

A Fiber Bundle Space Theory of Nonlocal Metamaterials

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Abstract

It is proposed that spacetime is not the most proper space to describe metamaterials with nonlocality. Instead, we show that the most general and suitable configuration space for doing electromagnetic theory in nonlocal domains is a proper function-space infinite-dimensional (Sobolev) vector bundle, a special case of the general topological structure known as fiber bundles. It appears that this generalized space explains why nonlocal metamaterials cannot have unique EM boundary conditions at interfaces involving spatially dispersive media.

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Abstract. The main goal of the present paper is to provide a fresh general view on the resurgent research area of nonlocal metamaterials. We explain how the concept of nonlocality had risen historically in the field of plasma and crystal optics and highlight the recent interest in using nonlocality as a new type of metamaterials. The paper will develop in increasing complexity the concept of nonlocality starting from general considerations, going through spatial dispersion, and ending up with an abstract but quite broad formulation that unveils the link between general topology and electromagnetic nonlocality in material media. It is shown that electromagnetic nonlocality naturally leads to a Banach (vector) bundle structure serving as an enlarged space on which electromagnetic processes take place. The added structures, essentially fiber space, model the topological microdomains of electromagnetic nonlocality and provide a fine-grained picture of field-matter interactions in nonlocal metamaterials. We use standard techniques borrowed from differential topology to construct the Banach bundle structure by paying careful attention to the relevant physics. The electromagnetic response tensor is then reformulated as a bundle homomorphism and the various expressions needed to connect the local topology to global domains are developed. The paper also introduces a discussion of various applications, including elucidating why nonlocal electromagnetic materials often require additional boundary conditions or extra input from microscopic theory in comparison with local electromagnetics.

1. Introduction

In classical electromagnetics, there are no nonlocal interactions or phenomena in vacuum because Maxwell's equations, which capture the ultimate content of the physics of electromagnetic fields, are essentially local differential equations [1]. An effect applied at point $\mathbf{r}$ in space will first be felt at the same location but then spread or propagate slowly into the *infinitesimally* immediate neighborhood.¹ Long-term disturbances such as electromagnetic waves propagate through both vacuum and material media by cascading these infinitesimal perturbations in outward directions (rays or propagation paths) emanating from the time-varying point source that originated the whole process. On the other hand, *nonlocal* interactions differ from this vacuum-like picture in allowing fields applied at position $\mathbf{r}'$ to influence the medium at *different* location $\mathbf{r}$, i.e., a location that is *not infinitesimally* close to

¹It is commonly accepted that Aharonov-Bohm effects, which lead to observable nonlocal electrodynamic effects, have their origin in quantum physics. In this paper we focus on *classical* field theory described using phenomenological models. Some authors, however, suggest that classical electromagnetics may induce nonlocal effects but such topic is outside the scope of the present paper.

the source position $\mathbf{r}'$. The distance $|\mathbf{r} - \mathbf{r}'|$ could be very small in most media (and certainly zero in vacuum), but in some types of materials, the so-called *nonlocal media*, observable response can be found such that the “radius of nonlocality” $|\mathbf{r} - \mathbf{r}'|$ is appreciably different from zero [2–4]. The main goal of the present paper is to explore the conceptual foundations of nonlocality in connection with applied electromagnetics, especially the potential of using nonlocal media as a new generation of metamaterials in antenna and circuit systems. Our main approach is conceptual and theoretical. While a massive amount of numerical and experimental data on all types of nonlocal materials abound in literature since the 1950s, the purpose of the present paper is attaining some clear understanding of the essentials of the subject, particularly in connection with the ability to build a very general formalism of nonlocal electromagnetics that can help foster new numerical methods and applications. The key motivation for our work is explicating the subtle but important difference between *infinitesimal interactions*, which characterize local electromagnetics, and interactions occurring in small *topological neighborhoods* around the observation point. We believe that this topological difference has not received the attention it deserves in the growing theoretical and methodological literature on nonlocal media. In particular, the author believes that a majority of present approaches to nonlocal metamaterials conflate the topologically nonlocal domain of neighborhoods and global domains. However, general topology and much of modern mathematical physics is based on clearly distinguishing the last two topological levels. It turns out that the standard formalism of local electromagnetics, which is based on spacetime points and their *differential* but not topological neighborhoods as the basic configuration space of the problem, is *not* the most natural or convenient framework for formulating the electromagnetics of nonlocal materials. This is mainly because the physics-based domain of electromagnetic nonlocality (to be defined precisely below), which captures the effective region of field-matter nonlocal interactions, is not usually built into the mathematical formalism of classical boundary value problems in applied electromagnetics. By investigating the subject, we will show that a natural space for conducting nonlocal metamaterials research is the *vector bundle* structure, more specifically, a Banach bundle [5] where every element in the fiber space is a vector field on the entire domain of nonlocality. The derivation of the various vector bundle structures starting from a generic phenomenological model of electromagnetic nonlocality is the main contribution of the present work.

We first provide a non-exhaustive and selective review of the development of nonlocal electromagnetic materials research.² Some of the physical phenomena that cannot be understood using local electromagnetic theory include spatial dispersion effects, superconductivity, natural optical activity [4]. Outside electromagnetics but within wave phenomena, there also exists processes that cannot be fully accounted for through simple local material models, for instance, we mention phase transitions and streaming birefringence [6]. By large, *spatial dispersion* has attracted most of the attention of the various research communities working on nonlocal electromagnetic materials. Indeed, few book-length researches on spatial dispersion already exist in literature, most notably [2–4, 7]. Moreover, several other researches conducted within condensed-matter physics and material science implicitly or explicitly assume that nonlocality is essentially based on microscopic (hence quantum) processes, and develop

²More information and proposals regarding engineering applications are given in Sec. 66.2, where additional references can be found.

an extensive body of work where the spatially dispersive dielectric tensor is deployed as the representative constitutive material relation [8–12]. On the other hand, one can also treat nonlocality without resort to spatial dispersion by modeling certain classes of material media as periodic structures [13] where the susceptibility tensor is derived from the symmetry of the overall structure [8, 14, 15] or from the lattice dynamics approach [16, 17]. For solving nonlocal problems, several methods exist in order to deal with the lack of universal model at the interface between a nonlocal material and another medium. The Additional Boundary Condition (ABC) approach adjoins new boundary conditions to the standard Maxwell’s equations in order to account for additional waves excited at the interface, which otherwise could not be explained by the standard local theory. All ABC formulations are inherently model-specific since they assume particular types of nonlocal media or postulate specific ABCs based on the physics and applications [18]. We note that this ABC formalism is not inevitable since there exist several boundary-condition free formulations, e.g., see [9, 10]. In the field of numerical methods, a number of discretization strategies for dealing with full-wave analysis in the presence of nonlocal materials were proposed. For example, an FDTD method was proposed in [19] to deal with metallic spatially dispersive objects. Surface integral equations for nonlocal plasmonic materials were also proposed in [20], with discretization done using the RWG basis functions. The idea of exploiting nonlocality to design and develop a new generation of metamaterials (MTMs) has also received a revival in recent years, though the key idea is not new and goes back to at least [4]. Examples include spatial dispersion in wire media [21–23], hyperbolic metamaterials [24], nanostructured nonlocal domains [25], plasma-based metamaterials [26], Chern metamaterials [27], and nonlocal uniaxial metamaterials [28]. Numerous homogenization theories for nonlocal MTM with averaging operations considered over multiple spatial scales have been reported, e.g., see [29], [22]. We note that the subject of electromagnetic metamaterials (with or without nonlocality) is enormous and it is beyond the scope of this paper to even summarize the main papers in the field. However, most publications (till recently) have focused on non-spatially dispersive and local scenarios. This situation has begun to change in recent years, and increasing numbers of reports appear to move from the old opinion that nonlocality is a “bug” to the more positive and fruitful perspective that nonlocality provides new dimensions to be exploited in metamaterial system design.

A particularly interesting direction of research in nonlocal media is the recent subject of *topological photonics*. The main idea was inspired by previous researches in Chern insulators and topological insulators [30], where the focus has been on electronic systems. There, it has already been observed that the nonlocal behaviour of the fermionic wavefunction may exhibit a rather interesting and nontrivial dependence on the entire configuration space of the system, in that case the momentum space (the wavevector $\mathbf{k}$ space). In addition to the already established role played by nonlocality in superconductors, quantum Hall effects are among the most intriguing physically observable phenomena that turned out to depend fundamentally on purely topological aspects of the electron wavefunction [31]. The major themes exhibited by electrons undergoing topological transition states include topological robustness of the excited edge (surface) states moving along a 2-dimensional interface under the influence of an external magnetic field. More recently, it was proposed that the same phenomenon may apply to photons (electromagnetics) [32], where the key idea was to use photonic crystals to emulate the periodic potential function experienced by electrons in fermionic systems. However, since photons are bosons, the transplantation

of the main theme of topological insulators to photons is not trivial and is currently generating great attention, see for example the extensive review article [33], which provides a very comprehensive literature survey of the field. One of the most important applications of topological photonics is the presence of “edge states,” which are topologically robust unidirectional surface waves excited on the interface between two metamaterials with topologically distinct invariants. Since edge states are immune to perturbations on the surface, they have been advocated for major new applications where topology and physics become deeply intertwined [34]. Topology can also be exploited to devise non-resonant metamaterials [35] and to investigate bifurcation transitions in media [36]. Another related exciting subject illustrating the synergy between topology, physics, and engineering is non-Hermitian dynamics, especially in light of recent work related to the origin of surface waves [37, 38], which is now being considered as essentially non-trivial topological effect.

The previous old and recent directions of research all point out toward a basic fact: topology and physics are destined to come closer to each other within the next decade. However, the unique feature in this convergence, though in itself is not totally new since Herman Weyl introduced topological thinking to physics in the 1920s, is the focus on *engineering* applications, in our case metamaterials and topology-based devices. For that reason, we propose that in addition to the now mainstream approach to topological materials where the focus is on the global dependence of the wavefunction on momentum (Fourier) space, there is a need to consider how materials can be assigned a direct structure in the *configuration* space, i.e., space-time or space-frequency. Our key observation is that EM nonlocality requires gathering information at *microdomains* (small regions around every point where the response is nonlocal), then aggregating these microdomains together in order to arrive into a global topological structure. The fundamental insight coming from topology is precisely how this process of “moving from the local to the global” can be enacted. We have found that a very efficient method to do this is the natural formulation of the entire problem in terms of a fiber bundle (described in details below, but also see the Appendix for review.) In other words, in contrast to most existing works on topological materials, we don’t first solve Maxwell’s equations to find the state function in the Fourier $\mathbf{k}$ -space then study topology over momentum space; instead, we work directly in spacetime (or space-frequency) and formulate the *dual* problem of the topology over a fiber bundle. It is the hope of the author that such new perspective may provide a complementary approach to the exciting subject of topological materials and help generate new insights into the physics and novel algorithms for the computation of suitable topological invariant characterizing complex material domains.

