A Higher-Order New Approach Numerical Method for Singularly Perturbed Parabolic Reaction-Diffusion Problems

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Abstract

In this study, a higher-order new approach numerical method for solving singularly perturbed parabolic reaction-diffusion problems has presented. To discretize time variable, we used the Crank-Nicolson method on uniform mesh and space variable, we used hybrid numerical method comprising a cubic spline tension method in the inner regions and a central difference method in the outer region on Shishkin mesh. The proposed method is proved to be uniformly convergent irrespective of the perturbation parameter. Three numerical examples are computed to validate the theoretical findings.

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A Higher-Order New Approach Numerical Method for Singularly Perturbed Parabolic Reaction-Diffusion Problems

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In this study, a higher-order new approach numerical method for solving singularly perturbed parabolic reaction-diffusion problems has presented. To discretize time variable, we used the Crank-Nicolson method on uniform mesh and space variable, we used hybrid numerical method comprising a cubic spline tension method in the inner regions and a central difference method in the outer region on Shishkin mesh. The proposed method is proved to be uniformly convergent irrespective of the perturbation parameter. Three numerical examples are computed to validate the theoretical findings.

KEYWORDS
Hybrid numerical method, Shishkin mesh, Tension spline, twin boundary layers.

1 | INTRODUCTION

Singularly perturbed problems arise in the modeling of fluid dynamics, elasticity, quantum mechanics, reaction diffusion process, chemical-reactor theory, plasma dynamics, meteorology, diffraction theory, aerodynamics, modeling of semi-conductor, hydrodynamics and many other allied areas. It is well-known fact that the solution of singular perturbation problems exhibits a multi-scale character (non-uniform behaviour), that is, there are thin transition layer(s) where the solution varies rapidly or jumps abruptly in some parts of the

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domain, which is known as boundary layer (inner) region while away from the layer(s) the solution behaves regularly and varies slowly, which is commonly known as outer region. In solving these types of problems using classical numerical methods on a uniform mesh, large oscillations may arise and pollute the solutions when the perturbation parameter becomes small in entire domain of interest due to the boundary layer behaviour. There is a vast literature about non-classical numerical methods. In the context of finite difference methods, we can group these methods into two. The first is fitted mesh finite difference methods and the second is fitted operator finite difference methods. Both types of methods have been used in the literature to solve singularly perturbed problems. In this study, we consider the following singularly perturbed parabolic reaction-diffusion problem

$$\mathcal{L}_\varepsilon y(x,t) \equiv y_t(x,t) - \varepsilon y_{xx}(x,t) + a(x,t)y(x,t) = f(x,t), \qquad (x,t) \in Q = \Omega_x^N \times \Omega_t^M = (0,1)\times(0,T], \tag{1.1}$$

subject to the initial condition

$$y(x,0) = \phi_b(x), \qquad 0 \le x \le 1, \tag{1.2}$$

and boundary conditions

$$\begin{cases} y(0,t) = \phi_l(t), & 0 \le t \le T, \\ y(1,t) = \phi_r(t), & 0 \le t \le T. \end{cases} \tag{1.3}$$

where $\varepsilon(0 < \varepsilon \ll 1)$ is perturbation parameter. The coefficient $a(x,t)$, the source function $f(x,t)$ and the boundary functions $\phi_b(x)$, $\phi_l(t)$ and $\phi_r(t)$ are sufficiently smooth and bounded where $a(x,t)$ is assumed to satisfy the condition $a(x,t) \ge \alpha > 0$, $(x,t) \in \bar{Q}$. The solution y of (1.1)–(1.3) is expected to exhibit twin layers of width $O(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$. Under suitable continuity and compatibility conditions on the data, the initial-boundary value problem (1.1)–(1.3) has unique solution $y(x,t)$.

Singularly perturbed problems of type (1.1) with initial-Dirichlet boundary conditions have been studied extensively in the literature using different numerical methods, to mention a few of recent studies, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. There are various numerical method exist in literature for singularly perturbed parabolic partial differential equations. Inspired by the preceding works and as far as the authors knowledge is concerned, the proposed new hybrid finite difference method have not been implemented for the problem of the type (1.1)–(1.3) so far. Therefore, our aim in this work is to provide a higher-order ε-uniform numerical method for the initial-boundary value problem (1.1)–(1.3). In this approach, the time derivative is discretized by Crank-Nicolson method, and the space derivative is discretized by a new approach hybrid numerical method with high order, which is the combination of cubic spline in tension method in the inner regions and central difference method in the outer region. The convergence analysis of the discrete problem is established very well. Three numerical experiments are conducted in order to validate our theoretical results.

The remaining parts of this study are outlined as follows. Section 2 is devoted to some properties of continuous problem and its bounds of derivatives. In Section 3, we fully discretize the problem. The parameter-uniform convergence analysis of the fully discrete problem is discussed in Section 4. In Section 5, three numerical examples are solved using the present method. The paper ends with a brief conclusion in Section 6.

Through out this paper C denotes a generic positive constant independent of ε, the meshes (x_i, t_j) and the step sizes $h_i, \Delta t$. The norm $\|.\|$ denotes the supremum norm.

2 | PROPERTIES OF CONTINUOUS PROBLEM

In this section, we present some bounds for the analytical solution $y(x,t)$ of (1.1)–(1.3) and its derivatives. The problem admits the following continuous maximum principle which ensures the stability of the solution for the problem in (1.1)–(1.3).

Lemma 2.1 (Continuous maximum principle) Let ξ be a sufficiently smooth function which satisfies $\xi \geq 0$ on ∂Q. Then $\mathcal{L}_\varepsilon \xi > 0$ on Q implies that $\xi \geq 0$, $\forall (x,t) \in \bar{Q}$.

Lemma 2.2 Let y be the solution of equation (1.1)–(1.3) then we have

$$\|y\| \leq \alpha^{-1}\|f\| + \max\{|\phi_b(x)|, |\phi_l(t)|, |\phi_r(t)|\}.$$

Theorem 2.3 Assume that the coefficient of the parabolic partial differential equation, and the initial boundary condition in (1.1)–(1.3) are sufficiently smooth, and satisfies the necessary compatibility condition stated in Theorem 3 [1]. Then, the problem (1.1)–(1.3) has unique solution $y(x,t) \in C_\lambda^4(\bar{Q})$, where

$$C_\lambda^4(\bar{\Omega}) = \min\left\{ y : \frac{\partial^{i+j} y}{\partial x^i \partial t^j} \in C_\lambda^0(\bar{\Omega}) \right\}$$

for all non negative integers i, j with $0 \leq i + 2j \leq 4$ here $C_\lambda^0(\bar{Q})$ is the set of holders continuous functions. Furthermore, the derivative of the solution u satisfy for all non negative integers i, j such that $0 \leq i + 2j \leq 4$,

$$\left\| \frac{\partial^{i+j} y}{\partial x^i \partial t^j} \right\| \leq C\varepsilon^{-i/2}.$$

We shall decompose the solution y as $y = v + w$, where v, w are respectively the smooth and singular components. The smooth and singular components v, and w satisfy the following bounds.

Theorem 2.4 Let $y(x,t)$ be a solution of problem (1.1)–(1.3), and assume that the coefficient of the parabolic PDE, and the initial and boundary value conditions given in (1.1)–(1.3) are sufficiently smooth, and satisfy the necessary compatibility conditions. Then, for all non-negative integers i, j such that $0 \leq i + 2j \leq 4$ we have

$$\begin{aligned} &\left| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right| \leq C(1 + \varepsilon^{1-i/2}), \quad \forall (x,t) \in \Omega, \\ &\left| \frac{\partial^{i+j} w_l}{\partial x^i \partial t^j} \right| \leq C\varepsilon^{-i/2} e^{-x/\sqrt{\varepsilon}}, \text{ and } \quad \left| \frac{\partial^{i+j} w_r}{\partial x^i \partial t^j} \right| \leq C\varepsilon^{-i/2} e^{-(1-x)/\sqrt{\varepsilon}}, \end{aligned}$$

where C is a constant independent of ε.

3 | DERIVATION OF NUMERICAL SCHEME

In this section, we first discretize time derivative by the Crank-Nicolson method and then we introduce a layer-adapted mesh of Shishkin type to discretize the space derivative using the hybrid numerical method comprising cubic spline difference method in the inner regions and central difference method in the outer region.

