numerical calculation. As a result, it has become clear that the saturation value S has the critical value S_C at which the mean-square stability of the adaptive system is lost. That is, when $S > S_C$, both the mean-square error (MSE) and mean-square deviation (MSD) converge, i.e., the adaptive system is mean-square stable. On the other hand, when $S < S_C$, the MSD diverges although the MSE converges, i.e., the adaptive system is not mean-square stable. In the latter case, the converged value of the MSE is a quadratic function of S and does not depend on the step size. Finally, S_C has been exactly derived by asymptotic analysis.

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APPENDIX A
DERIVATION OF MEANS AND VARIANCE-COVARIANCE MATRIX OF d AND y

From (2), (4), (6), (7), (13), (15), and (16), we obtain means, variances, and covariance of d and y as follows:

$$\langle d \rangle = \left\langle \sum_{i=1}^N g_i u(n-i+1) \right\rangle = \sum_{i=1}^N g_i \langle u(n-i+1) \rangle = 0, \quad (46)$$

$$\langle y \rangle = \left\langle \sum_{i=1}^N w_i u(n-i+1) \right\rangle = \sum_{i=1}^N w_i \langle u(n-i+1) \rangle = 0, \quad (47)$$

$$\begin{aligned} \langle d^2 \rangle &= \left\langle \left(\sum_{i=1}^N g_i u(n-i+1) \right)^2 \right\rangle = \left\langle \sum_{i=1}^N \sum_{j=1}^N g_i g_j u(n-i+1) u(n-j+1) \right\rangle \\ &= \sum_{i=1}^N g_i^2 \langle u(n-i+1)^2 \rangle = \sigma^2 \sum_{i=1}^N g_i^2 \xrightarrow{N \rightarrow \infty} \rho^2 \sigma_g^2, \end{aligned} \quad (48)$$

$$\begin{aligned} \langle y^2 \rangle &= \left\langle \left(\sum_{i=1}^N w_i u(n-i+1) \right)^2 \right\rangle = \left\langle \sum_{i=1}^N \sum_{j=1}^N w_i w_j u(n-i+1) u(n-j+1) \right\rangle \\ &= \sum_{i=1}^N w_i^2 \langle u(n-i+1)^2 \rangle = \sigma^2 \sum_{i=1}^N w_i^2 \xrightarrow{N \rightarrow \infty} \rho^2 Q, \end{aligned} \quad (49)$$

$$\begin{aligned} \langle dy \rangle &= \left\langle \left(\sum_{i=1}^N g_i u(n-i+1) \right) \left(\sum_{j=1}^N w_j u(n-j+1) \right) \right\rangle = \left\langle \sum_{i=1}^N \sum_{j=1}^N g_i w_j u(n-i+1) u(n-j+1) \right\rangle \\ &= \sum_{i=1}^N g_i w_i \langle u(n-i+1)^2 \rangle = \sigma^2 \sum_{i=1}^N g_i w_i \xrightarrow{N \rightarrow \infty} \rho^2 r. \end{aligned} \quad (50)$$

From (46)–(50), the covariance matrix of d and y is (14). Here, (47), (49), and (50) were derived on the basis of the assumption that the correlation between $w(n)$ and $u(n)$ is small [27], [39], [40]. This assumption is a standard assumption used to analyze many adaptive algorithms [1], [2].

APPENDIX B
DERIVATION OF (18)

$$\langle f(y)^2 \rangle = \int_{-\infty}^{\infty} dy f(y)^2 p(y) = \left(\int_{-\infty}^{-S} + \int_{-S}^S + \int_S^{\infty} \right) dy f(y)^2 \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \quad (51)$$

$$= 2 \left(\underbrace{\int_S^{\infty} dy S^2 \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right)}_{B1} + \underbrace{\int_0^S dy y^2 \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right)}_{B2} \right), \quad (52)$$

$$B1 = \int_S^{\infty} dy S^2 \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right) = \frac{S^2}{\sqrt{\pi}} \int_{\frac{S}{\sqrt{2\rho^2 Q}}}^{\infty} dy' \exp(-y'^2), \quad \text{where } y' = \frac{y}{\sqrt{2\rho^2 Q}} \quad (53)$$

$$= \frac{S^2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \int_0^{\frac{S}{\sqrt{2\rho^2 Q}}} dy \exp(-y^2) \right) \quad \left(\because \int_0^{\infty} dy \exp(-y^2) = \frac{\sqrt{\pi}}{2} \right) \quad (54)$$

$$= \frac{S^2}{2} \left(1 - \operatorname{erf}\left(\frac{S}{\sqrt{2\rho^2 Q}}\right) \right), \quad (55)$$