The main result of this paper is that every generic nonlocal domain can be topologically described by a Banach (infinite-dimensional) vector bundle $\mathcal{M}$. If two materials described by their corresponding vector bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ are juxtaposed, then one may use topological methods to combine them and to compare their topologies. The present paper focus mainly on the first part, how to construct the material bundle $\mathcal{M}$. Because of the complexity of the subject, and to make it more accessible to a wider audience involving physicists, engineers, and topologists, we have divided the argument into different stages with different flavors as follows. In Sec. 2, we kick off our presentation by introducing a general review of electromagnetic nonlocality targeting a wide audience of engineers and applied scientist. The key ingredients of nonlocal metamaterials/materials are illustrated using an abstract excitation-response model. This is followed in Sec. 3 by a more detailed description of

the special but important case of spatial dispersion, which tends to arise naturally in many investigations of nonlocal metamaterials. In Sec. 4, we begin the elucidation of the main topological ideas behind electromagnetic nonlocality, most importantly, the concept of EM *nonlocality microdomains*, which provides the key link between physics, engineering, and topology in this paper. The various physical and mathematical structures are spelled out explicitly, followed in Sec. 5 by a more careful construction of a very natural fiber bundle structure that appears to satisfy simultaneously both the physical and mathematical requirements of EM nonlocality. The material response function is then shown to be representable as a special fiber bundle homomorphism over the metamaterial base space. In Sec. 6, various applications to both fundamental theory and engineering are outlined in brief form. Some basic familiarity with vector bundles and Banach spaces is assumed, but since the target of the paper includes engineers and applied scientist, essential definitions and concepts will be reviewed briefly within the main formulation and references where more background on vector bundles can be found will be pointed out. The Appendix at the end provides a general guideline for readers who would like to learn more about the actually used mathematical background and should be consulted by those completely unfamiliar with differential topology. However, even such readers may benefit from reading the other portions directly. The paper intentionally avoids the rigid theorem-proof format to make it accessible to a wider audience. Most of the time we give only proof sketches and leave out straightforward but lengthy arguments. In general, only the very basic definitions of manifolds, vector bundles, Banach spaces are needed to comprehend this theory. The only place where the treatment is mildly technical is toward the end of Sec. 5 when the bundle homomorphism is constructed using partition of unity technique. That part can be skipped in first reading but the general concept can be grasped from the text.

2. Overview of Electromagnetic Nonlocal Materials

In order to introduce the concept of nonlocality in the simplest way possible, let us start in a scalar field theory setting. As mentioned in the introduction, vacuum classical fields cannot exhibit nonlocality, so in order to attain this phenomenon, one must consider fields in specialized domains. We then begin by reviewing the broad theory of such media. The main goal is to outline the main ingredients of the spacetime configuration space on which such theories are often formulated in literature. To further simplify the presentation, we work throughout this paper in the regime of linear response theory: all media are assumed to be linear with respect to field excitation. If the medium response is described by the function $R(\mathbf{r}, t)$, while the exciting field is $F(\mathbf{r}, t)$, then the most general response is given by an operator equation of the form

$$R(\mathbf{r}, t) = \mathcal{L} \{F(\mathbf{r}, t)\}, \quad (1)$$

where $\mathcal{L}$ is a the linear operator describing the medium, and is ultimately determined by the laws of physics relevant to the structure under consideration. Now, the entire physical process will occur in a spacetime domain. In nonrelativistic applications (like this paper), we intentionally separate and distinguish space from time. Therefore, let us consider a process of field-matter interaction where $t \in \mathbb{R}$, while spatially restricted to a small region $\mathbf{r} \in D \subset \mathbb{R}^3$, where D is an open set containing $\mathbf{r}$.³ Since the

³We assume the normal Euclidean topology on $\mathbb{R}^3$ for all spatial domains.

operator $\mathcal{L}$ is linear, one may argue (informally) that its associated Green's function $K(\mathbf{r}, \mathbf{r}'; t, t')$ must exist. Strictly speaking, this is not correct in general and one needs to prove the existence of the Green's function for every given linear operator on a case by case basis by actually constructing one.⁴ However, we will follow (for now) the common trend in physics and engineering by assuming that linearity alone is enough to justify the construction of the Green's function. If this is accepted, then we can immediately infer from the very definition of the Green's function itself that

$$R(\mathbf{r}, t) = \int_D \int_{\mathbb{R}} d^3 r' dt' K(\mathbf{r}, \mathbf{r}'; t, t') F(\mathbf{r}', t'). \quad (2)$$

The relation (2) represents the most general response function of a (scalar) material medium. It is in general valid for linear field-matter interaction regimes. The kernel (Green's) function $K(\mathbf{r}, \mathbf{r}'; t, t')$ is often called the *medium response function*. If we further assume that all of the material constituents of the medium are time-invariant (the medium is not changing with time), then the relation (2) maybe replaced by

$$R(\mathbf{r}, t) = \int_D \int_{\mathbb{R}} d^3 r' dt' K(\mathbf{r}, \mathbf{r}'; t - t') F(\mathbf{r}', t'), \quad (3)$$

where the only difference is that the kernel function's temporal dependence is replaced by $t - t'$ instead of two separated arguments. Such superficially small difference has nevertheless considerable consequences. Most importantly, working with (3) instead of (2), it becomes possible to apply the Fourier transform in time to simplify the formulation of the problem. Indeed, taking the temporal Fourier transform of both sides of (3) leads to

$$R(\mathbf{r}, \omega) = \int_D d^3 r' K(\mathbf{r}, \mathbf{r}'; \omega) F(\mathbf{r}'; \omega), \quad (4)$$

where the Fourier spectra of the fields are defined by

$$F(\mathbf{r}; \omega) := \int_{\mathbb{R}} dt F(\mathbf{r}; t) e^{-i\omega t}, \quad R(\mathbf{r}; \omega) := \int_{\mathbb{R}} dt R(\mathbf{r}; t) e^{-i\omega t}. \quad (5)$$

On the other hand, the medium response function's Fourier transform is given by the essentially equivalent formula

$$K(\mathbf{r}, \mathbf{r}'; \omega) := \int_{\mathbb{R}} d(t - t') K(\mathbf{r}, \mathbf{r}'; t - t') e^{-i\omega(t - t')}. \quad (6)$$

In this paper, we focus on time-invariant material media and hence work exclusively with frequency-domain expressions like (4). The generalization to the 3-dimensional (full-wave) electromagnetic picture is straightforward when the dyadic formalism is employed. The relation corresponding to (2) is

$$\mathbf{R}(\mathbf{r}, t) = \int_D d^3 r' \int_{\mathbb{R}} dt' \bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; t - t') \cdot \mathbf{F}(\mathbf{r}', t'), \quad (7)$$

where we have replaced the scalar fields $F(\mathbf{r})$ and $R(\mathbf{r})$ by vectors $\mathbf{F}(\mathbf{r}), \mathbf{R}(\mathbf{r}) \in \mathbb{R}^3$. The kernel function K , however, must be transformed into a *dyadic* function (tensor

⁴This is argued in details in [39]. In particular, the recently-introduced current Green's function of electromagnetic devices was inspired by finding a Green's function structure similar to that corresponding to nonlocal media [40], [41], [42].

of second rank) $\bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; t - t')$ [2, 43]. In the (temporal) Fourier domain, (7) becomes

$$\mathbf{R}(\mathbf{r}, \omega) = \int_D d^3r' \bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \mathbf{F}(\mathbf{r}'; \omega), \quad (8)$$

where

$$\bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) := \int_{\mathbb{R}} dt(t - t') \bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; t - t') e^{-i\omega(t - t')}, \quad (9)$$

$$\mathbf{F}(\mathbf{r}; \omega) := \int_{\mathbb{R}} dt \mathbf{F}(\mathbf{r}; t) e^{-i\omega t}, \quad \mathbf{R}(\mathbf{r}; \omega) := \int_{\mathbb{R}} dt \mathbf{R}(\mathbf{r}; t) e^{-i\omega t}. \quad (10)$$

The very essence of electromagnetic nonlocality can be captured by the mathematical structure of the basic relation (7). In words, it says that the field response $\mathbf{R}(\mathbf{r})$ is determined not only by the excitation field $\mathbf{F}(\mathbf{r}')$ applied at location $\mathbf{r}'$, but at all points $\mathbf{r}' \in D$. *Consequently, knowledge of the response at one point requires knowledge of the cause (excitation field) at an entire topologically local set D .* On the other hand, if the medium is local, then the material response function can be written as

$$\bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) = \bar{\mathbf{K}}_0(\omega) \delta(\mathbf{r} - \mathbf{r}'), \quad (11)$$

where $\bar{\mathbf{K}}_0$ is a constant tensor and $\delta(\mathbf{r} - \mathbf{r}')$ is the 3-dimensional Dirac delta function. In this case, (8) reduces to

$$\mathbf{R}(\mathbf{r}; \omega) = \bar{\mathbf{K}}_0(\omega) \cdot \mathbf{F}(\mathbf{r}; \omega), \quad (12)$$

which is the standard constitutive relation of linear electromagnetic materials. Clearly, (12) says that only the exciting field $\mathbf{F}(\mathbf{r})$ data at $\mathbf{r}$ is needed to induce a response at the same location. In a nutshell, locality implies that the natural configuration space of the electromagnetic problem is just the point-like spacetime manifold $D \subset \mathbb{R}^3$ or the entire Euclidean space $\mathbb{R}^3$. In addition, one may attach the “infinitesimally immediate neighborhood” to a given point $\mathbf{r}$ where a response is sought. Indeed, according to (12), only the exciting field at $\mathbf{r}$ is needed to compute the response, but since Maxwell’s equations will still need to be coupled with that local constitutive relation, then the fact that these equations are *differential* equations, the “largest” domain beside $\mathbf{r}$ needed to carry over the mathematical description of the field-matter interaction physics is just the infinitesimally close region. Conventional boundary-value problems in applied electromagnetics are formulated in this manner [39, 43, 44], i.e., with a 3-differential manifolds as the main problem space on which spatial fields live. Note that strictly speaking, the full configuration space in local electromagnetics (also called normal optics) is the 4-dimensional manifolds $D \times \mathbb{R}$ or $\mathbb{R}^4$ since either time t or the (temporal) radian frequency ω must be included to engender a full description of electromagnetic fields. However, nonlocal materials are most fundamentally a *spatial* type of materials/metamaterials where it the spatial structure of the field what carries most of the physics involved [39, 45]. For that reason, throughout this paper we investigate the required configuration spaces with focus mainly on the spatial degrees of freedom.

