

On the Calculation of 2D Green's Functions Via Sommerfeld Integrals and the Spatial Singularity Method

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Abstract

We provide a low-level review of the computation of Sommerfeld integration theory using the singularity expansion method (SEM) to analytically estimate the short-wavelength components of the 2-dimensional Green's function. The SEM is employed to replace the infinite tail of the spectral integral by a closed-form evaluation. The various steps in the SEM substitution and the calculations are elaborately presented and discussed with emphasis on giving the missing details often not included in the published literature.

On the Calculation of 2D Green's Functions Via Sommerfeld Integrals and the Spatial Singularity Method

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Abstract: We provide a low-level review of the computation of Sommerfeld integration theory using the singularity expansion method (SEM) to analytically estimate the short-wavelength components of the 2-dimensional Green's function. The SEM is employed to replace the infinite tail of the spectral integral by a closed-form evaluation. The various steps in the SEM substitution and the calculations are elaborately presented and discussed with emphasis on giving the missing details often not included in the published literature.

I. Introduction

Sommerfeld integrals arise in the problem of calculating Green's functions of electromagnetic sources in media containing open boundaries. For a source above stratified media, the spectral-domain Green's functions can be obtained in closed form. However, in order to obtain the corresponding spatial-domain Green's functions it is required to calculate the inverse Fourier transform. This process results in what is known as Sommerfeld integrals. The calculation of these integrals is very difficult due to the oscillatory nature of the integrand and the existence of branch points and surface pole (SWP) singularities. Discrete complex image theory (DCIT) or the spatial singularity expansion method (SEM) is a fast and efficient method that can be used to treat such problems [1-6]. Applications include computing the electromagnetic fields in cylindrical nanostructures [7,8], RF antenna system analysis and design [9], and near-field nano-optics [10].

II. Formulation of the Problem

In what follows, we assume familiarity with the concept of Green's function at the level of [11]. Consider the following spatial domain 2D Green's functions calculated using the inverse Hankel transform

$$G(\rho) = \frac{1}{2} \int_{-\infty}^{\infty} dk_{\rho} k_{\rho} J_0(k_{\rho} \rho) \tilde{G}(k_{\rho}) \quad (1)$$

where $\tilde{G}(k_{\rho})$ is the spectral-domain function. For review of the bessel functions $J_0(x)$ and the derivation of (1), see [11]. If the function $\tilde{G}(k_{\rho})$ is even then we can write the above integral as

$$G(\rho) = \int_0^{\infty} dk_{\rho} k_{\rho} J_0(k_{\rho} \rho) \tilde{G}(k_{\rho}) \quad (2)$$

The propagation constant k_z is given by

$$k_z = \sqrt{k_i^2 - k_{\rho}^2} \quad (3)$$

where k_i is the wavenumber in the i th medium. We now present the computation of the spectral integral above in the following hierarchical method.

II.a 1-Level Approach: The one-level approach consists of the following contour

$$C_{k_z} : k_z = k_i \left[-jt + \left(1 - \frac{t}{T_0} \right) \right] \quad 0 \leq t \leq T_0 \quad (4)$$

Fig. 1 shows how this contour is mapped from the complex k_z plane to the complex k_ρ . Here, we have

$$k_{\rho_{\max}} = k_i \sqrt{1 + T_0^2} \quad (5)$$

Thus, to insure that the deformed path C_ρ will include all the branch points, the surface wave poles (SWP), and the asymptotic behavior of the spectral domain GF when $k_\rho \rightarrow \infty$, it is important to choose $k_{\rho_{\max}}$ large enough.

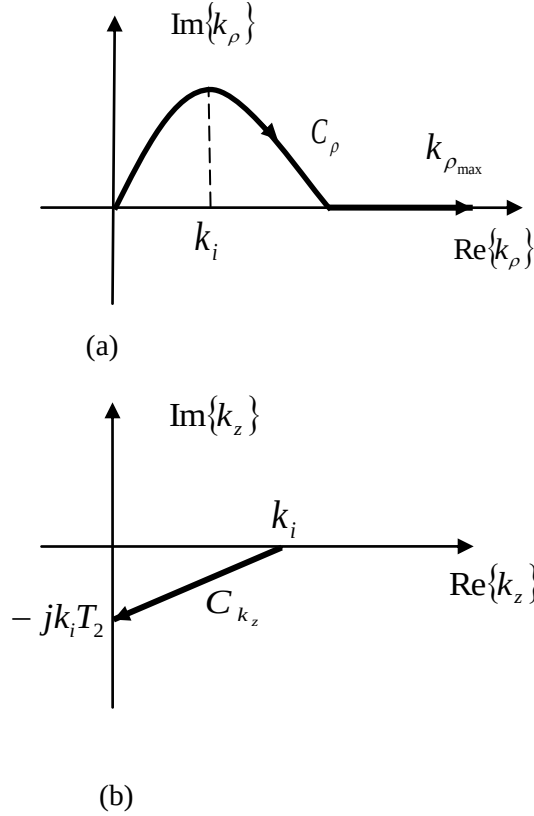


Fig. 1 1-level integration path in a) the k_ρ plane b) k_z plane

We will now approximate $jk_z \tilde{G}_{C_\rho}(k_\rho)$ using the singularity expansion method (SEM), powered by Prony's Method (PM) or the Generalized Pencil of Function (GPOF) Method