$$B2 = \int_0^S dy y^2 \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \quad (56)$$

$$= \left[-y \sqrt{\frac{\rho^2 Q}{2\pi}} \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \right]_0^S + \sqrt{\frac{\rho^2 Q}{2\pi}} \int_0^S dy \exp\left(-\frac{y^2}{2\rho^2 Q}\right), \quad \text{where we used integration by parts} \quad (57)$$

$$= -S \sqrt{\frac{\rho^2 Q}{2\pi}} \exp\left(-\frac{S^2}{2\rho^2 Q}\right) + \frac{\rho^2 Q}{2} \int_0^{\frac{S}{\sqrt{2\rho^2 Q}}} dy' \exp(-y'^2), \quad \text{where } y' = \frac{y}{\sqrt{2\rho^2 Q}} \quad (58)$$

$$= -S \sqrt{\frac{\rho^2 Q}{2\pi}} \exp\left(-\frac{S^2}{2\rho^2 Q}\right) + \frac{\rho^2 Q}{2} \operatorname{erf}\left(\frac{S}{\sqrt{2\rho^2 Q}}\right), \quad (59)$$

$$\therefore \langle f(y)^2 \rangle = 2(B1 + B2) = S^2 - S \sqrt{\frac{\rho^2 Q}{2\pi}} \exp\left(-\frac{S^2}{2\rho^2 Q}\right) + (\rho^2 Q - S^2) \operatorname{erf}\left(\frac{S}{\sqrt{2\rho^2 Q}}\right). \quad (60)$$

APPENDIX C DERIVATION OF (19)

$$\langle df(y) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ddd y df(y) p(d, y) \quad (61)$$

$$= \underbrace{\int_{-\infty}^{\infty} ddd \int_{-\infty}^{-S} dy (-S) p(d, y)}_{C1} + \underbrace{\int_{-\infty}^{\infty} ddd \int_{-S}^S dy y p(d, y)}_{C2} + \underbrace{\int_{-\infty}^{\infty} ddd \int_S^{\infty} dy S p(d, y)}_{C3}, \quad (62)$$

$$C2 = \int_{-\infty}^{\infty} ddd \int_{-S}^S dy y \frac{1}{2\pi \sqrt{\left| \rho^2 \begin{pmatrix} \sigma_g^2 & r \\ r & Q \end{pmatrix} \right|}} \exp\left(-\frac{\begin{pmatrix} d & y \end{pmatrix} \left(\rho^2 \begin{pmatrix} \sigma_g^2 & r \\ r & Q \end{pmatrix} \right)^{-1} \begin{pmatrix} d \\ y \end{pmatrix}}{2}\right) \quad (63)$$

$$= \int_{-\infty}^{\infty} ddd \int_{-S}^S dy y \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{Qd^2 - 2r dy + \sigma_g^2 y^2}{2\rho^2 (\sigma_g^2 Q - r^2)}\right) \quad (64)$$

$$= \int_{-\infty}^{\infty} ddd \int_{-S}^S dy y \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{Q\left(d - \frac{r}{Q}y\right)^2 + \left(\sigma_g^2 - \frac{r^2}{Q}\right)y^2}{2\rho^2 (\sigma_g^2 Q - r^2)}\right) \quad (65)$$

$$= \int_{-S}^S dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} ddd \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{\left(d - \frac{r}{Q}y\right)^2}{2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right)}\right) \quad (66)$$

$$= \int_{-S}^S dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} \sqrt{2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right)} dd' \left(\sqrt{2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right)} d' + \frac{r}{Q} y \right) \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp(-d'^2), \quad (67)$$

$$\text{where } d' = \frac{d - \frac{r}{Q}y}{\sqrt{2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right)}} \quad (68)$$

$$\begin{aligned} &= \int_{-S}^S dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} 2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right) ddd \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp(-d^2) \\ &\quad + \int_{-S}^S \frac{r}{Q} dy y^2 \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} \sqrt{2\rho^2 \left(\sigma_g^2 - \frac{r^2}{Q}\right)} dd \frac{1}{2\pi \rho^2 \sqrt{\sigma_g^2 Q - r^2}} \exp(-d^2) \end{aligned} \quad (69)$$

$$= \int_{-S}^S \frac{r}{Q} dy y^2 \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} dd \frac{1}{\pi\sqrt{2\rho^2 Q}} \exp(-d^2) \quad \left(\because \int_{-S}^S dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) = 0\right) \quad (70)$$