3. Spatial Dispersion in Homogeneous Nonlocal Materials

Spatial dispersion is currently considered by some researchers as one of the most promising routes toward nonlocal metamaterials [45]. It is by large the most intensely

investigated class of nonlocal media, receiving both theoretical and experimental treatments by various research groups since the early 1960s. In essence, the basic idea is to restrict the electromagnetics to the special but important case of media possessing transnational symmetry, a case that is obtained when the medium is homogeneous. In such situation, the material tensor function satisfies

$$\overline{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) = \overline{\mathbf{K}}(\mathbf{r} - \mathbf{r}'; \omega). \quad (13)$$

The spatial Fourier transforms are defined by

$$\overline{\mathbf{K}}(\mathbf{k}, \omega) := \int_{\mathbb{R}^3} d^3(r - r') \overline{\mathbf{K}}(\mathbf{r} - \mathbf{r}'; \omega) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (14)$$

$$\mathbf{F}(\mathbf{k}, \omega) := \int_{\mathbb{R}^3} d^3r \mathbf{F}(\mathbf{r}; t) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad \mathbf{R}(\mathbf{k}, \omega) := \int_{\mathbb{R}^3} d^3r \mathbf{R}(\mathbf{r}; t) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (15)$$

Inserting (13) into (8) and taking the spatial (3-dimensional) Fourier transform of both sides, the following relation is obtained

$$\mathbf{R}(\mathbf{k}, \omega) = \overline{\mathbf{K}}(\mathbf{k}, \omega) \cdot \mathbf{F}(\mathbf{k}, \omega), \quad (16)$$

The dependence of $\overline{\mathbf{K}}(\mathbf{k}, \omega)$ on the wavevector (“spatial frequency”) $\mathbf{k}$ in addition to the temporal frequency ω is the signature of spatial dispersion. As a spectral transfer function of the medium, $\overline{\mathbf{K}}(\mathbf{k}, \omega)$ includes all the information needed in order to obtain the nonlocal material domain’s response to arbitrary spacetime field excitation functions $\mathbf{F}(\mathbf{r}, t)$ through the inverse 4-dimensional Fourier transform [4]. In many treatments of the subject, the excitation field is taken as the electric field $\mathbf{E}(\mathbf{r}, t)$ while the response function is $\mathbf{D}(\mathbf{r}, t)$. In such formulation, the material tensor function $\overline{\mathbf{K}}(\mathbf{k}, \omega)$ takes into account both electric and magnetic effects [2–4, 8, 11], [9, 10, 12, 46]. This is different from the permittivity tensor often used in local electromagnetics, which is based on the multipole model of material media. A comparison between the two material response formalisms, the one based on $\overline{\mathbf{K}}(\mathbf{k}, \omega)$ and the multipole model is given in [39].

Historically, spatial dispersion had been under the radar since the 1950s, especially in connection with researches on the optical spectra of material domains. However, the first systematic treatment of the subject appeared in 1960s in the first edition of Ginzburg book on plasma physics, which was dedicated electromagnetic wave propagation in plasma media. The second edition of the book, published in 1970, contained a considerably extended treatment of the various mathematical and physical aspects of the electromagnetics of spatially dispersive media [2]. Media obtained by homogenizing arrays of wires, already very popular because of their connection with traditional (temporal) metamaterials, are known to exhibit spatial dispersion effects, though many researchers ignore that effect to focus on temporal dispersion [47–49]. Other types of periodic or large finite arrays of composed of unit cells like spheres and desks also exhibit spatial dispersion effects [50]. Nonlinear materials with observable nonlocality have also been investigated in the optical regime [51]. More recently, much of the reemergence of interest in spatial dispersion stems from the observation that the phenomenon cannot be ignored at the nanoscale, especially those of low-dimensional structures like carbon nanotubes [14] and graphene [52]. The subject was also introduced at a pedagogical level for applications involving current flow in spatially dispersive conductive materials like plasma and nanowires [53].

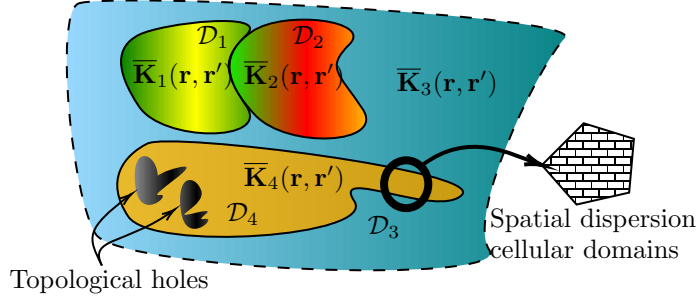


Figure 1: A generic depiction of an electromagnetic nonlocal metamaterial system. Each of the domains $\mathcal{D}_n$ is captured by a general linear nonlocal response function $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$.

Complex heterogeneous arrangements of various nonlocal materials can be realized by juxtaposing several subdomains where each is homogeneous, hence, can be described by a spatial dispersion profile $\bar{\mathbf{K}}(\mathbf{k}, \omega)$. The idea is that even materials inhomogeneous at a given spatial scale tend to become homogeneous at a finer spatial level, leading to a “grid like” spatially dispersive cellular building blocks at the lower level. In Fig. 1, we show a nonlocal metamaterial system with various multiscale structures. A large nonlocal domain, e.g., in this figure $\bar{\mathbf{K}}_3(\mathbf{r}, \mathbf{r}')$ acts like a “substrate” holding together several other smaller material constituents, such as $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$, $n = 1, 2, 4$. We envision that each nonlocal subdomain may possess its own especially tailored nonlocal response function profile serving one or several applications.⁵ By concatenating several regions, interfaces between subdomains with different material constitutive relations are created. We show here subdomains $\mathcal{D}_n$, $n = 1, 2, 3, 4$, and the possible material interfaces include $\mathcal{D}_1/\mathcal{D}_2$, $\mathcal{D}_1/\mathcal{D}_3$, $\mathcal{D}_2/\mathcal{D}_3$, $\mathcal{D}_3/\mathcal{D}_4$. In local electromagnetics, each material interface should be assigned a special electromagnetic boundary condition in order to ensure the existence of a unique solution to the problem [43]. However, as mentioned earlier, nonlocal electromagnetics introduces a tension absent in the local case. Indeed, additional boundary conditions are often invoked to handle the transition of fields along barriers separating different domains, like between two nonlocal domains or even one nonlocal and another local domain [4]. The topological fiber bundle theory to be developed in Sec. 5 will provide a clarification of why this is so since it turns out the traditional spacetime approach usually connected with local electromagnetics is not necessarily the most natural one. There is a need to examine in more details the existence of *multiple topological scales* in nonlocal metamaterials, and this paper will provide some initial insights such issue. For the time being, we note here three directly observable topological scales that don’t require the most detailed analysis to be given later. The first is the already mentioned separation between different nonlocal domains like $\mathcal{D}_1$ and $\mathcal{D}_2$. The second is the case captured by the inset in the right hand side in Fig. 1. Fine “microscopic” cells, each homogeneous and hence describable by a response function of the form $\bar{\mathbf{K}}(\mathbf{k}, \omega)$, can be combined to build up a complex effective nonlocal response tensor $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$ over its topologically global domain $\mathcal{D}_n$. Such juxtaposition at the microscopically local level leading to global behaviour is a classic example of multiscale physics, but here it is of further interest since *both* the constituent cells (rectangular “bricks” in the inset of Fig.

⁵Discussion of some possible engineering applications is given in Sec. 6.6.2.

1) and the global domain level $\mathcal{D}_n$ are already *electromagnetically* nonlocal. In other words, the terms ‘local’ and ‘global’ possess two different senses, one electromagnetic, the other spatio-geometric. Elucidating this subtle interconnection between the two senses will be one of the main objectives of the present work. Finally, the third directly observable topological scale is that connected to what we termed “topological holes” in Fig. 1. These are arbitrarily-shaped gaps, like holes, vias, etchings, etc, that are introduced purposefully to influence the the electromagnetic response by modifying the topology of the 3-dimensional material manifolds $\mathcal{D}_n$.

4. The Main Topological Structure of Nonlocal Electromagnetic Domains

Let the domain of nonlocality of the electromagnetic media, the region $D \subset \mathbb{R}^3$ in (8), be bounded. Corresponding to (1), a similar operator equation in the frequency domain can be assumed to represents the most general form of a nonlocal electromagnetic media, namely

$$\mathbf{R}(\mathbf{r}; \omega) = \mathcal{L}_\omega \{ \mathbf{F}(\mathbf{r}; \omega) \}, \quad (17)$$

where the nonlocal medium linear operator is itself a function of frequency.⁶ We are going to propose a change in the mathematical framework inside which electromagnetic nonlocality is usually defined. This will be done in two stages. First, in the present Section, we introduce the rudiments of the main physics-based micro-topological structure associated with EM nonlocality without going into considerable mathematical details. The aim is to familiarize ourselves with the minimal necessary physical picture and how it gives rise to a finer picture of the material domain compared with the more familiar (and much simpler) topological structure of local electromagnetics based on spacetime points. In the second stage, Sec. 5, a more careful mathematical picture is developed using the theory of topological fiber bundles. We eventually show that the EM nonlocal operator (17) can be constructed as Banach bundle map (homomorphism) over the 3-dimensional space of the material domain under consideration.

In conventional local electromagnetics, the boundary-value problem of multiple domains is formulated as a set of partial differential equations or integero-differential equations coupled with some boundary conditions dictating how fields change while crossing the various regions through which the equations hold [43, 44, 54]. This has been done traditionally by taking the electromagnetic response function $\overline{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega)$ as an essential key ingredient of the problem description, which can be exploited in two ways: First, the constitutive relations enter into the governing equations in each separate solution domain. Second, the constitutive relations themselves are used in order to construct the proper electromagnetic boundary conditions prescribing the continuity/discontinuity behaviour of the sought field solutions as they move across the various interfaces separating domains with different material properties. Unfortunately, it has been well known for a long time that it is not possible to formulate a universal electromagnetic boundary condition for nonlocal medium, especially for the case of spatial dispersion. This will be discussed in more detailed later when we treat the special but important case of spatial dispersion in Sec. 3 and again in Sec. 6. For now, we concentrate on understanding a deeper structure of the problem of

⁶For simplicity, we often drop the dependence on ω when the operator is understood from the context to be applying to the frequency domain.

nonlocality in electromagnetics. Later, we will suggest that this topological content of the theory may help resolving some of the major problems encountered in spatial dispersion and applications.

