Lp-Stability of a Class of Volterra Systems

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Abstract

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***Index Terms*—Lipschitz-norm, L_p-norm, nonconvex set, stability, Volterra series, Volterra system.**

I. INTRODUCTION

The stability of Volterra systems has been a focal research topic for both natural scientists [1] and engineers [2]–[4], with a renewed interest when novel applications and results emerge [5]. A state-of-the-art review on the use of Volterra series for modeling of nonlinear systems is presented in [6], which seems to suggest a declining interest in the stability analysis of Volterra systems in recent years.

This paper communicates new and explicit stability results for Volterra systems. Specifically, the L_p-stability of a general continuous-time nonlinear system, described by a Volterra series expansion, is studied. The equivalent discrete-time results follow in a straightforward manner by standard modifications to those for the continuous-time versions.

Following the seminal work of Boyd and Chua [7], on Volterra systems with fading memory, we are able to derive concise stability criteria. When a fading-memory Volterra system's memoryless output nonlinearity is not continuous, its Lipschitz-norm may not exist. However, it might still be bounded by some Lipschitz-continuous mapping, enabling one to derive a stability criterion. For the case of monomial domination, this approach produces nonconvex ϵ-balls in the signal spaces of interest for certain parameter values. On the other hand, imposing the more stringent requirement of Lipschitz-continuity on the output nonlinearity ensures the existence of its Lipschitz-norm, thus preventing nonconvex sets from entering the analysis. The stability criterion so obtained necessarily applies to a smaller class of systems. These two approaches yield different stability bounds, thus providing more versatility by allowing selection of the more appropriate of the two, for a given application.

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II. L_p-STABILITY OF VOLTERRA SYSTEMS

This paper focuses on the class of fading-memory Volterra systems, which includes a vast range of engineering applications. Furthermore, their specific structure allows concise analytical expressions to be obtained.

First, three distinct cases are considered: a monomial nonlinearity, a general monomially dominated nonlinearity and a Lipschitz-continuous nonlinearity. After a discussion of the results obtained, three more general cases are presented to pave the way toward analysis of multiple-input, multiple-convolution and multiple-output Volterra systems.

Case 1: Consider the following single-input single-convolution Volterra system: a Linear Time-Invariant (LTI) system with input signal x and impulse response $g \in L_1[0,\infty)$ that feeds into a memoryless nonlinearity of the form $(\cdot)^v$, for a given $0 < v < \infty$. Its output is described by

$$y = (g * x)^v \,, \tag{1}$$

where $*$ represents convolution. Even though the power function is memoryless, the output y possesses memory with respect to x due to the convolution featuring in (1). Young's convolution inequality [8, Th. 9.1] states that

$$\|g * x\|_p \leq \|g\|_1 \, \|x\|_p \,, \tag{2}$$

for all $1 \leq p \leq \infty$, with $\|\cdot\|_p$ the standard L_p-norm on $[0,\, \infty)$. Using inequality (2), one obtains

$$\begin{aligned} \|y\|_q^{\frac{1}{v}} \equiv \|(g*x)^v\|_q^{\frac{1}{v}} &= \Big(\int_0^\infty |(g*x)^v(\tau)|^q \,\mathrm{d}\tau\Big)^{1/vq} \\ &= \Big(\int_0^\infty |g*x|^{vq}(\tau) \,\mathrm{d}\tau\Big)^{1/vq} \\ &\equiv \|g*x\|_{vq} \\ &\leq \|g\|_1 \, \|x\|_{vq} \,, \end{aligned}$$

for all $1 \leq vq \leq \infty$, or alternatively, $1/v \leq q \leq \infty$. It follows that the Volterra system, described by (1), is finite-gain input-output stable from $L_{vq}[0,\infty)$ to $L_q[0,\infty)$ for every $v > 0$ and every q satisfying $1/v \leq q \leq \infty$, both fixed.

It is interesting to note that, for $v > 1$, the lower bound of q, namely $1/v$, is less than one. However, for $1/v < q < 1$, the L_q-norm's triangle inequality is violated, causing every ε-ball in the output signal space $L_q[0,\infty)$ to be *nonconvex* [9], [10]. On the other hand, if q is taken to be greater than or equal to one, then, for sufficiently small $0 < v < 1$, it follows that $vq < 1$ so that every ε-ball in the input signal space $L_{vq}[0,\infty)$ is *nonconvex*.

