Multivaluedness in Networks: Theory

Anton van Wyk [1]

[1]The University of the Witwatersrand

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Abstract

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As an application of these theoretical results, we demonstrate how these aid in the structural complexity reduction of directed complex networks.

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Michaël Anton(ie) van Wyk

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As an application of these theoretical results, we demonstrate how these aid in the structural complexity reduction of directed complex networks.

Index Terms—**Big data, cascaded systems, complex networks, counter-cascaded systems, distributed measurement systems, functional uniformization, immanence, mixed modeling, multivaluedness, multivalued function, multivalued relation, network analysis, network science, networked systems, neural networks, node rationalization, nonlinear systems, structural reduction, transcendence, single-valuedness, single-valued relation, well-defined mapping.**

I. Introduction

The area of Network Analysis and Complex Networks has rapidly expanded into a very active and vibrant field of research, with ever more fundamental theoretical results and novel applications being reported. In order to give a glimpse of the diverse nature of the objects of study, i.e. complex networks, note that size-wise, real-world networks range from a few nodes to billions of nodes and beyond, with some nodes interacting unilaterally and others bilaterally. Structure-wise, they range from highly homogeneously structured networks through to amorphously unstructured and even randomly structured networks. Character-wise, they vary from uniformly cooperative or competitive to heterogeneously mixed with cooperative and competitive factions contained within. Furthermore, the mathematical descriptions of nodes in a network range from uniform (identical) in some networks, to diverse (distinct) in others. For these reasons, graph-theoretical methods are indispensable for description and analysis of network problems. The most important characteristic of all real-world networks, is that they are in a constant state of flux, with at least some of these characteristics, if not all, evolving.

In the literature, the meaning of the term "network analysis" is rather diverse. Of particular interest here, is the extended definition of Zaidi [1], namely that it encapsulates the study of theory, methods and algorithms applicable to graph-based models representing interconnected real-world systems. From this perspective, the collection of interconnected elements of a finite element analysis of a distributed structure or physical field and a complex interconnection of nonlinear dynamical systems are instances of complex network analyses [2][3], the former undirected and the latter directed. Both an excellent account of the theory and overview of current research directions in complex networks, can be found in [4].

Even though directed complex networks might not always have explicit inputs (causes) and outputs (effects), there are always internal (i.e. local) inputs and outputs of interest when considering a single node or a collection of nodes. A deeper understanding of the global behavior and dynamics of a complex network usually requires a deeper understanding of the mechanisms of behavior at a more detailed level in the network. For this reason, oftentimes it requires one to relate two (sets of) effects $v \in V$ and $x \in X$, produced by the common cause $u \in U$ (as in Fig. 1), in order to gain deeper insight. This aspect, termed *counter-cascaded* systems, is the subject of this note. In particular, minimal underlying assumptions and an elementary set-theoretic argument, produce a necessary and sufficient condition for such a relation to be well-defined.

The outline of this paper is as follows: Section II presents the mathematical foundations: basic assumptions, definitions, the main result, its proof and some immediate consequences. By virtue of an example, Section III demonstrates the application of these results in structural reductions of directed complex networks, referred to as *functional uniformization* and *node rationalization*. The conclusion follows in Section IV.

II. Theory: Immanence versus Transcendence

In order to provide a definite and concrete context for the presentation and discussion, we consider (directed) complex networks of complexly interconnected nonlinear dynamical systems.[3] In such networks, we will study occurrences of counter-cascaded systems appearing in the framework shown in Fig. 1. Importantly, note that this framework includes *both* counter-cascaded ($M \neq I$) and cascaded ($M = I$) systems configurations; I denotes the identity mapping. Therefore, the

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Michaël A. van Wyk is with the School of Electrical and Information Engineering, The University of the Witwatersrand, Johannesburg, South Africa and City University of Hong Kong, Hong Kong SAR, China (e-mail: anton.vanwyk@wits.ac.za, mavanwyk@gmail.com).

[1] In the note, the term "system" refers to *any* object that can be satisfactorily approximated by some binary relation. The class of all dynamical systems is proper subclass of the class of all systems considered here.

[2] In Fig. 1 the systems T and M are *counter-cascaded systems* with common input u, considering outputs v and w; similarly are $N \circ T$ and M, considering outputs x and w.

[3] For economy of presentation, throughout, the *same* symbol is used for a system and its mathematical model; these are usually *not* the same.