3.1 | Time semi-discretization

Let the time domain $[0, T]$ be divided into M equal parts with uniform time step Δt such that

$$\Omega_t^M = \{t_j : t_j = j\Delta t, j = 0, 1, ..., M, \Delta t = T/M\},$$

where M denotes the number of mesh elements in the time direction. Uniform meshes with step size $\Delta t, \Omega_t^M$ with M mesh elements are used on the interval $[0, T]$. We utilize the Crank-Nicolson method to discretize the time derivative as follows and we obtain the system of linear ordinary differential equations

$$\begin{cases} Y(x,0) = \phi_b(x), & x \in \Omega, \\ \frac{Y^{j+1}-Y^j}{\Delta t} - \varepsilon\left(\frac{Y_{xx}^{j+1}+Y_{xx}^j}{2}\right) + \frac{(aY)^{j+1}+(aY)^j}{2} = \frac{f^{j+1}+f^j}{2}, & (x, t_{j+1}) \in (0,1)\times[0,T], \end{cases} \tag{3.1}$$

subject to the boundary conditions

$$\begin{cases} Y^{j+1}(0) = \phi_l(t_{j+1}), & 0 \le j \le M, \\ Y^{j+1}(1) = \phi_r(t_{j+1}), & 0 \le j \le M. \end{cases} \tag{3.2}$$

where $Y^{j+1} = Y(x, t_{j+1})$ and Y^{j+1} is the numerical solution at the $(j+1)$th time level. For each time step, equation (3.1)-(3.2) can be rewritten as

$$\frac{-\varepsilon Y_{xx}^{j+1}}{2} + \left(\frac{1}{\Delta t} + \frac{a^{j+1}}{2}\right)Y^{j+1} = \frac{\varepsilon Y_{xx}^j}{2} + \left(\frac{1}{\Delta t} - \frac{a^j}{2}\right)Y^j + \frac{f^{j+1}+f^j}{2}, \quad (x, t_{j+1}) \in (0,1)\times[0,T], \tag{3.3}$$

subject to the initial and boundary conditions, respectively

$$\begin{cases} Y(x,0) = \phi_b(x), & x \in \Omega, \\ \begin{cases} Y^{j+1}(0) = \phi_l(t_{j+1}), & 0 \le j \le M, \\ Y^{j+1}(1) = \phi_r(t_{j+1}), & 0 \le j \le M. \end{cases} \end{cases} \tag{3.4}$$

The local truncation error of the Crank-Nicolson method for the time semi-discretization is given by $e_{j+1} = y(x, t_{j+1}) - Y^{j+1}(x)$. This error measures the contribution of each time step to the global error of the time semi-discretization.

Lemma 3.1 If

$$\left|\frac{\partial^i y(x,t)}{\partial x^i}\right| \le C, \qquad (x,t) \in \bar{Q}, \quad 0 \le i \le 2$$

then the local error bound in the time direction is given by

$$\|e_{j+1}\|_\infty \le C(\Delta t)^3.$$

Proof We have

$$y(x,t_{j+1}) = y(x,t_{j+1/2}) + \frac{\Delta t}{2} y_t(x,t_{j+1/2}) + O(\Delta t^2), \tag{3.5a}$$

$$y(x,t_j) = y(x,t_{j+1/2}) + \frac{\Delta t}{2} y_t(x,t_{j+1/2}) + O(\Delta t^2), \tag{3.5b}$$

From (3.5), we have

$$\frac{y(x,t_{j+1}) - y(x,t_j)}{\Delta t} = y_t(x,t_j + \frac{\Delta t}{2}) + O(\Delta t^2) \tag{3.6}$$

Using (1.1) in (3.6), we have

$$\frac{y(x,t_{j+1}) - y(x,t_j)}{\Delta t} = \varepsilon y_{xx}(x,t_j + \frac{\Delta t}{2}) - a(x,t_j + \frac{\Delta t}{2}) y(x,t_j + \frac{\Delta t}{2}) + f(x,t_j + \frac{\Delta t}{2}) + O(\Delta t^2),$$

where

$$a(x,t_j + \frac{\Delta t}{2}) = \frac{a(x,t_{j+1}) + a(x,t_j)}{2} + O(\Delta t^2), \tag{3.7a}$$

$$b(x,t_j + \frac{\Delta t}{2}) = \frac{b(x,t_{j+1}) + b(x,t_j)}{2} + O(\Delta t^2), \tag{3.7b}$$

$$f(x,t_j + \frac{\Delta t}{2}) = \frac{f(x,t_{j+1}) + f(x,t_j)}{2} + O(\Delta t^2), \tag{3.7c}$$

$$y(x,t_j + \frac{\Delta t}{2}) = \frac{y(x,t_j+1) + y(x,t_j)}{2} + O(\Delta t^2). \tag{3.7d}$$

It can be seen that

$$\|e_{j+1}\| \le C_1(\Delta t^3). \tag{3.8}$$

The global error is the measure of the contribution of the local error estimate at each time step and is given by $e_j = y(x,t_j) - Y(x,t_j)$.

Lemma 3.2 Under the hypothesis of Lemma (3.1), the global error estimate at t_j is given by

$$\|E_j\|_\infty \le C\Delta t^2, \quad j \le T/\Delta t.$$

Proof Using the local error estimate up to the j^{th} time step, the global error estimate is given by

$$\begin{aligned}
\|E_j\|_\infty &= \|\sum_{p=1}^{j} e_p\|, \qquad \text{since } j \le T/\Delta t,\\
&\le \|e_1\| + \|e_2\| + \dots + \|e_j\|,\\
&\le C_1 j \Delta t^3, \qquad \text{using (3.8)}\\
&\le C_1 (j\Delta t)\Delta t^2,\\
&\le C_1 T \Delta t^2, \qquad \text{since } j\Delta t \le T,\\
&\le C\Delta t^2, \quad C = C_1 T,
\end{aligned}$$

where C is a positive constant and independent of Δt and ε.

3.2 | The Space Domian Discretization

Shishkin mesh is a piecewise uniform mesh such that the space domain $[0,1]$ is divided into three sub-intervals $\Omega_l = [0,\sigma), \Omega_c = [\sigma, 1-\sigma]$ and $\Omega_r = (1-\sigma, 1]$. We define a mesh transition parameter σ as

$$\sigma = \min\left\{\frac{1}{4}, \sigma_0\sqrt{\varepsilon}\ln N\right\},$$

where $\sigma_0 \geq 2/\sqrt{\alpha}$ is a positive constant. The mesh is equidistant on $[\sigma, 1-\sigma]$ with $N/2$ mesh elements and gradually divided into $N/4$ mesh elements on the intervals $[0,\sigma)$ and $(1-\sigma,1]$. Let $\bar{\Omega}_x^N = \{x_i\}_{i=0}^N$ be the set of mesh points. Now, a piecewise uniform mesh points is defined as

$$x_i = \begin{cases} ih, & 0 \leq i \leq N/4, \\ \sigma + (1 - N/4)H, & N/4 \leq i \leq 3N/4, \\ (1-\sigma) + (i - 3N/4)h, & 3N/4 \leq i \leq N. \end{cases}$$

where the mesh size in $[\sigma, 1-\sigma]$ is given by $H = 2(1-2\sigma)/N$ and in $[0,\sigma)$ and $(1-\sigma, 1]$, it is denoted by $h = 4\sigma/N$. We denote the local mesh sizes by $h_i = x_{i+1} - x_i$, for $i = 1, \cdots, N$ and the mesh diameter with $\hat{h}_i = (h_{i-1} + h_i)/2$. Next subsection gives the details of fully discrete problem.

3.3 | Fully Discretized Numerical Method

Discretizing problem (3.3) in the outer region $[\sigma, 1-\sigma]$ using the central difference method and obtain

$$\begin{aligned} -\frac{\varepsilon}{2\hat{h}_i}\left(\frac{Y_{i+1}^{j+1} - Y_i^{j+1}}{h_i} - \frac{Y_i^{j+1} - Y_{i-1}^{j+1}}{h_{i-1}}\right) + \left(\frac{1}{\Delta t} + \frac{a_i^{j+1}}{2}\right)Y_i^{j+1} = & \frac{\varepsilon}{2\hat{h}_i}\left(\frac{Y_{i+1}^{j} - Y_i^{j}}{h_i} - \frac{Y_i^{j} - Y_{i-1}^{j}}{h_{i-1}}\right) \\ & + \left(\frac{1}{\Delta t} - \frac{a_i^{j}}{2}\right)Y_i^j + \frac{f_i^{j+1} + f_i^j}{2}. \end{aligned} \tag{3.9}$$

After rearranging the terms in (3.9) for $i = \frac{N}{4}, \cdots, \frac{3N}{4}, j = 0, \cdots, M-1$, we obtain the following scheme in the outer region

$$r_i^- Y_{i-1}^{j+1} + r_i^c Y_i^{j+1} + r_i^+ Y_{i+1}^{j+1} = s_i^- Y_{i-1}^j + s_i^c Y_i^j + s_i^+ Y_{i+1}^j + \varsigma, \tag{3.10}$$

where the coefficients are given by

$$\begin{aligned} r_i^- &= \frac{-\varepsilon}{h_{i-1}(h_i + h_{i-1})}, & r_i^c &= \frac{\varepsilon}{h_i h_{i-1}} + \frac{a_i^{j+1}}{2} + \frac{1}{\Delta t}, & r_i^+ &= \frac{-\varepsilon}{h_i(h_i + h_{i-1})}, \\ s_i^- &= \frac{\varepsilon}{h_{i-1}(h_i + h_{i-1})}, & s_i^c &= \frac{-\varepsilon}{h_i h_{i-1}} + \frac{1}{\Delta t} - \frac{a_i^j}{2}, & s_i^+ &= \frac{\varepsilon}{h_i(h_i + h_{i-1})}, \\ \varsigma &= \frac{f_i^{j+1} + f_i^j}{2}. \end{aligned}$$

We discretized problem (3.3) in the inner regions $[0,\sigma)$ and $(1-\sigma,1]$ using the cubic spline in tension method. We have

$$\frac{-\varepsilon M_k^{j+1}}{2}+\left(\frac{1}{\Delta t}+\frac{a_k^{j+1}}{2}\right)Y_k^{j+1}=\frac{\varepsilon M_k^{j}}{2}+\left(\frac{1}{\Delta t}-\frac{a_k^{j}}{2}\right)Y_k^{j}+\frac{f_k^{j+1}+f_k^{j}}{2}. \tag{3.11}$$

where $M_i^{j+1}=(Y_{xx})_i^{j+1}$ for $k=i,i\pm 1$. Equation (3.11) can be written as

$$\frac{-\varepsilon M_i^{j+1}}{2}=\frac{\varepsilon M_i^{j}}{2}+\left(\frac{1}{\Delta t}-\frac{a_i^{j}}{2}\right)Y_i^{j}-\left(\frac{1}{\Delta t}+\frac{a_i^{j+1}}{2}\right)Y_i^{j+1}+\frac{f_i^{j+1}+f_i^{j}}{2}. \tag{3.12}$$