The consequences of a Lebesgue exponent[1] strictly between zero and one are far reaching concerning the operator theory

[1]Parameter p is called the *Lebesgue exponent* of the *Lebesgue space* L_p.

applicable to this Lebesgue space (see [11]). For example, any linear functional on this Lebesgue space is identically zero [12], implying that only nonlinear functionals on this space are worth studying.

Case 2: Consider a more general single-input single-convolution Volterra system, with a memoryless output nonlinearity described by a bounded and measurable real-valued function $h(\cdot)$ that is dominated by the monomial $(\cdot)^v$, that is,

$$|h(u)| \leq |u|^v, \qquad \text{for all} \quad u \in \mathbb{R}\,.$$

This implies that $h(0) = 0$ and so it follows immediately that

$$\|h(g * x)\|_q \leq \|g\|_1^v \, \|x\|_{vq}^v \,, \tag{3}$$

for all $1 \leq vq \leq \infty$, based on the results derived for Case 1 above.

Interestingly, in this last inequality, the norm of the nonlinear function $h(\cdot)$ does not appear on the right-hand side. From (3), it follows that the single-input single-convolution Volterra system with memoryless output nonlinearity being any measurable function $h(\cdot)$, dominated by the power-v function, is finite-gain input-output stable from $L_{vq}[0,\infty)$ to $L_q[0,\infty)$ for every $v > 0$ and every $1/v \leq q \leq \infty$, both fixed.

Note that the above remarks about the nonconvexity of ε-balls in the Lebesgue spaces also apply here.

Case 3: To further extend the previous results, assume $h(\cdot)$ to be a Lipschitz-continuous memoryless nonlinearity. Consider the Lipschitz-norm defined by

$$\|h\|_{\mathrm{Lip}(p,q)} = \|h\|_\infty + \sup_{u_1 \neq u_2} \frac{\|h(u_1) - h(u_2)\|_q}{\|u_1 - u_2\|_p}\,,$$

for fixed $1 \leq p,\, q \leq \infty$, where $\|\cdot\|_\infty$ denotes the standard sup-norm [13],[14]. Here, the subscript $\mathrm{Lip}(p,q)$ is introduced merely to explicitly indicate this norm's dependency on the domain and range spaces of $h(\cdot)$.

For every constant function $h(\cdot)$, the first term in $\|\cdot\|_{\mathrm{Lip}(p,q)}$ satisfies $\|h\|_\infty = |h(0)|$ while the second term there, i.e. the slope of $h(\cdot)$, is identically zero.[2] This yields the following expression for the Lipschitz-norm:

$$\|h\|_{\mathrm{Lip}(p,q)} = |h(0)| + \sup_{u_1 \neq u_2} \frac{\|h(u_1) - h(u_2)\|_q}{\|u_1 - u_2\|_p}\,.$$

For the purpose of the discussion that follows, it is noted that

$$\|h\|_{\mathrm{Lip}(p,q)} \geq |h(0)| + \frac{\|h(u_1) - h(u_2)\|_q}{\|u_1 - u_2\|_p}\,,$$

for arbitrary but fixed arguments u_1 and u_2 on the right-hand side of the inequality, yielding

$$\begin{aligned} &\|h(u_1) - h(u_2)\|_q \\ &\qquad \leq \big(\|h\|_{\mathrm{Lip}(p,q)} - |h(0)|\big)\|u_1 - u_2\|_p\,. \end{aligned} \tag{4}$$

[2]The quotient, in the expression for the Lipschitz-norm $\|h\|_{\mathrm{Lip}(p,q)}$ of $h(\cdot)$, is a *seminorm*, which vanishes for every constant function $h(\cdot)$.

Since the intermediate signals, u_1 and u_2, are the result of convolutions, the above inequality can be expressed as

$$\begin{aligned} &\|h(g*x_1) - h(g*x_2)\|_q \\ &\quad \leq \big(\|h\|_{\mathrm{Lip}(p,q)} - |h(0)|\big)\|g*x_1 - g*x_2\|_p \\ &\quad = \big(\|h\|_{\mathrm{Lip}(p,q)} - |h(0)|\big)\|g*(x_1 - x_2)\|_p \quad \text{(Linearity)} \\ &\quad \leq \big(\|h\|_{\mathrm{Lip}(p,q)} - |h(0)|\big)\|g\|_1\|x_1 - x_2\|_p\,, \end{aligned} \tag{5}$$

for some input signals x_1 and x_2, after applying (2).