$$jk_z \tilde{G}_{C_\rho}(k_\rho) \approx \sum_{n=1}^N b_n e^{\beta_n t} \quad (6)$$

The reason of including the multiplicative factor jk_z in equation (6) will be apparent when we use Sommerfeld identity later. Alternatively, the equation (6) can be re-written in the following form

$$\tilde{G}_{C_\rho}(k_\rho) = \sum_{n=1}^N b_n \frac{e^{\beta_n t}}{jk_z} = \sum_{n=1}^N a_n \frac{e^{-\alpha_n k_z}}{jk_z} \quad (7)$$

where it is easy to show that

$$\alpha_n = \frac{\beta_n T_0}{k_i (1 + jT_0)} \quad a_n = b_n e^{k_i \alpha_n} \quad (8)$$

Substituting (7) into (2) we get

$$\begin{aligned} G(\rho) &= \int_0^\infty dk_\rho k_\rho J_0(k_\rho \rho) \sum_{n=1}^N a_n \frac{e^{-\alpha_n k_z}}{jk_z} \\ &= \sum_{n=1}^N a_n \int_0^\infty dk_\rho k_\rho J_0(k_\rho \rho) \frac{e^{-\alpha_n k_z}}{jk_z} \end{aligned} \quad (9)$$

The following form of Sommerfeld identity will now be used

$$\frac{e^{-jkr}}{r} = \int_0^\infty dk_\rho k_\rho J_0(k_\rho \rho) \frac{e^{-jk_z |z|}}{jk_z} \quad (10)$$

Equation (9) become then

$$G(\rho) = \sum_{n=1}^N a_n \frac{e^{-jk_r}}{r_n} \quad (11)$$

where we have

$$r_n = \sqrt{\rho^2 - \alpha_n^2} \quad (12)$$

which is the distance – generally complex - of the n th image.

II.b 2-Level Approach: The two-level DCIT approach consists of sampling over the following two parameterized integration paths

$$\begin{aligned} C_1 : k_z &= -jk_i [T_2 + t_1] & 0 \leq t_1 \leq T_1 \\ C_2 : k_z &= k_i \left[-jt + \left(1 - \frac{t_2}{T_2} \right) \right] & 0 \leq t_2 \leq T_2 \end{aligned} \quad (13)$$

The corresponding path in the k_ρ complex plane is shown in Fig. 1. It is clear that a linear variation in the k_z plane shown in Fig. 1(a) is translated in the k_ρ plane to the deformed path shown in Fig. 1(b). Notice that $k_{\rho_{\max}} = k_i \sqrt{1 + T_2^2}$.

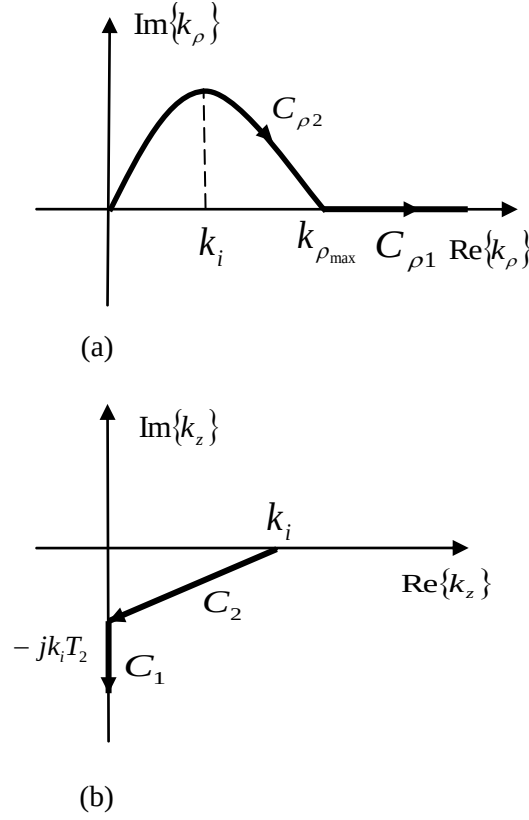


Fig. 2 2-level Integration path in a) the k_ρ plane b) k_z plane

The Prony's method (PM), or the Generalized Pencil of Function (GPOF) method, can be utilized to expand an arbitrary function in terms of sum of complex exponentials. We utilize now this method in order to calculate the Sommerfeld integral of equation (2).