From (3.12), we have

$$\begin{cases}\frac{-\varepsilon M_{i-1}^{j+1}}{2}=\frac{\varepsilon M_{i-1}^{j}}{2}+\left(\frac{1}{\Delta t}-\frac{a_{i-1}^{j}}{2}\right)Y_{i-1}^{j}-\left(\frac{1}{\Delta t}+\frac{a_{i-1}^{j+1}}{2}\right)Y_{i-1}^{j+1}+\frac{f_{i-1}^{j+1}+f_{i-1}^{j}}{2}.\\ \frac{-\varepsilon M_{i+1}^{j+1}}{2}=\frac{\varepsilon M_{i+1}^{j}}{2}+\left(\frac{1}{\Delta t}-\frac{a_{i+1}^{j}}{2}\right)Y_{i+1}^{j}-\left(\frac{1}{\Delta t}+\frac{a_{i+1}^{j+1}}{2}\right)Y_{i+1}^{j+1}+\frac{f_{i+1}^{j+1}+f_{i+1}^{j}}{2}.\end{cases} \tag{3.13}$$

From cubic spline in tension method, we have

$$\lambda_1 h_{i-1}M_{i-1}^{j+1}+\lambda_2(h_{i-1}+h_i)M_i^{j+1}+\lambda_1 h_i M_{i+1}^{j+1}=\frac{Y_{i+1}^{j+1}-Y_i^{j+1}}{h_i}-\frac{Y_i^{j+1}-Y_{i-1}^{j+1}}{h_{i-1}}. \tag{3.14}$$

Substituting (3.12) and (3.13) into (3.14) and after rearranging the terms for $i=1,\cdots,\frac{N}{4}-1$ and $\frac{3N}{4}+1,\cdots,N-1, j=0,\cdots,M-1$, we obtain the following difference scheme in the inner regions

$$r_i^- Y_{i-1}^{j+1}+r_i^c Y_i^{j+1}+r_i^+ Y_{i+1}^{j+1}=s_i^- Y_{i-1}^{j}+s_i^c Y_i^{j}+s_i^+ Y_{i+1}^{j}+\varsigma, \tag{3.15}$$

where the coefficients are given by

$$\begin{aligned}
r_i^- &=\frac{-\varepsilon}{2h_{i-1}(h_i+h_{i-1})}+\frac{\lambda_1 h_{i-1}}{(h_{i-1}+h_i)}\left(\frac{a_{i-1}^{j+1}}{2}+\frac{1}{\Delta t}\right), \qquad r_i^c=\frac{\varepsilon}{2h_i h_{i-1}}+\lambda_2\left(\frac{a_i^{j+1}}{2}+\frac{1}{\Delta t}\right),\\
r_i^+ &=\frac{-\varepsilon}{2h_i(h_i+h_{i-1})}+\frac{\lambda_1 h_i}{(h_{i-1}+h_i)}\left(\frac{a_{i+1}^{j+1}}{2}+\frac{1}{\Delta t}\right),\\
s_i^- &=\frac{\varepsilon}{2h_{i-1}(h_i+h_{i-1})}+\frac{\lambda_1 h_{i-1}}{(h_{i-1}+h_i)}\left(\frac{1}{\Delta t}-\frac{a_{i-1}^{j}}{2}\right), \qquad s_i^c=\frac{-\varepsilon}{2h_i h_{i-1}}+\lambda_2\left(\frac{1}{\Delta t}-\frac{a_i^{j}}{2}\right),\\
s_i^+ &=\frac{\varepsilon}{2h_i(h_i+h_{i-1})}+\frac{\lambda_1 h_i}{(h_{i-1}+h_i)}\left(\frac{1}{\Delta t}-\frac{a_{i+1}^{j}}{2}\right),\\
\varsigma &=\frac{\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(f_{i-1}^{j+1}+f_{i-1}^{j})+\frac{\lambda_2}{2}(f_i^{j+1}+f_i^{j})+\frac{\lambda_1 h_i}{2(h_{i-1}+h_i)}(f_{i+1}^{j+1}+f_{i+1}^{j}).
\end{aligned}$$

Combining the difference schemes (3.10) and (3.15), we obtain the following hybrid difference method

$$L_\varepsilon^{N,M}Y_i^{j+1}\equiv r_i^- Y_{i-1}^{j+1}-r_i^c Y_i^{j+1}+r_i^+ Y_{i+1}^{j+1}=s_i^- Y_{i-1}^{j}+s_i^c Y_i^{j}+s_i^+ Y_{i+1}^{j}+\varsigma, \tag{3.16}$$

for $i = 1, \cdots, N-1,\ j = 0, \cdots, M-1$ with the discrete initial and boundary conditions

$$\begin{cases} Y_i^0 = \phi_b(x_i), & 1 \le i \le N-1, \\ Y_0^{j+1} = \phi_l(t_{j+1}),\ \ Y_N^{j+1} = \phi_r(t_{j+1}), & 0 \le j \le M-1, \end{cases} \tag{3.17}$$

where the coefficients for $i = 1, \cdots, \frac{N}{4}-1$ and $\frac{3N}{4}+1, \cdots, N-1, j = 0, \cdots, M-1$ are given by

$$\begin{aligned}
r_i^- &= \frac{-\varepsilon}{2h_{i-1}(h_i+h_{i-1})} + \frac{\lambda_1 h_{i-1}}{(h_{i-1}+h_i)}\Big(\frac{a_{i-1}^{j+1}}{2} + \frac{1}{\Delta t}\Big), \quad r_i^c = \frac{\varepsilon}{2h_i h_{i-1}} + \lambda_2\Big(\frac{a_i^{j+1}}{2} + \frac{1}{\Delta t}\Big),\\
r_i^+ &= \frac{-\varepsilon}{2h_i(h_i+h_{i-1})} + \frac{\lambda_1 h_i}{(h_{i-1}+h_i)}\Big(\frac{a_{i+1}^{j+1}}{2} + \frac{1}{\Delta t}\Big),\\
s_i^- &= \frac{\varepsilon}{2h_{i-1}(h_i+h_{i-1})} + \frac{\lambda_1 h_{i-1}}{(h_{i-1}+h_i)}\Big(\frac{1}{\Delta t} - \frac{a_{i-1}^{j}}{2}\Big), \quad s_i^c = \frac{-\varepsilon}{2h_i h_{i-1}} + \lambda_2\Big(\frac{1}{\Delta t} - \frac{a_i^{j}}{2}\Big),\\
s_i^+ &= \frac{\varepsilon}{2h_i(h_i+h_{i-1})} + \frac{\lambda_1 h_i}{(h_{i-1}+h_i)}\Big(\frac{1}{\Delta t} - \frac{a_{i+1}^{j}}{2}\Big),\\
\varsigma &= \frac{\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(f_{i-1}^{j+1} + f_{i-1}^{j}) + \frac{\lambda_2}{2}(f_i^{j+1} + f_i^{j}) + \frac{\lambda_1 h_i}{2(h_{i-1}+h_i)}(f_{i+1}^{j+1} + f_{i+1}^{j}),
\end{aligned}$$

and the coefficients for $i = \frac{N}{4}, \cdots, \frac{3N}{4}, j = 0, \cdots, M-1$ are given by

$$\begin{aligned}
r_i^- &= \frac{-\varepsilon}{h_{i-1}(h_i+h_{i-1})}, \quad r_i^c = \frac{\varepsilon}{h_i h_{i-1}} + \frac{a_i^{j+1}}{2} + \frac{1}{\Delta t}, \quad r_i^+ = \frac{-\varepsilon}{h_i(h_i+h_{i-1})},\\
s_i^- &= \frac{\varepsilon}{h_{i-1}(h_i+h_{i-1})}, \quad s_i^c = \frac{-\varepsilon}{h_i h_{i-1}} + \frac{1}{\Delta t} - \frac{a_i^{j}}{2}, \quad s_i^+ = \frac{\varepsilon}{h_i(h_i+h_{i-1})}, \quad \varsigma = \frac{f_i^{j+1} + f_i^{j}}{2}.
\end{aligned}$$

Thomas algorithm is employed to solve (3.16) and (3.17), which is more efficient and reduce the calculation time over the usual matrix inverse method [16].

Remark 3.1 [17] The proposed method gives a second-order convergent solution for arbitrary λ_1 and λ_2 with $\lambda_1 + \lambda_2 = \frac{1}{2}$ and a fourth-order convergent solution for $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$.