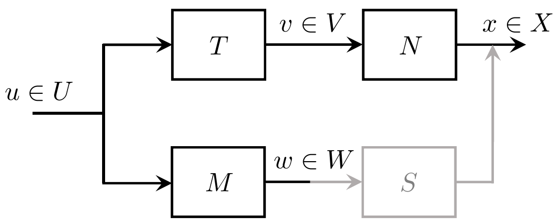

Fig. 1. Two counter-cascaded paths with the common cause, $u \in U$. The implicit relation S is shown in gray.

results obtained here are applicable to both of these configurations. Generally U, V, W and X can be very general sets, with $M : U \to W$, $T : U \to V$ and $N : V \to X$ mappings. For the selected context, unless stated otherwise, these mappings are nonlinear operators[4] with domains and ranges real vector spaces; typically T is a nonlinear operator describing some nonlinear system, with M and N nonlinear operators describing input and output ancillary systems, respectively, either other nonlinear systems or even identity operators. When they describe ancillary systems e.g. measurement systems, M and N are usually many-to-one mappings.

For the purpose of our presentation here, N is redundant and can be absorbed into T by replacing T with $N \circ T$. However, for applications of this work in other areas, it has a distinct and explicit purpose, as will be reported on in the future. Finally, the implicit relation $S \subset W \times X$ will either be single-valued (i.e. a mapping) or multivalued, depending on the problem considered.

Next follows an important definition:

Definition II.1. (Immanence, Transcendence) *In Fig. 1, the mapping T is called* immanent with respect (or relative) to the ordered pair of mappings (M, N)*, if for every element $w \in M(U)$, there exists an*[5] *element $x \in N(T(U))$ such that $T(M^{-1}(w)) \subseteq N^{-1}(x)$.*

If not immanent with respect to (M, N), then T is called transcendent *with respect to (M, N).*

Notes.

a. The statement "T is (M, N)-immanent" is often used to mean "T is immanent with respect to (M, N)" and similarly for transcendence.
b. Two nodes in a complex network, can be analyzed for immanence or transcendence *only if* they are affected by the very same cause.
c. Since collections of nodes can be clustered to form supernodes, which are themselves nodes, this definition and all subsequent results apply to supernodes without explicit further mention.

The main result and its proof follows next:

[4]In order to emphasize that the results presented here apply in much more general contexts, the term "mapping" will be used instead of "operator."

[5]If such x exists, then it is unique. To see this suppose that two such elements x_1 and x_2 exist, implying that $N^{-1}(x_1) \bigcap N^{-1}(x_2) \neq \emptyset$. Now, applying N to this nonempty intersection immediately yields $x_1 = x_2$.

Theorem II.2. (Well-Defined Mapping) *The mapping T is immanent relative to (M, N) if and only if $N \circ T \circ M^{-1}$ is well-defined (i.e. single-valued).*[6]

Note. Before proceeding with the proof, first observe that for each element $u \in U$, there exist elements $w_u := M(u)$ and $x_u := N(T(u))$. Next, we associate w_u and x_u by writing $x_u = S(w_u)$ for every $u \in U$. This can be compactly expressed as $S := N \circ T \circ M^{-1}$. Here, S defines a relation. If for every pair of distinct elements $u_1, u_2 \in U$ we have that $w_{u_1} = w_{u_2}$ implies that $x_{u_1} = x_{u_2}$, then S is well-defined.

Proof. We first prove the "only if" part. Suppose that T is immanent with respect to (M, N). Now, if S is not well-defined, then there exist at least two distinct elements $u, u' \in U$ such that $M(u) = M(u')$ but $x := N(T(u)) \neq N(T(u')) =: x'$. This contradicts the consequence of immanence, namely that $N(T(u)) = x$ for all $u \in M^{-1}(w)$ and consequently S is well-defined.

Conversely, to prove the "if" part, suppose T is transcendent with respect to (M, N). Then, for some $w \in W$, there are distinct elements $u, u' \in M^{-1}(w)$ for which $x := N(T(u)) \neq N(T(u')) =: x'$, implying that S is not well-defined because $S(w) = x$ and $S(w) = x'$ and yet $x \neq x'$. This concludes the converse via the contrapositive and completes the proof. $\square$

An equivalent statement of this result follows:

Theorem II.3. (Multivalued Relation) *The mapping T is transcendent relative to (M, N) if and only if $N \circ T \circ M^{-1}$ is not well-defined (i.e. multivalued).*

To our knowledge this result, identifying all those situations when the outputs of two counter-cascaded subsystems are functionally related (as well as when not), is a novel result.

Some immediate consequences of Theorem II.2 now follow.