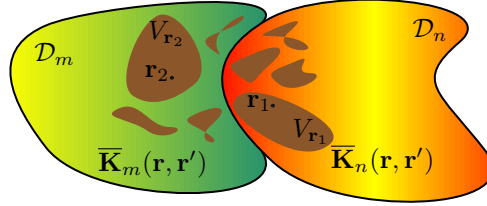


Figure 2: The micro-topological structure of nonlocal metamaterial systems includes more than just the 3-dimensional spatial domains $\mathcal{D}_n, n = 1, 2, \dots$. It is best captured by classes $\mathcal{V}(\mathcal{D}_n)$ composed of various open sets $V_{\mathbf{r}} \subset \mathcal{D}_n$ based at each point $\mathbf{r} \in \mathcal{D}_n$. On every such subset a vector field is defined, representing the EM excitation field. The collection of all vector fields on a given set $V_{\mathbf{r}}$ gives rise to a linear topological function space $\mathcal{F}(V_{\mathbf{r}})$. The topologies of the base spaces $\mathcal{D}_n$, the nonlocal micro-domains $V_{\mathbf{r}}$, and the function spaces $\mathcal{F}(V_{\mathbf{r}})$ collectively give rise to a total “macroscopic” topological structure considerably more complex than the base spaces $\mathcal{D}_n$.

A key starting observation is how nonlocality forces us to associate with every spacetime point $(\mathbf{r}, t)$, or frequency-space point $(\mathbf{r}; \omega)$, a topological neighborhood of $\mathbf{r}$, say D , such that $\mathbf{r} \in D$. For now, let us assume that D is just an open set in the topology of the Euclidean space $\mathbb{R}^3$ inherited from the standard Euclidean metric. By restricting D for now to be open, we avoid the problem of dealing with boundary or an interface. That is, the topological closure of D , denoted by $\text{cl}(D)$, is excluded from the domain of nonlocality. Let D be the maximal such topological neighbored for the problem under consideration.⁷ We now associate with each point $\mathbf{r}$ a smaller open set $V_{\mathbf{r}}$ such that $\mathbf{r} \in V_{\mathbf{r}}$, so $V_{\mathbf{r}} \subset D$ for all $\mathbf{r} \in D$. (The fact that D is assumed open makes this possible.) Now, instead of considering fields like $\mathbf{R}(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$ defined on the entire maximal domain of nonlocality D (which can grow “very large,”) we propose to reformulate the problem of nonlocal electromagnetic materials in a *topologically local form* by noting that the physics of field-matter interactions gives the EM response at location $\mathbf{r}$ due to excitation fields essentially confined within a “smaller domain” around $\mathbf{r}$, namely the open set $V_{\mathbf{r}}$. On the other hand, if the response at another *different* point $\mathbf{r}'$ (i.e., $\mathbf{r} \neq \mathbf{r}'$) is needed, then a new, generally different small open set $V_{\mathbf{r}'}$ will be required. That is, in general we allow that $V_{\mathbf{r}} \neq V_{\mathbf{r}'}$, even though we expect that typically there is some overlap between these two small local domains of electromagnetic nonlocality, i.e., $V_{\mathbf{r}} \cap V_{\mathbf{r}'} \neq \emptyset$, especially if $|\mathbf{r} - \mathbf{r}'|$ is small. The “smaller sets” $V_{\mathbf{r}}, \mathbf{r} \in D$, will be dubbed *nonlocal microdomains* or just *microdomains* in short. They explicate the fine micro-topological structure of nonlocal electromagnetic domains at a spatial scale different from that of the (topologically “larger”) material domain D itself.

Consider now the set of all sufficiently differentiable vector fields $\mathbf{F}(\mathbf{r})$ defined on $V_{\mathbf{r}}, \mathbf{r} \in D$, for all $\mathbf{r}' \in V_{\mathbf{r}}$. This set possesses an obvious complex vector space structure; namely, for any complex numbers $a_1, a_2 \in \mathbb{C}$, the sum $a_1 \mathbf{F}_1(\mathbf{r}') + a_2 \mathbf{F}_2(\mathbf{r}')$ is defined

⁷If D approaches asymptotically an unbounded region, the problem transforms into that of unbounded domain (bulk media), well treated in the basic literature on spatial dispersion.

on $V_{\mathbf{r}}$ whenever $\mathbf{F}_1(\mathbf{r}')$ and $\mathbf{F}_2(\mathbf{r}')$ are, while the null field plays the role of the origin. We denote this function space by $\mathcal{F}(V_{\mathbf{r}})$. It is possible to equip $\mathcal{F}(V_{\mathbf{r}})$ by a topology in order to measure the distance between any two fields defined on $V_{\mathbf{r}}$ [55,56]. Therefore, $\mathcal{F}(V_{\mathbf{r}})$ becomes a topological vector space. In general, depending on $\mathbf{r}$ and the topology of $V_{\mathbf{r}}$, each function space will have a different topology. A very convenient choice for this function space topology is that induced on the Sobolev space $W^{p,2}(V_{\mathbf{r}})$, $p \geq 2$, through its standard norm [57]. This is a topological vector space that is also a Hilbert space (and hence a Banach space.) In this way, each microdomain $V_{\mathbf{r}}$ induces an infinite-dimensional linear function space (Sobolev) $\mathcal{F}(V_{\mathbf{r}})$ indexed by the position $\mathbf{r} \in D$, with topology essentially determined by the geometry of $V_{\mathbf{r}}$. On the other hand, this latter geometry is obtained from the *physics* of field-matter interaction in nonlocal media. Consequently, the physical content of nonlocal materials is encoded at the *topologically micro-local* level of the following structure

$$D \ni \mathbf{r} \xrightarrow[\text{EM Nonlocality}]{\text{Physical Data}} V_{\mathbf{r}} \xrightarrow[\text{Sobolev Space}]{\text{Mathematical Data}} \mathcal{F}(V_{\mathbf{r}}). \quad (18)$$

If we define the collections of subsets

$$\mathcal{V}(D) := \{V_{\mathbf{r}} \subset D | \mathbf{r} \in D, V_{\mathbf{r}} \text{ is open}\}, \quad (19)$$

$$\mathcal{G}[\mathcal{V}(D)] := \{\mathcal{F}(V_{\mathbf{r}}) | \mathbf{r} \in D, \mathcal{F}(V_{\mathbf{r}}) : \text{Sobolev function space}\}, \quad (20)$$

then (18) can be summarized by the ordered triplet

$$D \times \mathcal{V}(D) \times \mathcal{G}[\mathcal{V}(D)]. \quad (21)$$

Each open domain in $D \subseteq \mathbb{R}^3$ will then be assigned a distribution $\mathcal{V}(D)$ of open sets $V_{\mathbf{r}}$, the micro-topological fine structure of electromagnetic nonlocality, which is determined solely by the field-matter interaction physics. In order to provide a complete description of this domain, we further emphasize that the sets forming the elements of $\mathcal{V}(D)$ constitute an open cover of D , that is, we have

$$D = \bigcup_{\mathbf{r} \in D} V_{\mathbf{r}}. \quad (22)$$

In this way, the model can accommodate excitation fields $\mathbf{F}(\mathbf{r})$ applied at every point in $\mathbf{r} \in D$. Next, the set $\mathcal{V}(D)$ induces the space $\mathcal{G}[\mathcal{V}(D)]$ of function spaces $\mathcal{F}(V_{\mathbf{r}})$, $\mathbf{r} \in D$, where each vector field is defined on elements chosen from the set $\mathcal{V}(D)$. It is interesting to see how in the proposed framework some kind of constructive “division of labour” is distributed between physics and mathematics in order to generate the various required structures. This is also the source of some potential difficulties hidden in the topological structure (21). Indeed, we will next try to smooth out the differences between the two main substructures $\mathcal{V}(D)$, which is controlled mostly by physics, and $\mathcal{G}[\mathcal{V}(D)]$, dominated by mathematical considerations, by unifying the entire total topological structure (21) using a vector bundle formulation.

It is now possible to provisionally construct the response function by working on the fundamental topological domain structure (21) instead of the global domain D . The response function $\mathbf{R}(\mathbf{r})$ will be re-expressed by the map

$$\mathbf{R} : D \times \mathcal{V}(D) \rightarrow \mathbb{C}^3, \quad (23)$$

where the codomain is taken to be $\mathbb{C}^3$ because the electric or magnetic response functions $\mathbf{D}$ or $\mathbf{B}$, respectively, are complex vector fields in the frequency domain.⁸ The value of the EM nonlocal response function due to excitation field $\mathbf{F}(\mathbf{r})$ applied at a microdomain $V_{\mathbf{r}}$ can be computed by means of

$$\mathbf{R}(\mathbf{r}; \omega) = \int_{V_{\mathbf{r}}} d^3r' \bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \mathbf{F}(\mathbf{r}'; \omega). \quad (24)$$

Although (24) appears to be only slightly different from (8), the similarity is superficial. In essence, the construction of the EM response functions via the map (23) amounts to *topological localization* of electromagnetic nonlocality where the EM response function is now no longer allowed to extend globally onto “large and complicated material domains.” Indeed, only the response to small or topological local domains, namely the microdomains $V_{\mathbf{r}}$, is admitted. On the other hand, in order to find the response field $\mathbf{R}(\mathbf{r})$ everywhere in D , one needs to use sophisticated topological techniques to extend the response from one point to another till it covers the entirety of D .⁹ In this manner, it becomes possible to provide an alternative, more detailed explication of the behaviour of the medium at topological interfaces¹⁰ and also explore the effect of the topology of the bulk medium itself on the allowable response functions and the production of non-trivial edge state, with obvious applications to nonlocal metamaterials.¹¹

5. Fiber Bundle Formalism for Electromagnetic Nonlocality

In order to investigate in depth the fundamental physico-mathematical constraints imposed on EM nonlocal metamaterials, the material domain D considered so far should be promoted to a manifold structure. There are several reasons why this is highly desirable. First, it provides a natural and obvious generalization of the basic structure (21) from the mathematical perspective. Second, engineers often need to insert metamaterials into specific device settings, hence the shape of the material becomes highly restricted. It is therefore important to develop efficient tools to deal with variations of geometric and topological degrees of freedom and how they could possibly impact the design process. Third, applied scientists and engineers are often interested in deriving fundamental limitation on metamaterials, e.g., what are the ultimate allowable response-excitation relations or constitutive response functions possible given this material domain topology or that. Fourth, sophisticated full-wave electromagnetic numerical solvers prefer working with local coordinates in order to handle complicated shapes, even if a global coordinate system is sometimes available, making the deployment of 3-manifold structures for describing the material domain D useful. Fifth, in topological photonics and materials [33], most applications seem to focus on lower-dimensional states of matter like those associated with quantum Hall effects and edge states (surface waves). There, new phenomena appear at materials

⁸In this paper, we don't worry much about the details of the electromagnetic model and for simplicity assume that only one vector field $\mathbf{F}$ acts as excitation and one response function $\mathbf{R}$ is induced. More complex media like bianisotropic domains and others can also be treated within this formulation. For example, if two response functions are needed, the codomain in (23) can be simply changed to $\mathbb{C}^6$.