4 | CONVERGENCE ANALYSIS

In this section, we derive the truncation at the inner regions, outer region and the transition points. Now, the truncation error in the inner regions, that is, on the subintervals $0 < i < N/4$ and $3N/4 < i < N$ is given by

$$T_{i,Y} = \Big(r_i^- Y_{i-1}^{j+1} + r_i^c Y_i^{j+1} + r_i^+ Y_{i+1}^{j+1}\Big) - \Big(s_i^- Y_{i-1}^{j} + s_i^c Y_i^{j} + s_i^+ Y_{i+1}^{j} + \varsigma\Big) \tag{4.1}$$

where $\varsigma = \frac{\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(f_{i-1}^{j+1} + f_{i-1}^{j}) + \frac{\lambda_2}{2}(f_i^{j+1} + f_i^{j}) + \frac{\lambda_1 h_i}{2(h_{i-1}+h_i)}(f_{i+1}^{j+1} + f_{i+1}^{j})$. From (3.3), we have

$$\begin{aligned}
&\frac{\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(f_{i-1}^{j+1} + f_{i-1}^{j}) + \frac{\lambda_2}{2}(f_i^{j+1} + f_i^{j}) + \frac{\lambda_1 h_i}{2(h_{i-1}+h_i)}(f_{i+1}^{j+1} + f_{i+1}^{j}) \\
&= -\frac{\varepsilon\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(Y_{xx})_{i-1}^{j+1} + \frac{\lambda_1 a_{i-1}^{j+1} h_{i-1}}{2(h_{i-1}+h_i)} Y_{i-1}^{j+1} - \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(Y_{xx})_{i-1}^{j} \\
&+ \frac{\lambda_1 a_{i-1}^{j} h_{i-1}}{2(h_{i-1}+h_i)} Y_{i-1}^{j} - \frac{\varepsilon\lambda_2}{2}(Y_{xx})_i^{j+1} + \frac{\lambda_2}{2} a_i^{j+1} Y_i^{j+1} - \frac{\varepsilon\lambda_2}{2}(Y_{xx})_i^{j} + \frac{\lambda_2}{2} a_i^{j} Y_i^{j} \\
&- \frac{\varepsilon\lambda_1 h_i}{2(h_{i-1}+h_i)}(Y_{xx})_{i+1}^{j+1} + \frac{\lambda_1 a_{i+1}^{j+1} h_i}{2(h_{i-1}+h_i)} Y_{i+1}^{j+1} - \frac{\varepsilon\lambda_1 h_i}{2(h_{i-1}+h_i)}(Y_{xx})_{i+1}^{j} + \frac{\lambda_1 a_{i+1}^{j} h_i}{2(h_{i-1}+h_i)} Y_{i+1}^{j}.
\end{aligned} \tag{4.2}$$

Using (4.2) in (4.1), we obtain

$$\begin{aligned}
T_{i,Y} &= \left(r_i^- Y_{i-1}^{j+1} + r_i^c Y_i^{j+1} + r_i^+ Y_{i+1}^{j+1}\right) - \left(s_i^- Y_{i-1}^{j} + s_i^c Y_i^{j} + s_i^+ Y_{i+1}^{j}\right) \\
&+ \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(Y_{xx})_{i-1}^{j+1} - \frac{\lambda_1 a_{i-1}^{j+1} h_{i-1}}{2(h_{i-1}+h_i)} Y_{i-1}^{j+1} + \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_{i-1}+h_i)}(Y_{xx})_{i-1}^{j} \\
&- \frac{\lambda_1 a_{i-1}^{j} h_{i-1}}{2(h_{i-1}+h_i)} Y_{i-1}^{j} + \frac{\varepsilon\lambda_2}{2}(Y_{xx})_i^{j+1} - \frac{\lambda_2}{2} a_i^{j+1} Y_i^{j+1} + \frac{\varepsilon\lambda_2}{2}(Y_{xx})_i^{j} - \frac{\lambda_2}{2} a_i^{j} Y_i^{j} \\
&+ \frac{\varepsilon\lambda_1 h_i}{2(h_{i-1}+h_i)}(Y_{xx})_{i+1}^{j+1} - \frac{\lambda_1 a_{i+1}^{j+1} h_i}{2(h_{i-1}+h_i)} Y_{i+1}^{j+1} + \frac{\varepsilon\lambda_1 h_i}{2(h_{i-1}+h_i)}(Y_{xx})_{i+1}^{j} - \frac{\lambda_1 a_{i+1}^{j} h_i}{2(h_{i-1}+h_i)} Y_{i+1}^{j}.
\end{aligned} \tag{4.3}$$

Using the Taylor series expansion for the terms $Y_{i-1}^{j+1}, Y_{i+1}^{j+1}$ up-to $O(h^7)$ in the space direction in (4.3), we obtain the following truncation error for any time level

$$T_{i,Y} = T_{0,i} Y_i(t) + T_{1,i}(Y_x)_i(t) + T_{2,i}(Y_{xx})_i(t) + T_{3,i}(Y_{xxx})_i(t) + T_{4,i}(Y_{xxxx})_i(t) + T_{5,i}(Y_{xxxxx})_i(t) + T_{6,i}(Y_{xxxxxx})_i(t), \tag{4.4}$$

where the coefficients are give by

$$\begin{aligned}
T_{0,i} &= r_i^- + r_i^c + r_i^+ - \frac{\lambda_1 a_{i-1}(t) h_{i-1}}{2(h_{i-1}+h_i)} - \frac{\lambda_2 a_i(t)}{2} - \frac{\lambda_1 a_{i+1}(t) h_i}{2(h_{i-1}+h_i)} \\
&- \left(s_i^- + s_i^c + s_i^+ + \frac{\lambda_1 a_{i-1}(t) h_{i-1}}{2(h_{i-1}+h_i)} + \frac{\lambda_2 a_i(t)}{2} + \frac{\lambda_1 a_{i+1}(t) h_i}{2(h_{i-1}+h_i)}\right).
\end{aligned} \tag{4.5a}$$

$$T_{1,i} = -h_{i-1} r_i^- + h_i r_i^+ + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^2}{2(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^2}{2(h_{i-1}+h_i)} + \left(h_{i-1} s_i^- - h_i s_i^+ + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^2}{2(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^2}{2(h_{i-1}+h_i)}\right). \tag{4.5b}$$

$$\begin{aligned}
T_{2,i} &= \left(\frac{h_{i-1}^2}{2} r_i^- + \frac{h_i^2}{2} r_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_i+h_{i-1})} + \frac{\varepsilon\lambda_2}{2} + \frac{\varepsilon\lambda_1 h_i}{2(h_i+h_{i-1})} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^3}{4(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^3}{4(h_{i-1}+h_i)}\right) \\
&+ \left(\frac{-h_{i-1}^2}{2} s_i^- - \frac{h_i^2}{2} s_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_i+h_{i-1})} + \frac{\varepsilon\lambda_2}{2} + \frac{\varepsilon\lambda_1 h_i}{4(h_i+h_{i-1})} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^3}{4(h_i+h_{i-1})} - \frac{\lambda_1 a_{i+1}(t) h_i^3}{4(h_{i-1}+h_i)}\right).
\end{aligned} \tag{4.5c}$$

$$\begin{aligned} T_{3,i} &= \Big(\frac{-h_{i-1}^3}{3!} r_i^- + \frac{h_i^3}{3!} r_i^+ - \frac{\varepsilon\lambda_1 h_{i-1}^2}{2(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^2}{2(h_i+h_{i-1})} + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^4}{12(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^4}{12(h_{i-1}+h_i)} \Big) \\ &\quad + \Big(\frac{h_{i-1}^3}{3!} s_i^- - \frac{h_i^3}{3!} s_i^+ + \varepsilon\lambda_1 \frac{h_i^2}{2(h_i+h_{i-1})} - \frac{\varepsilon\lambda_1 h_{i-1}^2}{2(h_{i-1}+h_i)} + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^4}{12(h_i+h_{i-1})} - \frac{\lambda_1 a_{i+1}(t) h_i^4}{12(h_i+h_{i-1})} \Big). \end{aligned} \tag{4.5d}$$

$$\begin{aligned} T_{4,i} &= \Big(\frac{h_{i-1}^4}{4!} r_i^- + \frac{h_i^4}{4!} r_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}^3}{4(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^3}{4(h_i+h_{i-1})} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^5}{48(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^5}{48(h_{i-1}+h_i)} \Big) \\ &\quad + \Big(\frac{-h_{i-1}^4}{4!} s_i^- - \frac{h_i^4}{4!} s_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}^3}{4(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^3}{4(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^5}{48(h_i+h_{i-1})} - \frac{\lambda_1 a_{i+1}(t) h_i^5}{48(h_{i-1}+h_i)} \Big). \end{aligned} \tag{4.5e}$$

$$\begin{aligned} T_{5,i} &= \Big(\frac{-h_{i-1}^5}{5!} r_i^- + \frac{h_i^5}{5!} r_i^+ - \frac{\varepsilon\lambda_1 h_{i-1}^4}{12(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^4}{12(h_i+h_{i-1})} + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^6}{5!2(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^6}{5!2(h_{i-1}+h_i)} \Big) \\ &\quad + \Big(\frac{h_{i-1}^5}{5!} s_i^- - \frac{h_i^5}{5!} s_i^+ - \frac{\varepsilon\lambda_1 h_{i-1}^4}{12(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^4}{12(h_{i-1}+h_i)} + \frac{\lambda_1 a_{i-1}(t) h_{i-1}^6}{5!2(h_i+h_{i-1})} - \frac{\lambda_1 a_{i+1}(t) h_i^6}{5!2(h_{i-1}+h_i)} \Big). \end{aligned} \tag{4.5f}$$

$$\begin{aligned} T_{6,i} &= \Big(\frac{h_{i-1}^6}{6!} r_i^- + \frac{h_i^6}{6!} r_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}^5}{48(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^5}{48(h_i+h_{i-1})} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^7}{6!2(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i+1}(t) h_i^7}{6!2(h_{i-1}+h_i)} \Big) \\ &\quad + \Big(\frac{-h_{i-1}^6}{6!} s_i^- - \frac{h_i^6}{6!} s_i^+ + \frac{\varepsilon\lambda_1 h_{i-1}^5}{48(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^5}{48(h_{i-1}+h_i)} - \frac{\lambda_1 a_{i-1}(t) h_{i-1}^7}{6!2(h_i+h_{i-1})} - \frac{\lambda_1 a_{i+1}(t) h_i^7}{6!2(h_{i-1}+h_i)} \Big). \end{aligned} \tag{4.5g}$$