For the case of $h(0) = 0$, with $h(\cdot)$ not necessarily power-v dominated, by setting $x_2 = 0$, one has

$$\|h(g*x_1)\|_q \leq \|h\|_{\mathrm{Lip}(p,q)}\, \|g\|_1\, \|x_1\|_p\,, \tag{6}$$

for all $1 \leq p,\, q \leq \infty$. Therefore, for arbitrary but fixed $1 \leq p,\, q \leq \infty$, a single-input single-convolution Volterra system with a memoryless output nonlinearity $h(\cdot)$ that has a finite Lipschitz-norm $\|\cdot\|_{\mathrm{Lip}(p,q)}$ and $h(0) = 0$, but is not necessarily dominated by the power-v function, is finite-gain input-output stable from $L_p[0,\infty)$ to $L_q[0,\infty)$.

At this juncture, a caution is not to underestimate the role of the assumption $h(0) = 0$. To see this, note that assuming $h(0) \neq 0$ and $x_2 \equiv 0$ in (5) gives

$$\begin{aligned} &\|h(g*x_1) - h(0)\, 1(\cdot)\|_q \\ &\qquad \leq \big(\|h\|_{\mathrm{Lip}(p,q)} - |h(0)|\big)\, \|g\|_1\, \|x_1\|_p\,, \end{aligned} \tag{7}$$

where $1(t) := 1$ for all $t \in [0,\infty)$. If one now attempts to apply the norm inequality $\|z_1\| - \|z_2\| \leq \|z_1 - z_2\|$, derived from the triangle inequality, to the left-hand side, then

$$\begin{aligned} &\|h(g*x_1)\|_q - |h(0)|\|1(\cdot)\|_q \\ &\qquad \leq \|h(g*x_1) - h(0)1(\cdot)\|_q\,. \end{aligned} \tag{8}$$

This is problematic because, for the standard $L_q[0,\infty)$ space, when $q \neq \infty$, one would have that $\|1(\cdot)\|_q = \infty$ because $1(\cdot)$ is not in the space $L_q[0,\infty)$. Therefore, one cannot isolate the term $\|h(g*x_1)\|_q$ by combining (5), (7) and (8) in the hope of deriving an upper bound for it.

However, for the case of $q = \infty$, one has $\|1(\cdot)\|_\infty = 1$ and, therefore, one can now isolate the term $\|h(g*x_1)\|_\infty$ by combining (5), (7) and (8), so as to obtain

$$\begin{aligned} &\|h(g*x_1)\|_\infty \\ &\qquad \leq \big(\|h\|_{\mathrm{Lip}(p,\infty)} - |h(0)|\big)\, \|g\|_1\, \|x_1\|_p + |h(0)|\,. \end{aligned}$$

Consequently, the Volterra system considered here is finite-gain input-output stable, even for $h(0) \neq 0$.

Now, returning to (6), with $h(0) = 0$ therein, and setting $p = q$, one obtains

$$\|h(g*x)\|_q \leq \|h\|_{\mathrm{Lip}(q,q)}\, \|g\|_1\, \|x\|_q\,, \tag{9}$$

for an arbitrary input signal $x \in L_q[0,\infty)$ with $1 \leq q \leq \infty$.

Comparing (9) with (3), it can be seen that the root (or powers) of v has been eliminated by including the factor consisting of the Lipschitz-norm of $h(\cdot)$ on the right-hand side. It is important to note that (9) and (3) give two alternative but *not* equivalent criteria for assessing the L_p-stability of a Volterra system. In fact, (3) does *not* require $h(\cdot)$ to be Lipschitz-continuous; it even applies to the extreme case of $h(\cdot)$ being L_q-integrable but nowhere-differentiable.

Case 4: Consider a two-input two-convolution Volterra system described by the equation

$$y := h(g_1 * x_1\,,\, g_2 * x_2)\,,$$

where the real-valued memoryless nonlinearity $h(\cdot,\cdot)$ is assumed to be Lipschitz-continuous and $g_1, g_2 \in L_1[0,\infty)$. The Lipschitz-norm considered here is

$$\|h\|_{\mathrm{Lip}(p_1,p_2,q)} := |h(0,0)| \quad + \\ + \sup_{(u_1,s_1)\neq(u_2,s_2)} \frac{\|h(u_1,s_1)-h(u_2,s_2)\|_q}{\|(u_1,s_1)-(u_2,s_2)\|_{p_1,p_2}}\,,$$

for fixed $1 \leq p_1,\, p_2,\, q \leq \infty$, and the direct-sum norm[3] $\|\cdot\|_{p_1,p_2}$, taken to be

$$\|(u_1,s_1)-(u_2,s_2)\|_{p_1,p_2} := \|u_1-u_2\|_{p_1} + \|s_1-s_2\|_{p_2}\,.$$