⁹This local-to-global extension application of differential topology is briefly discussed in Sec. 6 6.1.

¹⁰Boundary conditions in nonlocal metamaterials are treated (provisionally) in Sec. 6 6.1.

¹¹See the discussion of nonlocal and topological metamaterials applications in Sec. 6 6.2.

where the base space (material domain D) is a 2-surface, which is best described mathematically as a differential 2-manifold. For all these reasons, it is expedient to describe the domain D in the most general mathematical form, which in our case amounts to equipping it with a manifold structure. If we denote by $\mathcal{D}$ this 3-manifold, then, being a subset of $\mathbb{R}^3$, there is a natural differential structure defined on it, that inherited from the 3-dimensional Euclidean space itself. This differential 3-manifold structure will be presupposed in the remaining parts of this paper.

Let (U_i, ϕ_i) be a collection of charts (an atlas), labeled by $i \in I$, an index set, and equipping $\mathcal{D} \subset \mathbb{R}^3$ with a differential 3-manifold structure.¹² Our goal is to attach a vector fiber (linear function space in our case) at every point $\mathbf{r} \in \mathcal{D}$, namely the function space $\mathcal{F}(V_{\mathbf{r}})$. That would require finding suitable “compatibility laws” dictating how coordinates change between two intersecting charts U_i and U_j interact with each other. In particular, we will need later to find the law of mutual transformation of vectors in the fibers $\mathcal{F}(V_{\phi_i(\mathbf{r})})$ and $\mathcal{F}(V_{\phi_j(\mathbf{r})})$.¹³ The major technical problem facing us here is the following: Since the differential structure associated with charts $(U_i, \phi_i(\mathbf{r}))$ is fully determined by purely mathematical considerations alone, while the collection of microdomains $V_{\mathbf{r}}, \mathbf{r} \in \mathcal{D}$, is determined mainly by the physics of EM nonlocality, there is no direct and simple way to express the transformation of vectors in $\mathcal{F}(V_{\phi_i(\mathbf{r})})$ into vectors in $\mathcal{F}(V_{\phi_j(\mathbf{r})})$ because *several different coordinate patches other than U_i and U_j* , belonging to the differential 3-manifold $\mathcal{D}$ atlas, might be involved in building up the microdomain $V_{\mathbf{r}}$. This technical problem will be solved in this Section by using the technique of *partition of unity* borrowed from differential topology [5, 55, 58]. It will allow us to split each full microdomain $V_{\mathbf{r}}$ into several suitable sub-microdomain, which can be later be pieced up to give back the original EM nonlocality microdomain $V_{\mathbf{r}}$.

We start by noting that the collection $\mathcal{V}(\mathcal{D}) := \{V_{\mathbf{r}}, \mathbf{r} \in \mathcal{D}\}$ is an open cover of the manifold $\mathcal{D}$. Therefore, and since $\mathcal{D}$ has countable basis, it possesses a locally finite open cover subordinated to $\mathcal{V}(\mathcal{D})$ [5, 58]. In fact, an atlas $(U_i, \phi_i), i \in I$, with diffeomorphisms,

$$\phi_i : U_i \rightarrow \mathbb{R}^3, \quad (25)$$

exists such that the elements $\{U_i, i \in I\}$ constitutes the above mentioned locally finite subcover, while the images $\phi_i(U_i)$ are open balls centered around 0 in $\mathbb{R}^3$ with finite radius $a > 0$, which we denote by B_a [5]. Moreover, the sets $\phi_i^{-1}(B_{a/3}), i \in I$, cover $\mathcal{D}$ [58]. In this way, the physics-based open cover set $\mathcal{V}(\mathcal{D})$ provides a first step toward the construction of a completely mathematized description of the EM nonlocal topological micro-structure since the coordinate patches $(U_i, \phi_i), i \in I$, turn out to be subordinated to $\{V_{\mathbf{r}}, \mathbf{r} \in \mathcal{D}\}$ [5]. It is also known that there exists a partition of unity associated with the chart $(U_i, \phi_i), i \in I$, constructed above [5, 55, 56, 58]. This is a collection of functions

$$\psi_i : U_i \subset \mathcal{D} \rightarrow \mathbb{R} \quad (26)$$

satisfying the following requirements: $\psi_i(\mathbf{r}) \geq 0$ and each function is $C^p, p \geq 1$. The support of $\psi_i(\mathbf{r})$, i.e., the closure of the set such that $\psi_i(\mathbf{r}) \neq 0$, is contained within U_i . Finally, since the open cover $U_i, i \in I$, is locally finite, at each point $\mathbf{r} \in \mathcal{D}$, only

¹²For simplicity, we will refer to the points of the manifold $\mathcal{D}$ by $\mathbf{r}$, i.e., using the language of the global (ambient) Euclidean space $\mathcal{R}^3$.

¹³Here, the expression $\mathcal{F}(V_{\phi_i(\mathbf{r})})$ means the fiber space attached to the point whose coordinates are $\phi_i(\mathbf{r})$, i.e., the *function* space where all functions are expressed in terms of the language of the i th chart $(U_i, \phi_i(\mathbf{r}))$.

a finite number of U_i will intersect $\mathbf{r}$. Let the set of indices of those intersecting U_i s be $I_{\mathbf{r}}$. Then we require that

$$\sum_{i \in I_{\mathbf{r}}} \psi_i(\mathbf{r}) = 1, \quad (27)$$

where the sum is always convergent because $I_{\mathbf{r}}$ is finite. We may take the closure $\text{cl}\{\phi_i^{-1}(B_{a/3})\}$ to be the support of $\psi_i(\mathbf{r})$, which is denoted by $\text{supp}\{\psi_i(\mathbf{r})\}$, while $\psi_i(\mathbf{r}) = 0$ for all $\mathbf{r} \notin \text{supp}\{\psi_i(\mathbf{r})\}$ [5, 58, 59].

The motivation behind the deployment of the partition of unity technique and how it very naturally arises in connection with our fundamental EM nonlocal structure should now be clear. Initially, the physics-based collection of sets $\{V_{\mathbf{r}}, \mathbf{r} \in \mathcal{D}\}$, the EM nonlocal microdomains based on each point $\mathbf{r}$ in the nonlocal metamaterial $\mathcal{D}$, is given. Next, we introduce a differential atlas $(U_i, \phi_i(\mathbf{r})), i \in I$ on $\mathcal{D}$. Finally, the same atlas is linked to a set of functions $\psi_i(\mathbf{r})$ that can be used as “basis” to expand any differentiable function defined on open subsets in $\mathcal{D}$. The key idea, to be developed next, is that both the base manifold $\mathcal{D}$ and the nonlocal EM microdomain $V_{\mathbf{r}}$ are described locally by the same collection of charts, namely $(U_i, \phi_i(\mathbf{r}))$, which will permit us to construct a direct unified description of both the base manifold $\mathcal{D}$ and its fibers, i.e., the linear topological function spaces $\mathcal{F}(V_{\mathbf{r}})$, the latter being the models of the physical electromagnetic fields exciting the nonlocal material $\mathcal{D}$. The construction of a fiber bundle space for nonlocal electromagnetic materials (metamaterials) will be accomplished in two steps. First, we construct a tailored fiber bundle based on the partition of unity charts $(U_i, \phi_i(\mathbf{r}))$ introduced above. Second, the original physical structure (21) is recovered by gluing together various sub-microdomain U_i of each EM nonlocal microdomain $V_{\mathbf{r}}$.

We start with the first step and consider the $(U_i, \phi_i(\mathbf{r})), i \in I$, as our atlas on the 3-manifold $\mathcal{D}$. At each point $\mathbf{r} \in U_i$, we attached a linear topological space $\mathcal{F}(U_i)$ defined as the Sobolev space $W^{p,2}(U_i)$ of functions on the open set U_i , that is,

$$\mathcal{F}(U_i) := \{\psi_i(\mathbf{r})\mathbf{F}(\mathbf{r}), \mathbf{r} \in U_i \mid \text{Sobolev Space } W^{p,2}(U_i)\}, \quad (28)$$

where $\mathbf{F}(\mathbf{r})$ is a suitable $C^{p,2}$ vector field. However, because the C^p functions $\psi_i(\mathbf{r})$ have compact support $\text{supp}\{\psi_i\} \subset U_i$, it follows that $\mathcal{F}(U_i)$ is effectively a *local* Sobolev space on U_i . In fact, we can alternatively define a less complicated function space on U_i by

$$\mathcal{F}'(U_i) := \{\psi_i(\mathbf{r})\mathbf{F}(\mathbf{r}), \mathbf{r} \in U_i \mid C^p \text{ Sup-norm function space}\}, \quad (29)$$

where the sup norm is given by

$$\|\psi_i(\mathbf{r})\mathbf{F}(\mathbf{r})\| := \sup_{\mathbf{r} \in \text{supp}\{\psi_i\}} [\psi_i(\mathbf{r})\mathbf{F}(\mathbf{r})]. \quad (30)$$

In the case of $\mathcal{F}'(U_i)$, one may further consider only C^p vector fields $\mathbf{F}(\mathbf{r})$. The choice of which linear function space to work with depends on the applications. In what follows, we further simplify notation by writing $\mathcal{F}_i$ instead of $\mathcal{F}(U_i)$ whenever the partition of unity differential atlas coordinate patches U_i are used.