First, approximate the "tail" region of the spectral domain function, i.e. the region over the contour $C_{\rho1}$, in the following way

$$jk_z \tilde{G}_{C_{\rho1}}(k_\rho) \approx \sum_{n=1}^{N_1} b_{1n} e^{\beta_{1n} t_1} \quad (14)$$

Next, putting the above in a form suitable for the future use of Sommerfeld identity, we write

$$\tilde{G}_{C_{\rho1}}(k_\rho) = \sum_{n=1}^{N_1} b_{1n} \frac{e^{\beta_{1n} t_1}}{jk_z} = \sum_{n=1}^{N_1} a_{1n} \frac{e^{-\alpha_{1n} k_z}}{jk_z} \quad (15)$$

where it easy to find that

$$\alpha_{1n} = \frac{\beta_{1n}}{jk_i}, \quad a_{1n} = b_{1n} e^{-jk_i \alpha_{1n} T_2} \quad (16)$$

Next, the function approximated in (15) is subtracted from the original spectral domain function $\tilde{G}(k_\rho)$. The result will be non-zero only on the contour $C_{\rho2}$. The total integration can be then written as

$$\tilde{G}(k_\rho) = \int_{C_{\rho 2}} dk_\rho k_\rho J_0(k_\rho \rho) [\tilde{G}(k_\rho) - \tilde{G}_{C_{\rho 1}}(k_\rho)] + \int_{C_{\rho 1} + C_{\rho 2}} dk_\rho k_\rho J_0(k_\rho \rho) \tilde{G}_{C_{\rho 1}}(k_\rho) \quad (17)$$

Now, the spectral domain function in the first integral is approximated using PM or GPOF over the path $C_{\rho 2}$ as follow

$$jk_z [\tilde{G}(k_\rho) - \tilde{G}_{C_{\rho 1}}(k_\rho)] \approx \sum_{n=1}^{N_2} b_{2n} e^{\beta_{2n} t_2} \quad (18)$$

where this yield

$$\tilde{G}(k_\rho) - \tilde{G}_{C_{\rho 1}}(k_\rho) = \sum_{n=1}^{N_2} b_{2n} \frac{e^{\beta_{2n} t_2}}{jk_z} = \sum_{n=1}^{N_2} a_{2n} \frac{e^{-\alpha_{2n} k_z}}{jk_z} \quad (19)$$

and again it is easy to deduce that

$$\alpha_{2n} = \frac{\beta_{2n} T_2}{k_i (1 + jT_2)}, \quad a_{2n} = b_{2n} e^{k_i \alpha_{2n}} \quad (20)$$

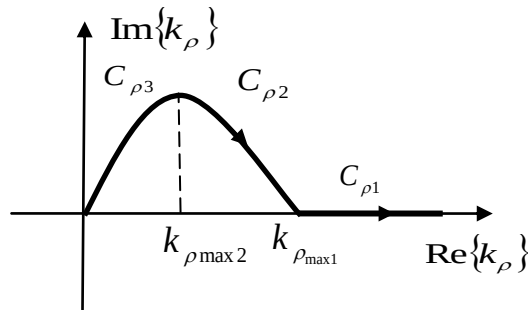
By substituting into equation (17) the approximations (15) and (19) and utilizing Sommerfeld identity in (10) we get

$$G(\rho) = \sum_{n=1}^{N_1} a_{1n} \frac{e^{\alpha_{1n} r_{1n}}}{r_{1n}} + \sum_{n=1}^{N_2} a_{2n} \frac{e^{\alpha_{2n} r_{2n}}}{r_{2n}} \quad (21)$$

$$r_{(1,2)n} = \sqrt{\rho^2 - \alpha_{(1,2)n}^2} \quad (22)$$

III.c 3-Level Formulation

The 3-level formulation follows the basic idea developed in the previous section. Fig. 3 illustrates the three paths



(a)