Using (3.16) in (4.5a), $T_{0,i} = 0$. Similarly, using (3.16) in (4.5b), $T_{1,i} = 0$. Simplifying (4.5c) using (3.16) gives

$$\begin{aligned} T_{2,i} &= -\frac{\varepsilon h_{i-1}^2}{4h_{i-1}(h_i+h_{i-1})} - \frac{\varepsilon h_i^2}{4h_i(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_{i-1}}{2(h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 h_i}{2(h_i+h_{i-1})} + \frac{\varepsilon\lambda_2}{2}, \\ &= \frac{-\varepsilon}{4} + \varepsilon\frac{\lambda_1}{2} + \varepsilon\frac{\lambda_2}{2} - \frac{\varepsilon}{4} + \varepsilon\frac{\lambda_1}{2} + \varepsilon\frac{\lambda_2}{2}, \\ &= \frac{-\varepsilon}{4} + \frac{\varepsilon}{2}(\lambda_1+\lambda_2) - \frac{\varepsilon}{4} + \frac{\varepsilon}{2}(\lambda_1+\lambda_2), \quad \text{since } \lambda_1+\lambda_2 = \frac{1}{2}, \\ &= 0. \end{aligned}$$

Again, simplifying (4.5d) using (3.16) yields

$$\begin{aligned} T_{3,i} &= \frac{\varepsilon h_{i-1} h_i (h_{i-1}^2 - h_i^2)}{12 h_{i-1} h_i (h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 (h_i^2 - h_{i-1}^2)}{2(h_i+h_{i-1})} + \frac{\varepsilon h_{i-1} h_i (h_{i-1}^2 - h_i^2)}{12 h_{i-1} h_i (h_i+h_{i-1})} + \frac{\varepsilon\lambda_1 (h_i^2 - h_{i-1}^2)}{2(h_i+h_{i-1})}, \\ &= \frac{\varepsilon}{12}(h_{i-1}-h_i) - \varepsilon\frac{\lambda_1}{2}(h_{i-1}-h_i) + \frac{\varepsilon}{12}(h_{i-1}-h_i) - \varepsilon\frac{\lambda_1}{2}(h_{i-1}-h_i), \\ &= \varepsilon(h_{i-1}-h_i)\Big(\frac{1}{6} - \lambda_1\Big). \end{aligned}$$

Sine $h_{i-1} = h_i = h$ on the interval $0 < i < N/4$ and $3N/4 < i < N$, we easily seen that $T_{3,i} = 0$. Equation (4.5e) is simplified as

$$\begin{aligned}
T_{4,i} &= \frac{-\varepsilon h_{i-1}^4}{48h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^4}{48h_i(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1 h_{i-1}^3}{4(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^3}{4(h_i + h_{i-1})} \\
&- \frac{\varepsilon h_{i-1}^4}{48h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^4}{48h_i(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1 h_{i-1}^3}{4(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1 h_i^3}{4(h_{i-1} + h_i)}, \\
&= \frac{-\varepsilon(h_{i-1}^3 + h_i^3)}{48(h_{i-1} + h_i)} + \frac{\varepsilon\lambda_1(h_{i-1}^3 + h_i^3)}{4(h_{i-1} + h_i)} + \frac{-\varepsilon(h_{i-1}^3 + h_i^3)}{48(h_{i-1} + h_i)} + \frac{\varepsilon\lambda_1(h_{i-1}^3 + h_i^3)}{4(h_{i-1} + h_i)}, \\
&= \frac{-\varepsilon(h_{i-1}^3 + h_i^3)}{24(h_{i-1} + h_i)} + \frac{\varepsilon\lambda_1(h_{i-1}^3 + h_i^3)}{2(h_{i-1} + h_i)}, \\
&= -\varepsilon\left(\frac{h_{i-1}^3 + h_i^3}{h_{i-1} + h_i}\right)\left[\frac{1}{4!} - \frac{\lambda_1}{2!}\right].
\end{aligned}$$

To obtain fourth-order method in the space direction, using Remark (3.1) for $\lambda_1 = \frac{1}{12}$, we obtain $T_{4,i} = 0$. Equation (4.5f) is simplified as

$$\begin{aligned}
T_{5,i} &= \frac{\varepsilon h_{i-1} h_i(h_{i-1}^4 - h_i^4)}{2\times 5! h_{i-1} h_i(h_i + h_{i-1})} - \frac{\varepsilon\lambda_1(h_{i-1}^4 - h_i^4)}{12(h_{i-1} + h_i)} + \frac{\varepsilon h_{i-1} h_i(h_{i-1}^4 - h_i^4)}{2\times 5! h_{i-1} h_i(h_i + h_{i-1})} - \frac{\varepsilon\lambda_1(h_{i-1}^4 - h_i^4)}{12(h_{i-1} + h_i)}, \\
&= \frac{\varepsilon(h_{i-1}^4 - h_i^4)}{5!(h_i + h_{i-1})} - \frac{\varepsilon\lambda_1(h_{i-1}^4 - h_i^4)}{6(h_{i-1} + h_i)}, \\
&= \varepsilon\left(\frac{h_{i-1}^4 - h_i^4}{h_{i-1} + h_i}\right)\left[\frac{1}{5!} - \frac{\lambda_1}{6}\right].
\end{aligned}$$

Sine $h_{i-1} = h_i = h$ on the interval $0 < i < N/4$ and $3N/4 < i < N$, we easily seen that $T_{5,i} = 0$. Equation (4.5g) is simplified as

$$\begin{aligned}
T_{6,i} &= \frac{-\varepsilon h_{i-1} h_i(h_{i-1}^5 + h_i^5)}{2\times 6! h_{i-1} h_i(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1(h_{i-1}^5 + h_i^5)}{48(h_{i-1} + h_i)} - \frac{\varepsilon h_{i-1} h_i(h_{i-1}^5 + h_i^5)}{2\times 6! h_{i-1} h_i(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1(h_{i-1}^5 + h_i^5)}{48(h_{i-1} + h_i)}, \\
&= -\varepsilon\frac{(h_{i-1}^5 + h_i^5)}{6!(h_i + h_{i-1})} + \frac{\varepsilon\lambda_1(h_{i-1}^5 + h_i^5)}{24(h_{i-1} + h_i)}, \\
&= -\varepsilon\left(\frac{h_{i-1}^5 + h_i^5}{h_{i-1} + h_i}\right)\left[\frac{1}{6!} - \frac{\lambda_1}{4!}\right].
\end{aligned}$$

Therefore, in the inner regions, that is, on the subintervals $0 < i < N/4$ and $3N/4 < i < N$, the truncation error in (4.4) reduces to

$$T_{i,Y} = -\varepsilon\left(\frac{h_{i-1}^5 + h_i^5}{h_{i-1} + h_i}\right)\left[\frac{1}{6!} - \frac{\lambda_1}{4!}\right](Y_{xxxxxx})_i(t) + O(N^{-5}). \tag{4.6}$$

Similarly, on the interval $N/4 < i < 3N/4$, the truncation error in the outer region is given by

$$T_{i,Y} = \left(r_i^- Y_{i-1}^{j+1} + r_i^c Y_i^{j+1} + r_i^+ Y_{i+1}^{j+1}\right) - \left(s_i^- Y_{i-1}^j + s_i^c Y_i^j + s_i^+ Y_{i+1}^j + \varsigma\right) \tag{4.7}$$

where $\varsigma = \frac{f_i^{j+1}+f_i^j}{2}$. From (3.3), we have

$$\frac{f_i^{j+1} + f_i^j}{2} = -\varepsilon\frac{(Y_{xx})_i^{j+1}}{2} + \frac{a_i^{j+1}}{2}Y_i^{j+1} - \varepsilon\frac{(Y_{xx})_i^j}{2} + \frac{a_i^j}{2}Y_i^j. \tag{4.8}$$

Using (4.8) in (4.7), we obtain

$$T_{i,Y} = \left(r_i^- Y_{i-1}^{j+1} + r_i^c Y_i^{j+1} + r_i^+ Y_{i+1}^{j+1}\right) - \left(s_i^- Y_{i-1}^j + s_i^c Y_i^j + s_i^+ Y_{i+1}^j\right) + \varepsilon\frac{(Y_{xx})_i^{j+1}}{2} - \frac{a_i^{j+1}}{2}Y_i^{j+1} + \varepsilon\frac{(Y_{xx})_i^j}{2} - \frac{a_i^j}{2}Y_i^j. \tag{4.9}$$

Using the Taylor series expansion for the terms $Y_{i-1}^{j+1}, Y_{i+1}^{j+1}$ up-to $O(h^5)$ in the space direction in (4.9), we obtain the following truncation error for any time level

$$T_{i,Y} = T_{0,i}Y_i(t) + T_{1,i}(Y_x)_i(t) + T_{2,i}(Y_{xx})_i(t) + T_{3,i}(Y_{xxx})_i(t) + T_{4,i}(Y_{xxxx})_i(t), \tag{4.10}$$

where the coefficients are give by

$$T_{0,i} = r_i^- + r_i^c + r_i^+ - \frac{a_i(t)}{2} - \left(s_i^- + s_i^c + s_i^+ - \frac{a_i(t)}{2}\right), \tag{4.11a}$$

$$T_{1,i} = -h_{i-1}r_i^- + h_i r_i^+ + \left(h_{i-1}s_i^- - h_i s_i^+\right), \tag{4.11b}$$

$$T_{2,i} = \left(\frac{h_{i-1}^2}{2!}r_i^- + \frac{h_i^2}{2!}r_i^+ + \frac{\varepsilon}{2}\right) + \left(\frac{-h_{i-1}^2}{2!}s_i^- - \frac{h_i^2}{2!}s_i^+ + \frac{\varepsilon}{2}\right), \tag{4.11c}$$

$$T_{3,i} = \left(\frac{-h_{i-1}^3}{3!}r_i^- + \frac{h_i^3}{3!}r_i^+\right) + \left(\frac{h_{i-1}^3}{3!}s_i^- - \frac{h_i^3}{3!}s_i^+\right), \tag{4.11d}$$

$$T_{4,i} = \left(\frac{h_{i-1}^4}{4!}r_i^- + \frac{h_i^4}{4!}r_i^+\right) + \left(\frac{-h_{i-1}^4}{4!}s_i^- - \frac{h_i^4}{4!}s_i^+\right). \tag{4.11e}$$