Corollary II.4. (Existence of a Unique Faithful Model) *For a given mapping T, a unique faithful* model *or* modeling mapping *S exists if and only if T is (M, N)-immanent.*

For modeling problems, if T is (M, N)-immanent, then there exists a unique mapping $w \mapsto S(w)$ which yields a unique faithful model of T, as perceived through M and N, that is, $S(w) = N \circ T \circ M^{-1}(w)$ for every $w \in W$. In this case, T can be fully explained by the combined abilities of the pair (M, N) and explains the use of the term "immanence" here. On the other hand, if T is (M, N)-transcendent, then $S = N \circ T \circ M^{-1}$ is a (multivalued) relation which *cannot* be described by any mapping, whatsoever, and hence *no* faithful model exist; a "full explanation" of T is beyond the combined abilities of the pair (M, N), justifying use of the term "transcendent."

Still concerning modeling problems, approximation of S becomes essential in the following situations: either T is

[6]Here, M^{-1} denotes the *preimage* of M.

(M, N)-transcendent, or T is (M, N)-immanent but the solution $N \circ T \circ M^{-1}$ is excluded from the allowable set of candidate model mappings, based on prior considerations. In these situations, the only remaining course of action is to select a qualifying approximation $\hat{S}_{opt} : W \to X$ that approximates S optimally according to some nonnegative-valued optimality criterion $J(e)$, with the error-relation defined by $X \supset e(t, \cdot) := S(t, \cdot) - \hat{S}(t)$, for all t.[7]

Corollary II.5. *Let M, T, N and S be as depicted in Fig. 1.*

a. *If M is given and T is of* canonical form $T = F \circ M$ *for some fixed $F : W \to V$, then T is (M, N)-immanent for every N.*
b. *If M is invertible then every T is (M, N)-immanent for every N.*
c. *If M is many-to-one, T is not of canonical form and N is one-to-one, then T is (M, N)-transcendent.*
d. *If the condition for (M, N)-immanence holds everywhere in $B \subseteq U$, then $T|B$ is $(M|B, N)$-immanent.*[8]
e. *If M is one-to-one, $N \circ T$ is many-to-one and T is (M, N)-immanent, then the mapping S is many-to-one.*
f. *If mappings M, N and S are given and there exists a T satisfying the identity $N \circ T = S \circ M$, then T is (M, N)-immanent. As a candidate solution, $N^{-1} \circ S \circ M$ is a well-defined mapping if and only if N is invertible.*

Notes.

a. If M is many-to-one and T is *not* of the canonical form $T = F \circ M$, then the (M, N)-immanence of T depends on the choice of N.
b. In Corollary II.5(f), if N is invertible, then the expression $N^{-1} \circ S \circ M$ gives an explicit formula for T. However, if N is not invertible then T satisfies the expression $N^{-1} \circ N \circ T = N^{-1} \circ S \circ M$ which is generally not solvable for T since $N^{-1} \circ N \neq I$. So, unless additional information about T is available, we can merely test candidate mappings T to determine if they satisfy this expression; if one does, it follows immediately that T is (M, N)-immanent.

Now, a little thought reveals the following to be true for configurations similar to that shown in Fig. 1, but with additional exogenous inputs entering:

Lemma II.6. (Resolution of Exogenous Inputs) *Suppose that, along one of the paths of a counter-cased pair, another cause enters. Proceed to adjoin this cause to the original cause and adjoin the identity operator to the up-stream (i.e. unaffected) portion of the affected path, both by direct-sum operations. After this transformation, the mappings M and T may now be defined as before.*

[7]When S is multivalued, then the difference in the expression for $e(t, \cdot)$ refers to the value $\hat{S}(t)$ subtracted from every element in the set $S(t, \cdot)$. When S is single-valued, $e(t, \cdot) \equiv e(t)$ is a real number for each t.

[8]Here, $T|B$ is the *restriction* of T to the subset B of its domain U.

In the case of counter-cascaded systems, there are two possible directions to be considered for immanence or transcendence. The next definition expands on the previous definition to cover both possibilities. For this, N is removed by choosing it to be the identity mapping.