$$= \frac{2r}{Q\sqrt{2\pi\rho^2 Q}} \int_0^S dy y^2 \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \quad \left(\because \int_{-\infty}^{\infty} dd \exp(-d^2) = \sqrt{\pi}\right) \quad (71)$$

$$= \frac{2r}{Q\sqrt{2\pi\rho^2 Q}} \left(\left[-\rho^2 Q y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \right]_0^S - \int_0^S dy (-\rho^2 Q) \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \right), \quad (72)$$

where we used integration by parts

$$= \frac{2r}{Q\sqrt{2\pi\rho^2 Q}} \left(-\rho^2 S Q \exp\left(-\frac{S^2}{2\rho^2 Q}\right) + \rho^2 Q \int_0^{\frac{S}{\sqrt{2\rho^2 Q}}} \sqrt{2\rho^2 Q} dy' \exp(-y'^2) \right), \quad \text{where } y' = \frac{y}{\sqrt{2\rho^2 Q}} \quad (73)$$

$$= -r S \rho \sqrt{\frac{2}{\pi Q}} \exp\left(-\frac{S^2}{2\rho^2 Q}\right) + \rho^2 r \operatorname{erf}\left(\frac{S}{\sqrt{2\rho^2 Q}}\right), \quad (74)$$

$$C3 = \int_{-\infty}^{\infty} ddd \int_S^{\infty} dy S \frac{1}{2\pi\sqrt{\left|\rho^2 \begin{pmatrix} \sigma_g^2 & r \\ r & Q \end{pmatrix}\right|}} \exp\left(-\frac{\begin{pmatrix} d & y \end{pmatrix} \begin{pmatrix} \rho^2 \begin{pmatrix} \sigma_g^2 & r \\ r & Q \end{pmatrix} \end{pmatrix}^{-1} \begin{pmatrix} d \\ y \end{pmatrix}}{2}\right) \quad (75)$$

$$= \int_{-\infty}^{\infty} ddd \int_S^{\infty} dy S \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{Qd^2 - 2r dy + \sigma_g^2 y^2}{2\rho^2(\sigma_g^2 Q - r^2)}\right) \quad (76)$$

$$= \int_{-\infty}^{\infty} ddd \int_S^{\infty} dy S \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{Q\left(d - \frac{r}{Q}y\right)^2 + \left(\sigma_g^2 - \frac{r^2}{Q}\right)y^2}{2\rho^2(\sigma_g^2 Q - r^2)}\right) \quad (77)$$

$$= S \int_S^{\infty} dy \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} ddd \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp\left(-\frac{\left(d - \frac{r}{Q}y\right)^2}{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)}\right) \quad (78)$$

$$= S \int_S^{\infty} dy \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} \sqrt{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)} dd' \left(\sqrt{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)} d' + \frac{r}{Q}y \right) \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp(-d'^2), \quad (79)$$

$$\text{where } d' = \frac{d - \frac{r}{Q}y}{\sqrt{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)}} \quad (80)$$

$$= S \int_S^{\infty} dy \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} 2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right) ddd \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp(-d'^2) \\ + S \int_S^{\infty} \frac{r}{Q} dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \int_{-\infty}^{\infty} \sqrt{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)} dd \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \exp(-d'^2) \quad (81)$$

$$= \frac{Sr}{Q} \int_S^{\infty} dy y \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \sqrt{2\rho^2\left(\sigma_g^2 - \frac{r^2}{Q}\right)} \frac{1}{2\pi\rho^2\sqrt{\sigma_g^2 Q - r^2}} \sqrt{\pi} \quad (82)$$

$$\left(\because \int_{-\infty}^{\infty} ddd \exp(-d'^2) = 0, \quad \int_{-\infty}^{\infty} dd \exp(-d'^2) = \sqrt{\pi}\right) \quad (83)$$

$$= \frac{Sr}{Q} \frac{1}{\sqrt{2\pi\rho^2 Q}} \left[(-\rho^2 Q) \exp\left(-\frac{y^2}{2\rho^2 Q}\right) \right]_S^{\infty} = \frac{Sr\rho}{\sqrt{2\pi Q}} \exp\left(-\frac{S^2}{2\rho^2 Q}\right), \quad (84)$$

$$C1 = \int_{-\infty}^{\infty} ddd \int_{-S}^{-S} dy (-S) p(d, y) = C3, \quad \text{where we used integration by substitution } y' = -y, \quad d' = d, \quad (85)$$

$$\therefore \langle df(y) \rangle = C1 + C2 + C3 = \rho^2 r \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right). \quad (86)$$