In the remaining parts of the Section, an outline of the direct construction of a fiber (Banach) bundle over $\mathcal{D}$ is given, where our purpose is to build into every point $\mathbf{r} \in U_i$ a *fiber space* $\mathcal{F}_i$. Let the fiber bundle be denoted by $\mathcal{M}$, which is called the *total bundle space*. We define this space as the disjoint union of all spaces $\mathcal{F}_i$ as follows

$$\mathcal{M} := \{(\mathbf{r}, \mathcal{F}_i) \mid \forall i \in I, \mathbf{r} \in U_i\}. \quad (31)$$

Associated with $\mathcal{M}$ is a surjective map

$$p : \mathcal{M} \rightarrow \mathcal{D}, \quad (32)$$

which “projects” the fiber onto its corresponding point in the base manifold $\mathcal{D}$, i.e., $p((\mathbf{r}, \mathcal{F})) := \mathbf{r}$. More rigorously, the *fiber* of $\mathcal{M}$ at $\mathbf{r} \in \mathcal{M}$ is *defined* as the set $p^{-1}(\mathbf{r})$ provided the map p is given as part of the bundle data.¹⁴ The map p is called the *projection* of the vector bundle $\mathcal{M}$ onto its *base space* $\mathcal{D}$. Moreover, from now on we will also use the notation $\mathcal{F}_{\mathbf{r}}$ to denote the fiber $p^{-1}(\mathbf{r})$. By construction, it should be clear that $p^{-1}(\mathbf{r}) = \mathcal{F}_i$ iff $\mathbf{r} \in U_i$. Locally, $\mathcal{M}$ appears like a product space $U_i \times \mathcal{F}_i$. The map p should behave *locally* as a conventional projection operator, i.e., in a local domain U_i , the total space $\mathcal{M}$ is isomorphic to $U_i \times \mathcal{F}_i$, and $p(U_i \times \mathcal{F}_i)$ should be isomorphic to U_i . We next introduce the linear function space X_i defined by

$$X_i := \{ \psi_i [\phi_i^{-1}(\bar{x})] \mathbf{F} [\phi_i^{-1}(\bar{x})], \bar{x} \in B_a | \text{ Sobolev function space} \}, \quad (33)$$

which is the Sobolev space of $W^{p,2}(B_a)$ functions on the Euclidean 3-ball B_a . Here, each function is defined with respect to the local coordinates $\bar{x} := \phi^{-1}(\mathbf{r})$, where $\mathbf{r} \in U_i$. In fact, it should be straightforward to deduce from the above that there exists maps

$$\tau_i : p^{-1}(U_i) \rightarrow U_i \times X_i, \quad (34)$$

for all $i \in I$, that are *isomorphisms* (diffeomorphism in our case), which may be written as

$$\forall i \in I : p^{-1}(U_i) \cong U_i \times \mathcal{F}_i. \quad (35)$$

The fact that (34) is such an isomorphism follows from the definitions of the spaces $\mathcal{F}_i$ and X_i by (28) and (33), respectively, and from the fact that each ϕ_i is a diffeomorphism from U_i into $\mathbb{R}^3$ (or equivalently the unit 3-ball B_a with radius a .) We further note that by construction the diffeomorphism τ_i satisfies

$$\text{proj}_1 \circ \tau_i = p, \quad (36)$$

where proj_1 is the standard projection map defined by $\text{proj}_1(x, y) := x$. Finally, if we restrict τ_i to $p^{-1}(\mathbf{r})$, the resulting map

$$\tau_i|_{p^{-1}(\mathbf{r})} : p^{-1}(\mathbf{r}) \rightarrow \{\mathbf{r}\} \times X_i \quad (37)$$

is a (linear) topological vector space isomorphism from $\mathcal{F}_{\mathbf{r}}$ to X_i , i.e.,

$$\forall i \in I, \mathbf{r} \in U_i : \mathcal{F}_{\mathbf{r}} \cong X_i. \quad (38)$$

The charts (U_i, τ_i) are called *trivialization covering* of the vector bundle $\mathcal{M}$. They provide a coordinate representation of local patches of the vector bundle. The global topology of the bundle, however, is rarely trivial. Since here all maps are C^p smooth, τ_i are also called *smooth* trivialization maps. The details of the diffeomorphism (35) and the topological vector space isomorphism (38) are straightforward but lengthy and the full proofs are omitted.

Consider two patches U_i and U_j with $U_i \cap U_j \neq \emptyset$. By restricting τ_i and τ_j to $U_i \cap U_j$, two diffeomorphisms

$$\tau_i : p^{-1}(U_i \cap U_j) \rightarrow (U_i \cap U_j) \times X_i, \quad \tau_j : p^{-1}(U_i \cap U_j) \rightarrow (U_i \cap U_j) \times X_j \quad (39)$$

¹⁴This is how fiber bundles are often introduced in the mathematical literature. However, in this paper, we *construct* the bundle data starting with the physics-based topological structure (21).

are obtained, which in turns implies that

$$(U_i \cap U_j) \times X_i \cong (U_i \cap U_j) \times X_j, \quad (40)$$

or $X_i \cong X_j$ as expected. In particular, it can be shown that the composition map

$$\tau_j \circ \tau_i^{-1} : (U_i \cap U_j) \times X_i \rightarrow (U_i \cap U_j) \times X_j \quad (41)$$

possesses the simple form

$$\tau_j \circ \tau_i^{-1}(\mathbf{r}, \mathbf{F}) = (\mathbf{r}, g(\mathbf{r})\mathbf{F}). \quad (42)$$

Here, $\mathbf{F} \in X_i$ and $g \in L(X_i, X_j)$, where $L(X_i, X_j)$ is the set of linear operators from X_i to X_j . In particular, $g(\mathbf{r})$ is a C^p -Banach space isomorphism. The smooth maps $\tau_j \circ \tau_i^{-1}$ are called the *vector bundle transition maps* and are essential for building up global data by gluing together local data. We have now succeeded in directly constructing a specialized smooth Banach vector bundle $(\mathcal{M}, \mathcal{D}, \tau, p)$ consisting of a total fiber bundle space $\mathcal{M}$, base 3-manifold $\mathcal{D}$, a set of smooth trivialization charts $\tau_i, i \in I$, and a projection map p . The base manifolds $\mathcal{D}$ itself is described by a differential atlas (U_i, ϕ_i) associated to the partition of unity $(U_i, \psi_i), i \in I$.

At this point, we need to describe how the *evaluation* process of the electromagnetic response field (23) may be enacted. The most obvious method is to introduce a *new* vector bundle with the base space being the same base space $\mathcal{D}$, but with the fibers now taken as the complex Hilbert space $\mathbb{C}^3$. This is a well-known vector bundle, which we denote by $\mathcal{R}$, and call the *range vector bundle*. Formally, the structure of this vector bundle is written as $(\mathcal{R}, \mathcal{D}, \tau', p')$, where τ' and p' are its own smooth trivialization and projection maps. The *source vector bundle* is taken as $\mathcal{M}$. The physical process of exciting a nonlocal electromagnetic medium can be understood as follows: i) The material domain is mathematically modeled by the Banach bundle $\mathcal{M}$. The response of the medium is to be sought at some point $\mathbf{r} \in \mathcal{D}$. ii) The bundle structure $\mathcal{M}$ will associate a linear function space at $\mathbf{r}$, namely the fiber $p^{-1}(\mathbf{r})$, which is a Banach space of functions defined on the region U_i . iii) A vector bundle homomorphism (to be formally defined shortly) will map one element of this fiber function space, namely, the particular excitation field $\mathbf{F}(\mathbf{r}), \mathbf{r} \in U_i$, to its value in the fiber isomorphic to $\mathbb{C}^3$ at $\mathbf{r}$ in the range vector bundle $\mathcal{R}$.

Formally, a (smooth) bundle homomorphism over a common base space $\mathcal{D}$ shared between the two vector bundles $\mathcal{M}$ and $\mathcal{R}$ is defined as a (smooth) map

$$\mathcal{L} : \mathcal{M} \rightarrow \mathcal{R} \quad (43)$$

satisfying $p' \circ \mathcal{L} = p$. Moreover, the restriction of $\mathcal{L}$ to each fiber $p^{-1}(\mathbf{r})$ induces a linear operator on the vector space of that fiber [58]. Such mapping is best illustrated graphically through the simple commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathcal{L}} & \mathcal{R} \\ & \searrow p \quad \swarrow p' & \\ & \mathcal{D} & \end{array}$$

We note that because $\mathcal{M}$ and $\mathcal{R}$ share the same base manifold $\mathcal{D}$, the action of the map $\mathcal{L}$ is effectively reduced to how it acts on each fiber $p^{-1}(\mathbf{r})$ as a linear operator. Since the Banach space X_i is isomorphic to $p^{-1}(\mathbf{r})$, we will express $\mathcal{L}$ by giving its

expression locally in each element $U_i \subset \mathcal{D}$ of the open cover $\{U_i | i \in I\}$. In particular, we define the local action using the source and range bundles' trivialization maps τ_i and τ'_i by

$$\tau'_i \circ \mathcal{L}_\omega \circ \tau_i^{-1} : U_i \times X_i \rightarrow U_i \times \mathbb{C}^3, \quad (44)$$

with

$$\tau'_i \circ \mathcal{L}_\omega \circ \tau_i^{-1} := (\mathbf{r}, \mathcal{L}_{i,\omega} \mathbf{F}), \quad \mathbf{F} \in X_i, \quad (45)$$

where

$$\mathcal{L}_{i,\omega} : X_i \rightarrow \mathbb{C}^3 \quad (46)$$

is the linear operator defined by

$$\mathcal{L}_{i,\omega}(\ast) = \int_{U_i} d^3 r' \overline{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) \cdot (\ast), \quad (47)$$

in which “ $\ast$ ” stands for an element of the smooth Banach function space X_i . The map $\mathcal{L}$ then will leave every point in the base space unchanged, while it maps each smooth function on U_i (component of the total electromagnetic excitation field, see below) to its vector value in $\mathbb{C}^3$ at $\mathbf{r} \in U_i$.