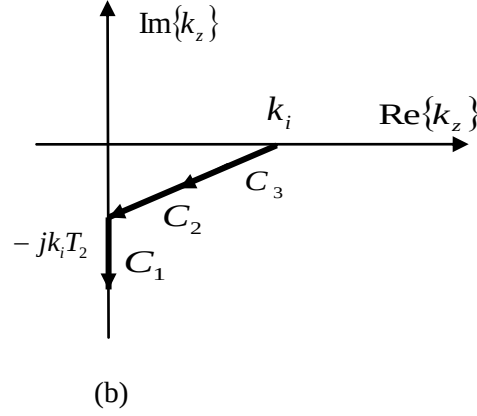


Fig. 3 3-level Integration path in a) the k_ρ plane b) k_z plane

The integration path in the region $[0, k_{\rho \max 1}]$ is divided into two parts, $[0, k_{\rho \max 2}]$ and $[k_{\rho \max 2}, k_{\rho \max 1}]$. The reason of doing this is the fact that for some problems the spectral domain Green's functions may vary considerably in one region while being smooth in other regions. Since the PM or the GPOM work only for uniform sampling, they can not be utilized directly in sampling the function unless a huge number of points is used (may be thousands). To avoid this, we divide the main region $[0, k_{\rho \max 1}]$ into two sub-regions where higher number of samples will be used in each region. Because the sub-regions are smaller than the original interval, the “higher” number of samples here is still less what would be required if the entire region is sampled once using uniform sampling. The suggested contour path will be given in the following way

$$\begin{aligned}
 C_1 : k_z &= -jk_i [T_2 + t_1] & , 0 \leq t_1 \leq T_1 \\
 C_2 : k_z &= k_i \left[-j(t_2) + \left(1 - \frac{t_2}{T_2} \right) \right] & , T_1 \leq t_2 \leq T_2 \\
 C_3 : k_z &= k_i \left[-jt_3 + \left(1 - \frac{t_3}{T_2} \right) \right] & , 0 \leq t_3 \leq T_3
 \end{aligned} \tag{23}$$

We first approximate the spectral domain function in the first region as follow

$$f_1(k_\rho) = \tilde{G}_{C_{\rho 1}}(k_\rho) = \sum_{n=1}^{N_1} b_{1n} \frac{e^{\beta_{1n} t_1}}{jk_z} = \sum_{n=1}^{N_1} a_{1n} \frac{e^{-\alpha_{1n} k_z}}{jk_z} \quad , k_\rho \in [0, k_{\rho \max 2}] \tag{24}$$

where α_{1n} and a_{1n} are given by equations (16).

Next, we approximate the following function in the interval $[k_{\rho \max 2}, k_{\rho \max 1}]$ as follow

$$f_2(k_\rho) = \tilde{G}_{C_{\rho 2}}(k_\rho) = \sum_{n=1}^{N_2} b_{2n} \frac{e^{\beta_{2n} t_2}}{jk_z} = \sum_{n=1}^{N_2} a_{2n} \frac{e^{-\alpha_{2n} k_z}}{jk_z} \quad , k_\rho \in [k_{\rho \max 2}, k_{\rho \max 1}] \tag{25}$$

where α_{2n} and a_{2n} are given by (20).

We now approximate the following function over the interval $[k_{\rho\max 2}, \infty]$

$$f_3(k_\rho) = \tilde{G}(k_\rho) - f_1(k_\rho) - f_2(k_\rho) = \sum_{n=1}^{N_3} b_{3n} \frac{e^{\beta_{3n} t_2}}{jk_z} = \sum_{n=1}^{N_3} a_{3n} \frac{e^{-\alpha_{3n} k_z}}{jk_z}, \quad k_\rho \in [0, k_{\rho\max 2}] \quad (26)$$

where α_{2n} and a_{2n} are also given by (20).

The inverse Hankel transformation can be written then as

$$\tilde{G}(k_\rho) = \int_{C_\infty} dk_\rho k_\rho J_0(k_\rho \rho) [\tilde{G}(k_\rho) - f_2(k_\rho) - f_1(k_\rho)] + \int_{C_\infty} dk_\rho k_\rho J_0(k_\rho \rho) [f_1(k_\rho) + f_2(k_\rho)] \quad (27)$$

Now, since

$$\tilde{G}(k_\rho) - f_1(k_\rho) - f_2(k_\rho) = 0, \quad k_\rho \in [0, k_{\rho\max 2}]$$

we can write (27) as

$$G(\rho) = \int_{C_3} dk_\rho k_\rho J_0(k_\rho \rho) [\tilde{G}(k_\rho) - f_2(k_\rho) - f_1(k_\rho)] + \int_{C_\infty} dk_\rho k_\rho J_0(k_\rho \rho) [f_1(k_\rho) + f_2(k_\rho)] \quad (28)$$

which is the Green's function in the spatial domain expressed in terms of mixed spectral and closed-form analytical expression.

III. Conclusion

We provided a pedagogically illuminating reexamination of how to compute Sommerfeld integrals using the singularity expansion method. The derivation was given with high degree of details to explain how each integration interval was handled in the spectral domain. The expression (28) is much faster to evaluate than a fully-fledged numerical calculation over the entire spectral domain. The reduction of cost clearly comes from the fact that the short-wavelength components have been approximated analytically using the deeper SEM poles and residues information buried in the high spatial frequency band. Physically, it appears that the SEM data (poles and residues) represent special spectral points of major singularities that capture the overall (global) behavior of the Green's functions [12].

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