Using (3.16), we can easily seen that $T_{0,i} = 0, T_{1,i} = 0, T_{2,i} = 0$. Using (3.16) in (4.11d) and simplified to obtain

$$\begin{aligned} T_{3,i} &= \frac{\varepsilon h_{i-1}^3}{3!h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^3}{3!h_i(h_i + h_{i-1})} + \frac{\varepsilon h_{i-1}^3}{3!h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^3}{3!h_i(h_i + h_{i-1})}, \\ &= \frac{\varepsilon(h_{i-1}^3 - h_i^3)}{3!h_{i-1}(h_i + h_{i-1})} + \frac{\varepsilon(h_{i-1}^3 - h_i^3)}{3!h_{i-1}(h_i + h_{i-1})}. \end{aligned}$$

Sine $h_{i-1} = h_i = H$ on the interval $N/4 < i < 3N/4$, we easily seen that $T_{3,i} = 0$. Equation (4.11e) is simplified to obtain

$$\begin{aligned} T_{4,i} &= \frac{-\varepsilon h_{i-1}^4}{4!h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^4}{4!h_i(h_i + h_{i-1})} - \frac{\varepsilon h_{i-1}^4}{4!h_{i-1}(h_i + h_{i-1})} - \frac{\varepsilon h_i^4}{4!h_i(h_i + h_{i-1})}, \\ &= \frac{-\varepsilon}{12}\left(\frac{h_{i-1}^3 + h_i^3}{h_{i-1} + h_i}\right). \end{aligned}$$

Therefore, for outer region, that is, on the interval $N/4 < i < 3N/4$, the truncation error in (4.10) reduces to

$$T_{i,Y} = \frac{-\varepsilon}{12}\left(\frac{h_{i-1}^3 + h_i^3}{h_{i-1} + h_i}\right)(Y_{xxxx})_i(t) + O(N^{-3}). \tag{4.12}$$

Finally, the truncation error at the transition points is bounded by

$$|T_{i,Y}| \le \begin{cases} C(N^{-3} + N^{-\sigma_0\sqrt{\alpha}}), & i = N/4, 3N/4 \text{ and } H \ge \sqrt{\varepsilon}, \\ C(N^{-1}\varepsilon + N^{-\sigma_0\sqrt{\alpha}}), & i = N/4, 3N/4 \text{ and } H < \sqrt{\varepsilon}. \end{cases}$$

Theorem 4.1 Let $Y(x_i, t)$ be a hybrid scheme to the solution of (3.16)–(3.17) and $y(x, t)$ is the solution to the continuous problem. Then, the parameter-uniform error estimate satisfies the bound

$$|T_{i,Y}| \le \begin{cases} CN^{-4}(\ln N)^4, & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ CN^{-2}(\ln N)^2, & N/4 \le i \le 3N/4. \end{cases}$$

where C is a constant independent of ε and N.

Proof The truncation error is given by

$$T_{i,Y} = \begin{cases} -\varepsilon\left(\frac{h_{i-1}^5+h_i^5}{h_{i-1}+h_i}\right)\left[\frac{1}{6!} - \frac{\lambda_1}{4!}\right](Y_{xxxxxx})_i(t) + O(N^{-5}), & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ \frac{-\varepsilon}{12}\left(\frac{h_{i-1}^3+h_i^3}{h_{i-1}+h_i}\right)(Y_{xxxx})_i(t) + O(N^{-3}), & N/4 < i < 3N/4. \end{cases} \tag{4.13}$$

Since the argument depends on whether $\sigma = 1/4$ or $\sigma = 2\sqrt{\varepsilon}\ln N < 1/4$, there arises two cases.
Case (i): When $\sigma = 1/4$, the mesh is uniform with spacing $1/N$, that is, $h_i = 1/N$ and $2\sqrt{\varepsilon}\ln N \ge 1/4$ gives $\varepsilon^{-1/2} \le C\ln N$. From this, we get $\varepsilon^{-1} \le (C\ln N)^2$. In this case, we use a classical analysis to prove convergence. Using the classical bound in Theorem (2.3) together with (4.13) yields

$$|T_{i,Y}| \le \begin{cases} \varepsilon\left(\frac{h_{i-1}^5+h_i^5}{h_{i-1}+h_i}\right)\left[\frac{1}{6!} - \frac{\lambda_1}{4!}\right]C\varepsilon^{-4}, & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ \frac{\varepsilon}{12}\left(\frac{h_{i-1}^3+h_i^3}{h_{i-1}+h_i}\right)C\varepsilon^{-2}, & N/4 < i < 3N/4. \end{cases} \tag{4.14}$$

Since $h_{i-1} = h_i = 1/N$, it follows from (4.14) that

$$|T_{i,Y}| \le \begin{cases} C\varepsilon N^{-4}(C\varepsilon^{-4}), & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ C\varepsilon N^{-2}(C\varepsilon^{-2}), & N/4 < i < 3N/4. \end{cases} \tag{4.15}$$

Using the fact that $\varepsilon^{-1} \le (C\ln N)^2$, we obtain

$$|T_{i,Y}| \le \begin{cases} CN^{-4}\ln^4 N, & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ CN^{-2}\ln^2 N, & N/4 < i < 3N/4. \end{cases} \tag{4.16}$$

Case (ii): If Ω_i lies in the inner regions, then mesh spacing $h_i = \frac{4\sigma}{N}$ and $\sigma = 2\sqrt{\varepsilon}\ln N$. Using the bounds in the

Theorem (2.4) together with the estimates in (4.13), we have

$$|T_{i,Y}| \leq \begin{cases} \varepsilon\left(\frac{h_{i-1}^5+h_i^5}{h_{i-1}+h_i}\right)\left[\frac{1}{6!}-\frac{\lambda_1}{4!}\right]C\varepsilon^{-4}(e^{-x_i/\sqrt{\varepsilon}}+e^{-(1-x_i)/\sqrt{\varepsilon}}), & 0<i<\frac{N}{4}, \frac{3N}{4}<i<N, \\ \frac{\varepsilon}{12}\left(\frac{h_{i-1}^3+h_i^3}{h_{i-1}+h_i}\right)C(1+\varepsilon^{-3}), & N/4<i<3N/4. \end{cases} \tag{4.17}$$

Since $e^{-x_i/\sqrt{\varepsilon}} \leq e^{-\sigma/\sqrt{\varepsilon}} = e^{-2\ln N} = N^{-2}$ and $e^{-(1-x_i)/\sqrt{\varepsilon}} \leq e^{-\sigma/\sqrt{\varepsilon}} = e^{-2\ln N} = N^{-2}$ and considering $\lambda_1 = \frac{1}{12}$, we have

$$|T_{i,Y}| \leq \begin{cases} C\varepsilon\left(\frac{h_{i-1}^5+h_i^5}{h_{i-1}+h_i}\right)(C\varepsilon^{-4}N^{-2}), & 0<i<\frac{N}{4}, \frac{3N}{4}<i<N, \\ C\varepsilon\left(\frac{h_{i-1}^3+h_i^3}{h_{i-1}+h_i}\right)(C(1+\varepsilon^{-3})), & N/4<i<3N/4. \end{cases} \tag{4.18}$$

Since $h_i = h_{i-1} = 8\sqrt{\varepsilon}N^{-1}\ln N$ and the fact that $\sqrt{\varepsilon} \leq CN^{-1}$, we obtain the estimate from (4.17)

$$|T_{i,Y}| \leq \begin{cases} CN^{-4}\ln^4 N, & 0<i<\frac{N}{4}, \frac{3N}{4}<i<N, \\ C\varepsilon\left(\frac{h_{i-1}^3+h_i^3}{h_{i-1}+h_i}\right)(C(1+\varepsilon^{-3})), & N/4<i<3N/4. \end{cases} \tag{4.19}$$

On the other hand, for the subinterval $[\sigma, 1-\sigma]$, that is, for the outer region the mesh spacing is $h_i = 2N^{-1}(1-2\sigma) = 2N^{-1} - C\sqrt{\varepsilon}N^{-1}\ln N \leq CN^{-1}$. Using this in (4.19) and the fact that $\sqrt{\varepsilon} \leq CN^{-1}$ gives us

$$|T_{i,Y}| \leq \begin{cases} CN^{-4}\ln^4 N, & 0<i<\frac{N}{4}, \frac{3N}{4}<i<N, \\ CN^{-2}, & N/4<i<3N/4. \end{cases} \tag{4.20}$$