The final step is tying up together the fundamental source Banach bundle $\mathcal{M}$, range bundle $\mathcal{R}$ and the EM nonlocal microdomain physics space (21). As it stands, the latter was developed for one domain D , not a manifold $\mathcal{D}$, but the essential ingredients of the *physics* of nonlocal EM field-matter interaction are encoded in the geometrical construction of the collection of microdomains $V_{\mathbf{r}}, \mathbf{r} \in D$, and the excitation fields $\mathbf{F}(\mathbf{r})$ defined on them, i.e., the sets $\mathcal{V}(D)$ and the function spaces $\mathcal{G}(D)$. So far, the vector bundle homomorphism $\mathcal{L}$ introduced above takes care of excitation fields supported on the open sets $U_i, i \in I$. However, these are *mathematical* fundamental building blocks used to construct the source vector bundle $\mathcal{M}$. The question now is to how to extend the description of nonlocal EM response operators for excitation fields applied to the entire cluster of EM nonlocality microdomains $\{V_{\mathbf{r}} | \mathbf{r} \in \mathcal{D}\}$. As mentioned before, it is the partition of unity $(U_i, \psi_i), i \in I$, what will make this expansion of the topological formulation possible. To see this, let us consider an electromagnetic field $\mathbf{F}(\mathbf{r})$ interacting with a nonlocal medium extended over the manifold $\mathcal{D}$. Our goal is to compute the response field $\mathbf{R}(\mathbf{r})$, that is, at point $\mathbf{r}$. The fundamental idea of EM nonlocality is that to know the response at one point $\mathbf{r}$, one must know the excitation field in a entire open set $V_{\mathbf{r}}$, a topological neighborhood of $\mathbf{r}$, and that in general this microdomain will change depending on the position $\mathbf{r}$. The goal now is to find $\mathbf{R}(\mathbf{r})$ using the vector bundle map $\mathcal{L}$ (43) given the data i) region $V_{\mathbf{r}}$ and ii) vector field $\mathbf{F}(\mathbf{r})$ acting on $V_{\mathbf{r}}$. We exploit the properties of the partition of unity functions ψ_i to expand the field $\mathbf{F}(\mathbf{r})$ over all patches U_i that cover $V_{\mathbf{r}}$ and write

$$\mathbf{F}(\mathbf{r}'; \omega) = \sum_{i \in I_{\mathbf{r}}} \psi_i(\mathbf{r}') \mathbf{F}_i(\mathbf{r}'; \omega), \quad (48)$$

where (27) was used. The truncated function $\mathbf{F}_i$ is equal to $\mathbf{F}(\mathbf{r})$ only if $\mathbf{r} \in U_i$ and zero elsewhere, i.e.,

$$\mathbf{F}_i(\mathbf{r}'; \omega) := \begin{cases} \mathbf{F}(\mathbf{r}'; \omega), & \mathbf{r} \in U_i, \\ 0, & \mathbf{r} \notin U_i. \end{cases} \quad (49)$$

Recall that $I_{\mathbf{r}}$ is defined as the indices $i \in I$ of all U_i having the point $\mathbf{r}$ in their common set intersection. In other words, each function $\psi_i(\mathbf{r}') \mathbf{F}_i(\mathbf{r}'; \omega)$ is a smooth component

of the total excitation field $\mathbf{F}$ with support contained fully *inside* the coordinate patch U_i . Thus, the vector bundle map constructed in (43) can be applied to each such component field. From (44)-(46) and (48), the following can be deduced

$$\mathbf{R}(\mathbf{r}; \omega) = \sum_{i \in I_r} \mathcal{L}_{i, \omega} [\psi_i(\mathbf{r}') \mathbf{F}_i(\mathbf{r}'; \omega)]. \quad (50)$$

Finally, using (47), we arrive at our main expression

$$\mathbf{R}(\mathbf{r}; \omega) = \sum_{i \in I_r} \int_{U_i} d^3 r' \bar{\mathbf{K}}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \psi_i(\mathbf{r}') \mathbf{F}_i(\mathbf{r}'; \omega). \quad (51)$$

The relation (51) indicates that the source bundle $\mathcal{M}$, $\mathcal{R}$, and the response map $\mathcal{L}$ provide a skeleton through which the total response to any EM excitation field defined on an arbitrary EM nonlocality microdomain can be computed. In this way, the vector bundle formalism for electromagnetic nonlocality is essentially complete and the connection between the purely mathematical fiber space and the physical microdomain structures is secured by (51).

6. Applications and Future Work

We provide a general outline of several possible applications, where some of them highlights the theory developed in this paper. Issues pertinent to fundamental considerations (physical and mathematical) and engineering functions are taken up in Secs. 6.1 and 6.2, respectively. All of the coming discussions is kept very brief and only the essential main ideas are given. Some of these applications are expected to be investigated more extensively in the near future.¹⁵

6.1. Fundamental Theory

Limitations on Nonlocal Metamaterials. Maps like $\mathcal{L}$ (43) can be reformulated in the space of *vector bundle sections* [55, 58, 59], a subject that is extremely well developed in classic differential topology. In fact, the electromagnetic response function $\mathbf{R}$ itself can sometimes be obtained by working directly with the source bundle $\mathcal{M}$. For example, under some conditions, this can be achieved by replacing each fiber X_i by $X_i \times \mathbb{C}^3$. In this way, the entire electromagnetic nonlocal response problems becomes identical to the investigation of how vector bundle sections interact with the topology of the underlying base manifold $\mathcal{D}$. There is an extremely large literature in differential topology and geometry focused on this problem, especially how local information can be propagated to extend into global structures [5, 56, 58]. The author believes that by starting from local data in a given nonlocal metamaterial domain, e.g., the global shape of the device, the distribution of topological holes, etc, one may then use existing techniques borrowed from differential topology, e.g., the theory of characteristic classes, to determine the allowable EM response functions permissible in principle at the global level. Engineers are typically interested in knowing in advance what the best (or worst) performance measures obtainable from specific topologies are, and hence reformulating the electromagnetics of nonlocal metamaterials

¹⁵No attempt at providing anything even remotely close to a comprehensive review of the large literature on nonlocal metamaterials is given here and the list is inherently selective.

in terms of vector bundles could be of help in this respect since it opens a pathway toward a synergy between general topology, physics, and engineering in the field of metamaterials.

Numerical Methods. Traditional full-wave numerical methods are sometimes deployed to deal with nonlocal EM materials, often using the additional boundary conditions framework, in spite of the latter’s lack of complete generality. By formulating the source space of field-matter interaction in terms of a Banach bundle, it should be possible to reformulate Maxwell’s equations to act on this extended geometric space instead of the conventional spacetime framework. Some of the advantages expected from such reformulation is the ability to resolve the issue of generalized boundary condition (more on this below). Moreover, since every point belonging to a fiber space is in itself a smooth function defined on an entire material sub-microdomain, by building a new system of discretized recursive equations approximating the behaviour of electromagnetic solutions living in the enlarged spaces $\mathcal{M}$ and $\mathcal{R}$, one may expect deeper understanding of the physics of nonlocality since the topology of the nonlocal interaction regime is explicitly encoded into the geometry of the new expanded solution space $\mathcal{M}$ itself. It is also possible that such numerical methods may emerge as more computationally efficient and broader in applicability than the conventional methods rooted in local electromagnetics. One reason for this is that the Banach vector bundle formulation introduced in this paper is quite natural and appears to reflect the underlying physics of nonlocal metamaterials in a direct manner.

Electromagnetic Boundary Conditions. The well-known tension between nonlocal electromagnetics and material interfaces has been already mentioned several times above. Here, we provide some application of the fiber bundle theory of Sec. 5 aiming at elucidating the nature of the tension and to suggest some possible new formulation of the problem. The starting point is Fig. 2, where a zoomed-in topological picture based on the general structures explicated in Sec. 4 is given. The focus now is on the *interface* between two generic nonlocal domains $\mathcal{D}_m$ and $\mathcal{D}_n$. In traditional local electromagnetics, the constitutive relation material tensor $\bar{\mathbf{K}}_n$ is exploited to deduce conditions dictating how various electromagnetic field components behave as they cross the $\mathcal{D}_m/\mathcal{D}_n$ interface. However, even if each $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$ was to be treated as that belonging to a spatially dispersive domain, i.e., by replacing it by $\bar{\mathbf{K}}_n(\mathbf{r} - \mathbf{r}')$, the presence of a boundary completely destroys the translational symmetry of the structure on which the very form of the spatial dispersion nonlocal response tensor $\bar{\mathbf{K}}_n(\mathbf{r} - \mathbf{r}')$ is based. This was very clearly explained in [4], with several proposals for a solution of the electromagnetic problem. For example, since close to the interface it is very obvious that the material tensor must be reverted back to the most general nonlocal form, namely $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$, it was then proposed that one may use the latter form only in a thin region containing the interface on both sides. Outside this region, a gradual transition or a continuous profile (tapering) is introduced to transition from the forms $\bar{\mathbf{K}}_n(\mathbf{r}, \mathbf{r}')$ and $\bar{\mathbf{K}}_m(\mathbf{r}, \mathbf{r}')$ to the spatially dispersive forms $\bar{\mathbf{K}}_n(\mathbf{r} - \mathbf{r}')$ and $\bar{\mathbf{K}}_m(\mathbf{r} - \mathbf{r}')$ characteristic of “bulk” homogeneous material domains [4]. Another proposal is to keep the spatial dispersion profiles $\bar{\mathbf{K}}_n(\mathbf{r} - \mathbf{r}')$ and $\bar{\mathbf{K}}_m(\mathbf{r} - \mathbf{r}')$ everywhere but introduce specialized additional boundary conditions (ABCs) at the interface suitable for the problem at hand. Although this latter approach is neither consistent mathematically nor physically (because of the breakdown of symmetry caused by the presence of an interface), it nevertheless remains popular because – at least in outline – nonlocal electromagnetics is thereby held up in a form as close

as possible to familiar local electromagnetic theory methods, especially numerical techniques such as Finite Element Method (FEM), Method of Moment (MoM), and Finite Difference Time-Domain Method (FDTD), i.e., established full-wave algorithms where it is quite straightforward to replace one boundary condition by another without essentially changing much of the code.¹⁶ However, both of these approaches discussed above require a considerable input from *microscopic* theory, mainly to determine the tapering transition region in the case of the first, and the ABCs themselves in the second. That motivated the third approach, called, the ABC-free formalism, where the relevant microscopic theory was utilized right from the beginning in order to formulate and solve Maxwell's equations. For example, in [9, 10], a global Hamiltonian of the matter-field system is constructed and Maxwell's equations are derived accordingly. In [11], the rim zone (field attached to matter) is investigated using different physical assumptions to understand the transition from nonlocal material domains to vacuum going through the entire complex near-field zone. We believe that main common conclusion from all these different formulations is that in nonlocal electromagnetics *it is not possible in general to formulate the electromagnetic problem at a fully phenomenological level*. In other words, microscopic theory appears to be in demand more often than in the case of systems involving only local materials. However, since all existing solutions use the traditional spatial manifold $\mathcal{D}$ as the main configuration space, the question now is whether the alternative formulation proposed in this paper, the extended fiber bundle approach, may provide some additional insights.

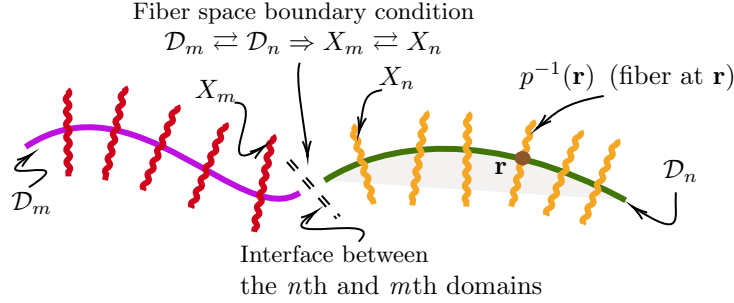


Figure 3: An abstract representation of the fiber bundle structure behind Fig. 2.