Definition II.7. (Bi-immanence, Bi-transcendence) *In Fig. 1, if T is (M, I)-immanent and M is (T, I)-immanent, then T and M are called* bilaterally immanent *or* bi-immanent*. Similarly, if T and M are (M, I)-transcendent and (T, I)-transcendent, respectively, then T and M are called* bilaterally transcendent *or* bi-transcendent*.*

Note. Considering both directions for two counter-cascaded systems, in principle, all four of the following cases are possible: immanent-immanent (I-I), immanent-transcendent (I-T), transcendent-immanent (T-I) and transcendent-transcendent (T-T). For the case I-I the mapping relating the outputs is a bijection while, for the case T-T, there is no mapping that relates the two outputs in either direction. The case I-T implies that such a mapping exists in one direction but not in the other; similarly for the case T-I.

III. Application: Node Rationalization

The results of the previous section will now be applied to structural reduction in directed complex networks. This demonstrates the use of Theorem II.2 and its consequences as a high level signals-and-systems type result.

We start by introducing the necessary terminology. The process of expressing node mappings in factored form, with the right-most factors chosen from as few as possible unique ones, will be referred to as *functional uniformization*. Furthermore, the process of minimizing the number of nodes in a functionally uniformized network, by merging as many nodes as is possible to share common right-most factors and (node) inputs, will be termed *node rationalization*. This form of structural transformation of a network results in a reduction in the number of nodes, with each of the newly formed nodes having either multiple inputs or multiple outputs or both. The resulting network has fewer connected dynamical systems, represented by nodes, with vector (i.e. parallelized) edges joining them.

Consider the simple yet general four-node directed geographic network with immediate neighbor interaction, shown in Fig. 2. As before, the same symbol is used to present both the node and the mathematical model describing its behavior, i.e. its mathematical description. For example, A identifies the upper-most node in Fig. 2; it also represents the mathematical mapping $A(\cdot, \cdot)$ that describes this node's behavior.

Unless additional information is available, no structural reduction of this network is possible. So, suppose that C is (A, I)-immanent. Then, according to Corollary II.4, there exists a modeling mapping $\overline{C}$, as indicated in Fig. 3.[9] Following

[9]In order to simplify matters, during the intermediate steps that follow, we relate back to earlier theoretical results by using the systems representation employed before.

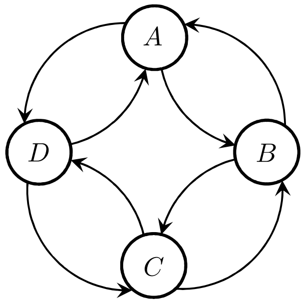

Fig. 2. A directed geographical network immediate neighbor interaction.

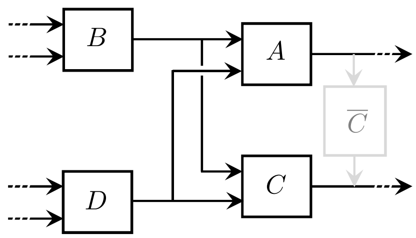

Fig. 3. Node C is (A, I)-immanent with model $\overline{C}$.

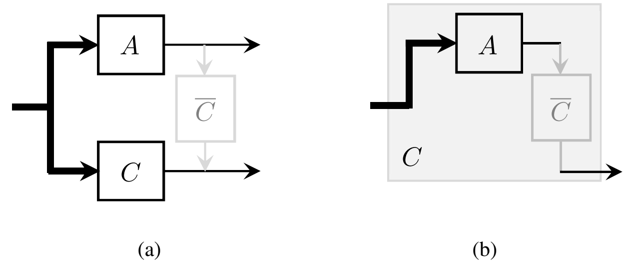

Fig. 4. Node C is (A, I)-immanent with model $\overline{C}$. (a) Nodes A and C with the common vector input. (b) Node $C = \overline{C} \circ A$ by virtue of Corollary II.4.

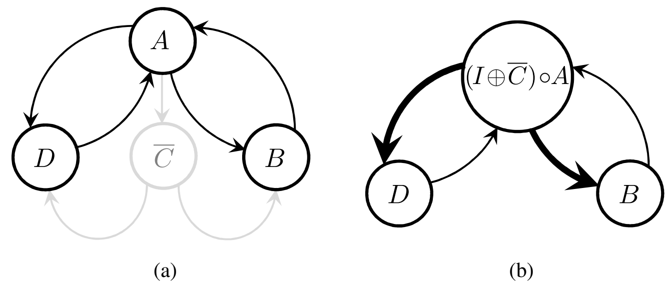

Fig. 5. Reduced network: (a) With node C represented by $\overline{C} \circ A$ thus sharing the existing node A. (b) With the outputs of the new node consolidated.