It follows from classical estimates that the truncation error at the transition points is given by

$$\begin{aligned} |T_{N/4,Y}| &\leq \left|\varepsilon\left(\frac{\partial^2}{\partial x^2}-\delta^2\right)y\right|, \\ &\leq C\varepsilon(x_{i+1}-x_{i-1})|y_i'''|, \text{ if } x_i=\sigma,\ x_i=1-\sigma. \end{aligned}$$

For $x_i = \sigma$ and $N^{-2} \leq \varepsilon$ and $x_{i+1} - x_{i-1} \leq 4N^{-1}$, we obtain

$$\begin{aligned} |T_{N/4,Y}| &\leq CN^{-1}\varepsilon\left(1+\varepsilon^{-1/2}+\varepsilon^{-1/2}\left[1+\varepsilon^{-1}\left\{e^{-x/\sqrt{\alpha/\varepsilon}}+e^{-(1-x)/\sqrt{\alpha/\varepsilon}}\right\}\right]\right), \\ &\leq CN^{-1}\left(\varepsilon+2\varepsilon^{1/2}+N^{-\sigma_0\sqrt{\alpha}}\right), \\ &\leq CN^{-1}\sqrt{\varepsilon}+CN^{-\sigma_0\sqrt{\alpha}}, \quad \text{since } \sqrt{\varepsilon} \leq CN^{-1} \text{and } \sigma_0 \geq 2/\sqrt{\alpha}, \\ &\leq CN^{-2}. \end{aligned}$$

When $\varepsilon \le N^{-2}$, we obtain

$$
\begin{aligned}
|T_{N/4,Y}| &\le 2\varepsilon \max_{x_{i-1}\le x\le x_{i+1}} |y''| \\
&\le C\Big(\varepsilon + \max_{x_{i-1}\le x\le x_{i+1}} \Big\{e^{-x/\sqrt{\alpha/\varepsilon}} + e^{-(1-x)/\sqrt{\alpha/\varepsilon}}\Big\}\Big), \\
&\le C\varepsilon + CN^{-\sigma_0\sqrt{\alpha}}, \\
&\le CN^{-1}\sqrt{\varepsilon} + CN^{-\sigma_0\sqrt{\alpha}}, \quad \text{since } \sqrt{\varepsilon} \le CN^{-1} \text{and } \sigma_0 \ge 2/\sqrt{\alpha}, \\
&\le CN^{-2}.
\end{aligned}
$$

The same analysis holds for the errors in the case when $x_i = 1 - \sigma$. Combining the above estimates for both the cases, we have

$$
|T_{i,Y}| \le \begin{cases} CN^{-4}(\ln N)^4, & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ CN^{-2}(\ln N)^2, & N/4 \le i \le 3N/4. \end{cases}
$$

This completes the proof.

Remark 4.1 If $\lambda_1 = \frac{1}{12}$, then the truncation error bound in space direction is given by

$$
|T_{i,Y}| \le \begin{cases} CN^{-4}(\ln N)^4, & 0 < i < \frac{N}{4}, \frac{3N}{4} < i < N, \\ CN^{-2}(\ln N)^2, & N/4 \le i \le 3N/4. \end{cases}
$$

Remark 4.2 If $\lambda_1 = \frac{1}{6}$, then the truncation error bound in the space direction is given by

$$
|T_{i,Y}| \le CN^{-2}(\ln N)^2, \qquad 1 < i < N.
$$

We have the following main convergence theorem.

Theorem 4.2 Let $y(x,t)$ be the solution of continuous problem and $Y(x_i,t)$ be the solution of the discrete problem. Then, the parameter-uniform error estimate for $\lambda_1 = \frac{1}{12}$ is given by

$$
|T_{i,Y}| \le \begin{cases} C(\Delta t^2 + N^{-4}(\ln N)^4), & 0 \le i < \frac{N}{4}, \frac{3N}{4} < i \le N, \\ C(\Delta t^2 + N^{-2}(\ln N)^2), & N/4 \le i \le 3N/4. \end{cases}
$$

Similarly, for $\lambda_1 = \frac{1}{6}$, the parameter-uniform error estimate is given by

$$
|T_{i,Y}| \le C(\Delta t^2 + N^{-2}(\ln N)^2), \qquad 0 \le i \le N,
$$

where C is a constant independent of ε and the mesh parameters N and Δt.

Proof The proof follows from the bound given in Lemma (3.2) for time and Theorem (4.1) for space.

Now, we illustrate the theoretical findings in the previous sections in practice via numerical experiments.

5 | NUMERICAL EXPERIMENTS

In this section, we carry out numerical experiment in order to corroborate the applicability of the proposed method. Since the exact solution for the first two examples are not known, we use the double mesh principle to calculate maximum point-wise errors using the following formula

$$E_{\varepsilon}^{N,M} = \max_{0\le i\le N; t\in[0,T]} \left|Y^{N,M}(x_i,t_j) - Y^{2N,2M}(x_i,t_j)\right|,$$

where $Y^{N,M}(x_i,t_j)$ denotes the numerical solution obtained at (N,M) mesh points where $Y^{2N,2M}(x_i,t_j)$ denotes the numerical solution at $(2N,2M)$ mesh points. Whereas the exact solution for the third example is known, we use the following formula to calculate the maximum point-wise errors.

$$E_{\varepsilon}^{N,M} = \max_{0\le i\le N; t\in[0,T]} \left|y(x_i,t_j) - Y^{N,M}(x_i,t_j)\right|,$$

where $Y^{N,M}(x_i,t_j)$ denotes the numerical solution obtained at (N,M) mesh points where $y(x_i,t_j)$ denotes the exact solution at (N,M) mesh points. The numerical ε-uniform rate of convergence and ε-uniform maximum point-wise errors were calculated using the following formulas, respectively

$$r^{N,M} = \log_2\left(\frac{E^{N,M}}{E^{2N,2M}}\right) \quad \text{and} \quad E^{N,M} = \max_{\varepsilon} E_{\varepsilon}^{N,M}.$$

Example 5.1 Consider singularly perturbed reaction-diffusion problem

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon\frac{\partial^2 y}{\partial x^2} + \frac{1+x^2}{2}y = t^3, & (x,t)\in(0,1)\times(0,1], \\ y(x,0)=0, & x\in[0,1], \\ y(0,t)=0, \quad y(1,t)=0, & t\in[0,1]. \end{cases}$$

Example 5.2 Consider singularly perturbed reaction-diffusion problem

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon\frac{\partial^2 y}{\partial x^2} + \frac{1+x^2}{2}y = e^t - 1 + \sin(\pi x), & (x,t)\in(0,1)\times(0,1], \\ y(x,0)=0, & x\in[0,1], \\ y(0,t)=0, \quad y(1,t)=0, & t\in[0,1]. \end{cases}$$

Example 5.3 Consider singularly perturbed reaction-diffusion problem

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon\frac{\partial^2 y}{\partial x^2} + (1+xe^{-t})y = f(x,t), & (x,t)\in(0,1)\times(0,1], \\ y(x,0)=0, & x\in[0,1], \\ y(0,t)=0, \quad y(1,t)=0, & t\in[0,1]. \end{cases}$$

where the function $f(x,t)$ is chosen from the exact solution

$$y(x,t) = (1-e^{-t})\Big(\frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x)\Big).$$

TABLE 1 $E_{\varepsilon}^{N,M}, E^{N,M}$ and $r^{N,M}$ for Example 5.1 by taking different values λ_1 and λ_2.

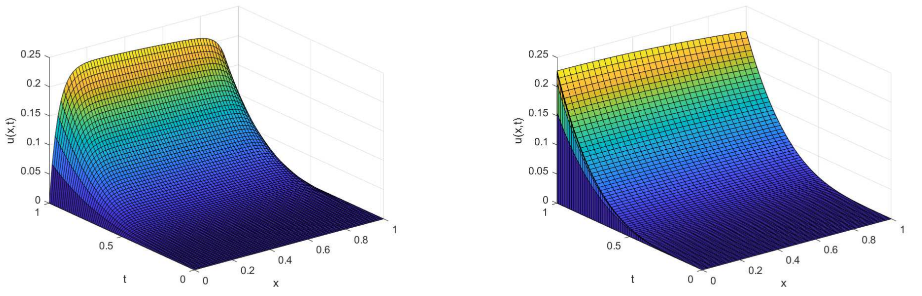
$\varepsilon \downarrow$	$N = 32$ $M = 20$	$N = 64$ $M = 80$	$N = 128$ $M = 320$	$N = 256$ $M = 1280$	$N = 512$ $M = 5120$
$\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$					
10^{-2}	3.2061e-04	1.9830e-05	1.2227e-06	7.4955e-08	2.1541e-08
10^{-4}	1.4930e-03	2.1583e-04	2.5044e-05	2.6872e-06	2.6825e-07
10^{-6}	1.4876e-03	2.1495e-04	2.4943e-05	2.6756e-06	2.6714e-07
10^{-8}	1.4871e-03	2.1486e-04	2.4933e-05	2.6744e-06	2.6703e-07
10^{-10}	1.4871e-03	2.1486e-04	2.4933e-05	2.6744e-06	2.6702e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	1.4930e-03	2.1583e-04	2.5044e-05	2.6872e-06	2.6825e-07
$r^{N,M}$	2.7902	3.1074	3.2203	3.3245	-
$\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{3}$					
10^{-2}	1.3294e-03	2.9970e-04	7.3178e-05	1.8186e-05	4.5431e-06
10^{-4}	1.0369e-02	3.6253e-03	1.1162e-03	3.6037e-04	1.1285e-04
10^{-6}	1.0369e-02	3.6210e-03	1.1162e-03	3.6037e-04	1.1285e-04
10^{-8}	1.0369e-02	3.6206e-03	1.1162e-03	3.6037e-04	1.1285e-04
10^{-10}	1.0369e-02	3.6206e-03	1.1162e-03	3.6037e-04	1.1285e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	1.0369e-02	3.6253e-03	1.1162e-03	3.6037e-04	1.1285e-04
$r^{N,M}$	1.5161	1.6995	1.6310	1.6751	-

TABLE 2 $E_\varepsilon^{N,M}, E^{N,M}$ and $r^{N,M}$ for Example 5.2 by taking different values λ_1 and λ_2.