For arbitrary direct-sum signals (u_1, s_1) and (u_2, s_2), one has

$$\|h(u_1,s_1)-h(u_2,s_2)\|_q \leq \\ \big(\|h\|_{\mathrm{Lip}(p_1,p_2,q)} - |h(0,0)|\big)\big(\|u_1-u_2\|_{p_1} + \|s_1-s_2\|_{p_2}\big)\,.$$

Substituting

$$u_i := g_1 * x_{1i} \quad \text{and} \quad s_i := g_2 * x_{1i}\,, \qquad i = 1,\,2\,,$$

into this inequality then yields

$$\|h(g_1 * x_{11}\,,\, g_2 * x_{21}) - h(g_1 * x_{12}\,,\, g_2 * x_{22})\|_q \\ \leq \big(\|h\|_{\mathrm{Lip}(p_1,p_2,q)} - |h(0,0)|\big)\big(\|g_1\|_1\,\|x_{11}-x_{12}\|_{p_1} \\ + \|g_2\|_1\,\|x_{21}-x_{22}\|_{p_2}\big)\,,$$

where the linearity of convolution and inequality (2) were used.

Now, assuming that $h(0,0) = 0$ holds and then setting $x_{12} = x_{22} \equiv 0$, finally gives

$$\|h(g_1 * x_1\,,\, g_2 * x_2)\|_q \leq \|h\|_{\mathrm{Lip}(p_1,p_2,q)}\,\|g_1\|_1\,\|x_1\|_{p_1} \\ + \|h\|_{\mathrm{Lip}(p_1,p_2,q)}\,\|g_2\|_1\,\|x_2\|_{p_2}, \quad (10)$$

for arbitrary input signals $x_1 \in L_{p_1}[0,\infty)$ ($1 \leq p_1 \leq \infty$, fixed) and $x_2 \in L_{p_2}[0,\infty)$ ($1 \leq p_2 \leq \infty$, fixed) and for arbitrary $1 \leq q \leq \infty$, fixed. Thus, if the Lipschitz-norm of $h(\cdot)$ is finite, then this Volterra system is finite-gain input-output stable from the direct-sum space $L_{p_1}[0,\infty) \oplus L_{p_2}[0,\infty)$ to $L_q[0,\infty)$.

Case 5: If the inputs are joined by setting $x_2 = x_1$ in Case 4, then one obtains a single-input two-convolution Volterra system. From (10), it then follows that

$$\|h(g_1 * x_1, g_2 * x_1)\|_q \leq \|h\|_{\mathrm{Lip}(p_1,p_2,q)}\,\|g_1\|_1\,\|x_1\|_{p_1} \\ + \|h\|_{\mathrm{Lip}(p_1,p_2,q)}\,\|g_2\|_1\,\|x_1\|_{p_2}\,,$$

where, in general, $p_1 \neq p_2$. Under the assumption that $h(\cdot)$ has a finite Lipschitz-norm, the resulting Volterra system is finite-gain input-output stable from $L_{p_1}[0,\infty) \cap L_{p_2}[0,\infty)$ to $L_q[0,\infty)$. If, in addition $p_2 = p_1 \equiv p$, then (10) reduces to

$$\|h(g_1 * x_1\,,\, g_2 * x_2)\|_q \\ \leq \|h\|_{\mathrm{Lip}(p_1,p_2,q)}\,\big(\|g_1\|_1 + \|g_2\|_1\big)\,\|x_1\|_p$$

and this Volterra system is finite-gain input-output stable from $L_p[0,\infty)$ to $L_q[0,\infty)$.

[3] There are many other direct-sum norms to choose from.

Now, imposing power-v domination extends the earlier results.

Case 6: The last two more general cases easily generalize to a Volterra system with an arbitrary number of inputs, an arbitrary number of convolutions associated with each input, and an arbitrary number of outputs for both the cases of power-v domination and of Lipschitz-continuity of the output nonlinear memoryless mapping.

Finally, note that the exact same arguments presented above apply to the discrete-time case, thus producing equivalent results.

III. Conclusion

This paper presented some new and explicit stability results for Volterra systems when the output nonlinearity possesses a Lipschitz-norm as well as for the more general case of a measurable, monomially dominated output nonlinearity. These two approaches yield alternative but not equivalent stability criteria, thus providing more versatility by allowing selection of the more appropriate of the two for a given application.

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