Lemma II.6, we can adjoin the inputs feeding into nodes A and C to obtain a common vector input feeding into both A and C, as depicted by the bold line in Fig. 4(a). Fig. 4(b) shows that the mapping C can be replaced by the composition $\overline{C} \circ A$ as follows from Corollary II.4. This means that we can now replace node C of the network with a "node" $\overline{C}$ which has a single input, fed by the output of node A as shown in Fig. 5(a). The output of node $\overline{C}$ then replaces the output of node C, feeding into nodes B and D (Fig. 5(a)). To reduce this network to a three-node network requires us to merge A and $\overline{C}$ into a single node with mathematical description $(I \oplus \overline{C}) \circ A$ yielding a vector output which feeds into nodes B and D via the bold edges in the graph shown in Fig. 5(b). The symbol $\oplus$ represents the direct sum operation.

Now, if there were no further immanence present in the network, then Fig. 5(b) shows the simplest network to which the original network can be structurally reduced, using node rationalization.

Next, in addition, assume that D is (B, I)-immanent. Once again, according to Corollary II.4, there exists a modeling mapping $\overline{D}$ such that $D = \overline{D} \circ B$. Following the same procedure as above, the network can now be reduced to the form shown in Fig. 6(a)—effectively the two-node uniformized network shown in Fig. 6(b). To see this simply define the two nodes to have the mathematical descriptions[10] $(I \oplus \overline{C}) \circ A$ and $(I \oplus \overline{D}) \circ B$, respectively, resulting in the interconnecting edges to become vector-valued. The final result consists of only two coupled systems. This reduction is striking, considering the generality of the mathematical descriptions of the four nodes of the original network.

To summarize, if our example complex network was completely void of immanence, i.e. all counter-cascaded node pairs were transcendent, then by Theorem II.2, the original network would not in any way have been structurally reducible using node rationalization. In other words, except perhaps for cosmetic changes, Fig. 2 would then represent the *simplest* form possible for this network and imposing node rationalization in such a case, would yield unavoidable and unresolvable modeling errors, an inevitable consequence of transcendence.

A note about bi-immanence and bi-transcendence is in order: for our example here, if bi-immanence was present, then we would have had the option to interchange the roles of relevant nodes, thus giving us more options while producing equivalent results.

In conclusion we point out that, in this example application, we stretched the presence of immanence to the limit in order to demonstrate the compactness of representation produced by the node rationalization. However, in real-world complex network investigations, node rationalization will usually only be applied selectively to expose important latent properties that would otherwise have gone unnoticed.

IV. Conclusion

Based on the novel concepts of immanence and transcendence, this note presented a theoretical result in Theorem II.2,

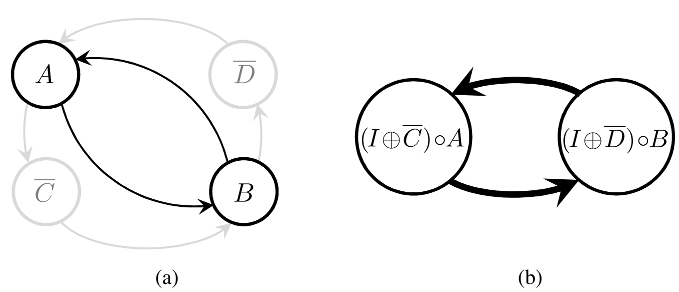

Fig. 6. Uniformized network: (a) With the explicit factorizations $\overline{C} \circ A$ and $\overline{D} \circ B$ explicitly shown. (b) With the outputs of the new nodes consolidated.

[10]For economy of presentation and for readability, we represent the two identity mappings I_A and I_B, operating on the ranges of A and B, respectively, using the same symbol, namely I.

specifically a necessary and sufficient condition for implicit multivaluedness when relating two outputs (effects) of nodes of a directed network in a counter-cascaded configuration. Subsequent corollaries provide further useful results for determining multivaluedness, given specific conditions. The configuration framework considered for this theoretical development, includes both the cases of counter-cascaded and cascaded node configurations which implies that these results can be applied to both these.

For the particular class of modeling problems considered by Corollary II.4, transcendence is an adverse characteristic. However, some work currently in progress suggests that transcendence is not necessarily always undesirable.

Next, a systems-and-signals type application, namely network uniformization via node rationalization, applied to a simple yet very general four-node geographic complex network, was presented. This demonstrated these results' potential to contribute toward the arsenal of tools for studying complex networks as well as network and systems in general.

Further work is in progress to apply these theoretical results to distributed measurement systems and mixed modeling in networks, big data and neural networks and analysis of signal processing algorithms. Also underway, is work to extend these theoretical results to include the case of noise contamination in networks.

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