$\varepsilon \downarrow$	$N = 32$ $M = 20$	$N = 64$ $M = 80$	$N = 128$ $M = 320$	$N = 256$ $M = 1280$	$N = 512$ $M = 5120$
$\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$					
10^{-2}	1.6830e-04	1.4261e-05	1.8287e-06	3.9577e-07	1.0124e-07
10^{-4}	1.7124e-03	2.2853e-04	2.7441e-05	2.9485e-06	2.9566e-07
10^{-6}	1.6986e-03	2.2650e-04	2.7204e-05	2.9238e-06	2.9318e-07
10^{-8}	1.6973e-03	2.2630e-04	2.7180e-05	2.9214e-06	2.9293e-07
10^{-10}	1.6971e-03	2.2628e-04	2.7178e-05	2.9211e-06	2.9291e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	1.7124e-03	2.2853e-04	2.7441e-05	2.9485e-06	2.9566e-07
$r^{N,M}$	2.9056	3.0580	3.2183	3.3180	-
$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$					
10^{-2}	2.1362e-03	5.1533e-04	1.2767e-04	3.1846e-05	7.9598e-06
10^{-4}	1.7877e-02	5.9003e-03	1.9218e-03	6.2063e-04	1.9548e-04
10^{-6}	1.7877e-02	5.8870e-03	1.9207e-03	6.2064e-04	1.9548e-04
10^{-8}	1.7877e-02	5.8857e-03	1.9207e-03	6.2064e-04	1.9548e-04
10^{-10}	1.7877e-02	5.8856e-03	1.9207e-03	6.2064e-04	1.9548e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	1.7877e-02	5.9003e-03	1.9218e-03	6.2064e-04	1.9548e-04
$r^{N,M}$	1.5992	1.6183	1.6306	1.6667	-

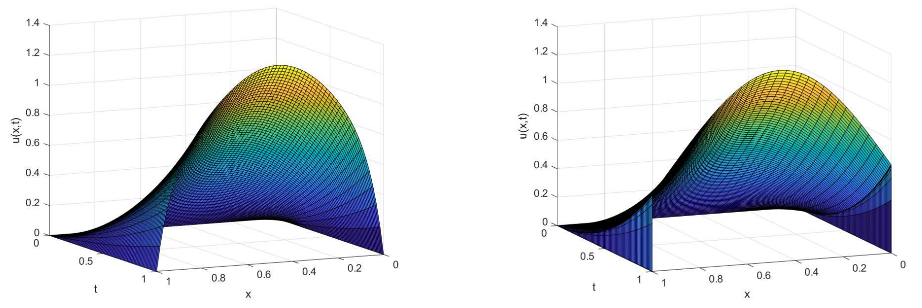
TABLE 3 $E_\varepsilon^{N,M}, E^{N,M}$ and $r^{N,M}$ for Example 5.3 by taking different values λ_1 and λ_2.

$\varepsilon \downarrow$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
	$M = 20$	$M = 80$	$M = 320$	$M = 1280$	$M = 5120$
$\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$					
10^{-2}	1.7107e-04	4.3569e-05	1.0945e-05	2.7392e-06	6.8497e-07
10^{-4}	2.1995e-04	3.2123e-05	5.3836e-06	1.0814e-06	2.4712e-07
10^{-6}	3.1893e-04	6.2553e-05	1.3368e-05	2.6919e-06	4.5630e-07
10^{-8}	3.3294e-04	6.8598e-05	1.6096e-05	3.8895e-06	9.3073e-07
10^{-10}	3.3436e-04	6.9229e-05	1.6394e-05	4.0343e-06	1.0011e-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	3.3452e-04	6.9299e-05	1.6427e-05	4.0507e-06	1.0092e-06
$r^{N,M}$	2.2712	2.0768	2.0198	2.0050	-
$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$					
10^{-2}	5.4732e-04	1.3431e-04	3.3265e-05	8.3053e-06	2.0755e-06
10^{-4}	5.4209e-03	1.7450e-03	5.9812e-04	1.9483e-04	6.1688e-05
10^{-6}	5.4261e-03	1.7470e-03	5.9871e-04	1.9502e-04	6.1741e-05
10^{-8}	5.4266e-03	1.7472e-03	5.9877e-04	1.9503e-04	6.1746e-05
10^{-10}	5.4266e-03	1.7472e-03	5.9878e-04	1.9504e-04	6.1747e-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E^{N,M}$	5.4266e-03	1.7472e-03	5.9878e-04	1.9504e-04	6.1747e-05
$r^{N,M}$	1.6350	1.5449	1.6183	1.6593	-



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-8}$

FIGURE 1 Surface plot of Example (5.1) for $N = 64, M = 80$ and $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$.



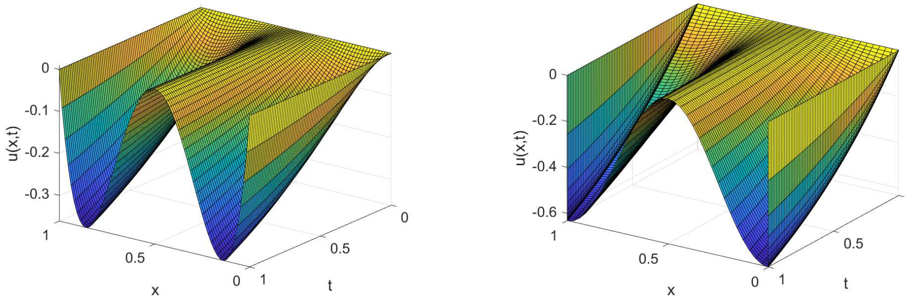
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-8}$

FIGURE 2 Surface plot of Example (5.2) for $N = 64, M = 80$ and $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$.

The maximum point-wise and ε−uniform errors with their respective rate of convergences are presented in Tables (1), (2) and (3) for Examples (5.1), (5.2) and (5.3), respectively. From these Tables, one can observe that the proposed method gives higher-order accuracy using $\lambda_1 = \frac{1}{12}$ and $\lambda_2 = \frac{5}{12}$ and almost second-order accuracy using $\lambda_1 = \frac{1}{6}$ and $\lambda_2 = \frac{1}{3}$, which confirms Remark (3.1). Numerical simulations in terms of surface plots in Figures (1), (2) and (3) for Example (5.1), (5.2) and (5.3) show the formation of boundary layers at the two end points as $\varepsilon \to 0$. The maximum point-wise errors for Examples (5.1), (5.2) and (5.3) are plotted using log-log scale as can be seen in Figures (4) showing the ε-uniform convergence. From all the tables of values, we deduce that when the mesh points increases the maximum point-wise errors decreases.

6 | CONCLUSION

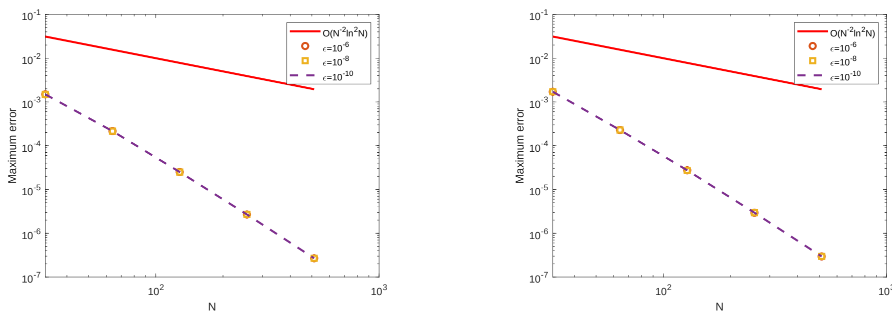
A higher-order new numerical method for singularly perturbed parabolic reaction-diffusion problems is presented in this study. The time variable is discretized using the Crank-Nicolson method on a uniform mesh, and the space variable is discretized using the hybrid difference method on a Shishkin mesh, which consists the cubic spline in tension method in the inner regions and the classical finite difference method in the outer



(a) $\varepsilon = 10^{-2}$

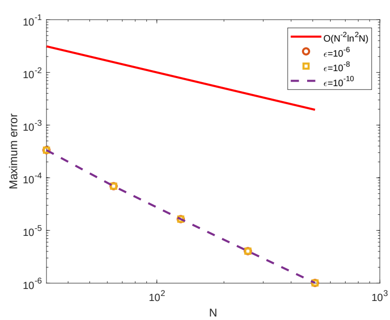
(b) $\varepsilon = 10^{-8}$

FIGURE 3 Surface plot of Example (5.3) for $N = 64, M = 80$ and $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$.



(a) Example (5.1).

(b) Example (5.2).



(c) Example (5.3).

FIGURE 4 Loglog plots for all the three Examples using the respective Tables.

region. The proposed method's stability and convergence analysis are very well established. The numerical results show that the proposed method is a higher-order $\varepsilon-$uniformly convergent method. Three numerical examples are solved to validate the theoretical results. The numerical results shown in the tables and figures show that the proposed method works well.

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Author contributions

All authors conceived of the study, drafted and approved the final manuscript.

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Conflict of interest

The authors declare no potential conflict of interests.

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