APPENDIX D
DERIVATION OF (31)

$$\langle y(f(y)) \rangle = \int_{-\infty}^{\infty} dy y f(y) p(y) = \underbrace{\int_{-\infty}^{-S} dy y (-S) p(y)}_{D1} + \underbrace{\int_{-S}^S dy y^2 p(y)}_{D2} + \underbrace{\int_S^{\infty} dy y S p(y)}_{D3}, \quad (87)$$

$$D2 = -S \sqrt{\frac{2\rho^2 Q}{\pi}} \exp \left(-\frac{S^2}{2\rho^2 Q} \right) + \rho^2 Q \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right), \quad (\because (59)) \quad (88)$$

$$D1 = D3 = S \int_S^{\infty} dy y \frac{1}{\sqrt{2\pi\rho^2 Q}} \exp \left(-\frac{y^2}{2\rho^2 Q} \right) = \frac{S}{\sqrt{2\rho^2 Q}} \left[-\rho^2 Q \exp \left(-\frac{y^2}{2\rho^2 Q} \right) \right]_S^{\infty} = S \sqrt{\frac{\rho^2 Q}{2\pi}} \exp \left(-\frac{S^2}{2\rho^2 Q} \right). \quad (89)$$

$$\therefore \langle yf(y) \rangle = D1 + D2 + D3 = \rho^2 Q \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right). \quad (90)$$

APPENDIX E
DERIVATION OF (38)

When $S < S_C$, since $Q \rightarrow \infty$ in the limit $t \rightarrow \infty$, from (32), (37), and (39), we obtain

$$\frac{dr}{dt} = \sigma_g \cos \theta \frac{d\sqrt{Q}}{dt} = \mu \rho^2 \left(\sigma_g^2 - \sigma_g \cos \theta \sqrt{Q} \operatorname{erf} \left(\frac{S}{\sqrt{2\rho^2 Q}} \right) \right) \simeq \mu \rho^2 \left(\sigma_g^2 - \sigma_g \cos \theta \sqrt{Q} \frac{2}{\sqrt{\pi}} \frac{S}{\sqrt{2\rho^2 Q}} \right), \quad (91)$$

$$\therefore \frac{d\sqrt{Q}}{dt} \simeq \mu \rho^2 \left(\frac{\sigma_g}{\cos \theta} - \sqrt{\frac{2}{\pi\rho^2}} S \right). \quad (92)$$

On the other hand, from (33), (37), (39), and (40), we obtain

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d(\sqrt{Q})^2}{dt} = 2\sqrt{Q} \frac{d\sqrt{Q}}{dt} \\ &\simeq \mu \rho^2 \left(\mu \left(\rho^2 Q - 2\rho^2 \sigma_g \sqrt{Q} \cos \theta - S^2 \right) - 2Q \right) \frac{2}{\sqrt{\pi}} \frac{S}{\sqrt{2\rho^2 Q}} \\ &\quad - \mu^2 \rho^2 S \sqrt{\frac{2\rho^2 Q}{\pi}} \left(1 - \frac{S^2}{2\rho^2 Q} \right) \\ &\quad + \mu \rho^2 \left(\mu \left(\rho^2 \sigma_g^2 + S^2 + \sigma_\xi^2 \right) + 2\sigma_g \sqrt{Q} \cos \theta \right). \end{aligned} \quad (94)$$

The right-hand side of the formula that is obtained by dividing both sides of (94) by $2\sqrt{Q}$ should be equal to the right-hand side of (92). Therefore, solving the equation for $\cos \theta$, we obtain

$$\cos \theta = 1. \quad (95)$$

APPENDIX F
DERIVATION OF (41)

When $S < S_C$, $Q \rightarrow \infty$ and $\cos \theta = 1$ in the limit $t \rightarrow \infty$. Therefore, from (92), we obtain

$$\frac{d\sqrt{Q}}{dt} \simeq \mu \rho^2 \left(\sigma_g - \sqrt{\frac{2}{\pi\rho^2}} S \right), \quad (96)$$

$$\therefore \sqrt{Q} \simeq \mu \rho^2 \left(\sigma_g - \sqrt{\frac{2}{\pi\rho^2}} S \right) t + \text{Const.} \quad (97)$$

On the other hand, from (37), we obtain

$$r \simeq \sigma_g \sqrt{Q}. \quad (98)$$

By substituting (39), (40), (97), and (98) into (21) and arranging the expression, we obtain the MSE as

$$\langle e^2 \rangle = S^2 - 2\sigma_g \rho \sqrt{\frac{2}{\pi}} S + \sigma_g^2 \rho^2 + \sigma_\xi^2. \quad (99)$$

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