We provide a provisional elucidation of the topological nature of electromagnetics across material interfaces by nothing that in Fig. 2, not only the behaviour of the fields $\mathbf{F}(\mathbf{r})$ in the two domains is relevant, but also the entire local topological microdomains $V_{\mathbf{r}}$ clustered inside, and in particular *how these topological regions with fields on them behave as they move across the boundary*. In general topology, *boundaries* are defined fully in terms of the behaviour of open sets. We will build on this key concept in order to show the the problem of nonlocal electromagnetics across interfaces may be reformulated. First, Fig. 3 provides a finer or more structured picture of the topological structure of Fig. 2 based on replacing the spaces $\mathcal{D}_m$ and $\mathcal{D}_n$ by the corresponding Banach bundles $\mathcal{M}_m$ and $\mathcal{M}_n$, respectively. The thick horizontal curved lines represent the bases spaces $\mathcal{D}_m$ and $\mathcal{D}_n$, while the wavy vertical lines stands for the fibers spaces X_m and X_n attached to each point in the corresponding base manifolds. The double discontinuous lines at the “junction” of the two base spaces $\mathcal{D}_m$ and $\mathcal{D}_n$ indicate the joining together of the two vector bundles $\mathcal{D}_m$ and $\mathcal{D}_n$. It is clear now

¹⁶This is more obvious in FEM and FDTD than MoM.

that since the two nonlocal material domains possess an *extra structure*, namely that of the fiber spaces attached to each point in the base space, *we must also indicate how the various elements belonging to the Banach function spaces, the fields defined on the microdomains $V_{\mathbf{r}}$ in Fig. 2, behave as they cross the boundary separating $\mathcal{D}_m$ and $\mathcal{D}_n$.* One obvious way to do this is to introduce a *bundle homomorphism* [58] between the two vector bundles $\mathcal{M}_m$ and $\mathcal{M}_n$ over the interface submanifold $\partial\mathcal{D}_{mn}$ separating $\mathcal{D}_m$ and $\mathcal{D}_n$.¹⁷ The goal of this bundle homomorphism is to serve as a “boundary condition operator” acting on the fiber bundle nonlocal domains $\mathcal{M}_m$ and $\mathcal{M}_n$ instead of $\mathcal{D}_m$ and $\mathcal{D}_n$. We will not go here into constructing this operator in details, but provide some additional remarks to illustrate the main idea. The traditional boundary condition applied in the base space will be denoted by $\mathcal{D}_m \rightleftharpoons \mathcal{D}_n$ and is usually spelled out in the form

$$\lim_{\mathbf{r} \rightarrow \partial\mathcal{D}_{mn}} \{\mathbf{F}_m(\mathbf{r}) - \mathbf{F}_n(\mathbf{r})\} = \Gamma_{b_1}[\mathbf{F}_m(\mathbf{r}), \mathbf{F}_n(\mathbf{r})], \quad (52)$$

$$\lim_{\mathbf{r} \rightarrow \partial\mathcal{D}_{mn}} \{\mathbf{R}_m(\mathbf{r}) - \mathbf{R}_n(\mathbf{r})\} = \Gamma_{b_2}[\mathbf{R}_m(\mathbf{r}), \mathbf{R}_n(\mathbf{r})], \quad (53)$$

where $\partial\mathcal{D}_{mn}$ is the boundary between $\mathcal{D}_m$ and $\mathcal{D}_n$. Here, Γ_{b_1} and Γ_{b_2} are “base space boundary functions.” On the other hand, the fiber bundle elements, i.e., functions defined on the microdomains $V_{\mathbf{r}}$, are mapped by

$$X_m \rightleftharpoons X_n : \quad \lim_{\mathbf{r} \rightarrow \partial\mathcal{D}_{mn}} (X_m - X_n) = \Gamma_f[X_m, X_n], \quad (54)$$

where Γ_f is a different fiber space boundary function. The full formulation is more complex because the boundary condition operator must also be proved to be compatible with the fiber bundle structures and so the entire *global topology* of $\mathcal{M}_m$ and $\mathcal{M}_n$ will interact with the effective final *electromagnetic* boundary condition resulting from this process. The main conclusion related to us here is that the existence of *extra* or *additional* structures in the fiber bundle space of electromagnetic nonlocality *makes the need for additional boundary conditions or information coming from the microscopic topological structure very natural. The fiber bundle formalism of nonlocal metamaterials does capture the physics of nonlocal domains joined together through interfaces.* The full formulation of the proposed fiber bundle boundary condition homomorphism is beyond the scope of this paper but it is hoped that this initial insight can at least clarify the subject and stimulate further research in the fundamental theory of nonlocal metamaterials.

6.2. Engineering Applications

Topological photonics. One of the main applications of the proposed vector bundle formalism is that it opens the door for a new way to investigate the topological structure of materials. It has already been noticed that nonlocal EM response is essential in topological photonics, e.g., see [27, 33]. Indeed, since in topological photonics the wavefunction of bosons, usually the Bloch state, is examined over the entirety of momentum space (usually the Brillouin zone), then it is the dependence of the EM response on $\mathbf{k}$ what is at stake, which naturally brings in nonlocal issues. Since now using our theory we can associate with every nonlocal material a concrete fiber bundle reflecting all the rich information about the topological microdomains and the global shape of the material plus the impact of the boundaries separating

¹⁷This mathematical object is similar to the nonlocal response map $\mathcal{L}$ introduced by (43).

various material domains, it is natural to examine whether a topological classification of the corresponding fiber bundles may lead to a new way to characterize the topology of materials other than the Chern invariants used extensively in literature. The advantage of our theory in this case is that the complicated topological and geometrical aspects of the boundaries and inhomogeneity in nonlocal media can be encoded very efficiently in the local structure of the material fiber bundle. Using standard techniques in differential topology [55], it should be possible to propagate this local information to the global domain (the entirety of the system), for example by computing fiber bundle topological invariants like its homology groups [59]. Our approach is then a “dual” to the standard approach since we work on an enlarged configuration space (spacetime or space-frequency), while the mainstream approach operates in the momentum space of the wavefunction.

Digital communications. Nonlocal metamaterials offer a very wide range of potential applications in wireless communications and optical fibers. The basic idea is to introduce specially engineered nonlocal domains either as part of the communication channel (e.g., optical fibers, plasmonic circuits, microwave transmission lines) [52], or as a control structure integrated with existing antennas [39]. Spatial dispersion was also used as a method to engineer wave propagation characteristics in material domains, e.g., see [60] for applications to high-efficiency modulation of free-space EM waves. A general linear partial equation explicating how spatial and temporal dispersion can be jointly exploited to produce zero distortion (e.g., constant negative group velocity) was derived and solved in [61]. The main idea originated from the fact that distortion in communication systems emerge from nonconstant group velocity $\mathbf{v}_g := \nabla_{\mathbf{k}}\omega$. Since $\mathbf{v}_g$ is a strong function of the dependence of the material response tensor $\bar{\mathbf{K}}(\mathbf{k}, \omega)$ on both $\mathbf{k}$ and ω , *dispersion management equations* can be derived for several applications. For example, it was proved in [61] that in simple isotropic spatially dispersive media with high-symmetry, one may obtain exact solutions where the group velocity is constant at an entire frequency band.

Electromagnetic Metamaterials. It was proposed as early as in the 1960s that EM nonlocality can be exploited to produce materials with very unusual properties. For example, in [4], negative refraction materials were noted as one possible application of spatial dispersion where the path toward attaining this goes through controlling the direction of the group velocity vector. Since in nonlocal media power does not flow along the Poynting vector [2], new (higher-order) effects were shown to be capable of generating arbitrary group velocity profiles by carefully controlling the spatial and temporal dispersion profiles. Overall, the ability of spatial dispersion to induce higher-order corrections to power flow is a unique added advantage enjoyed by nonlocal metamaterials exhibiting spatial dispersion in addition to normal dispersion. This extra spatial degrees of freedom provided by space was researched, reviewed and highlighted many publications, including for example [15, 23, 25, 36, 39, 45, 49, 62–66].

Near-Field Engineering and Energy. Another interesting application of nonlocality in electromagnetic media is near-field engineering, a subject that has not yet received the attention it deserves but has seen some activity lately especially with connection to the spatio-temporal structure of the electromagnetic near field energy, e.g., see [42, 67–85]. It was observed in [62] that a source radiating in homogeneous, unbounded isotropic spatial dispersive medium may exhibit several unusual and interesting phenomena due to the emergence of extra poles in the radiation Green’s function of such domains. Both longitudinal and transverse waves are possible (dispersion relations), and the dispersion engineering equations relevant

to finding suitable modes capable of engineering desired radiation field patterns are relatively easy to set and solve. For example, by carefully controlling the modes of the radiated waves, it is possible to shape the near field profile, including total confinement of the field around the antenna even when losses is very small, opening the door for applications like energy harvesting, storage, and retrieval in such media. This subject, however, has been explored only for simple materials so far and mainly at the theoretical level [39].

7. Conclusion

We have provided a general theoretical and conceptual investigation of nonlocal metamaterials aiming at achieving several goals. First, the subject was reviewed from a new perspective with the intention of introducing it to a wide audience, including engineers, applied physicists, and mathematicians. The various essential ideas behind EM nonlocality were viewed in new light using an abstract field-response model in three dimensions. Next, the fine-grained topological micro-structure of nonlocal metamaterials was explicated in details. We introduced EM nonlocality microdomains and showed that they present an important structural topological feature of the physics of nonlocal media. After that, it was proved using differential topology that a natural fiber bundle structure serving as a source space can be constructed. The source fiber bundle was shown to have all the required properties of standard fiber bundles while faithfully reflecting the physics of EM nonlocality microdomains. Eventually, and using the technique of partition of unity, it was proved that the source fiber bundle can be used to construct and compute the material response function over arbitrary microdomains. The new fiber bundle formulation suggests that EM nonlocality can be formulated in an alternative way compared with other existing methods that borrow heavily from the electromagnetics of local media. Most importantly, EM nonlocality forces us to consider an entire infinite-dimensional Banach space attached to each point in the conventional 3-dimensional space on which the material is defined. This extra or additional structure provides a natural explanation of why traditional boundary conditions often fail to account for the physics of nonlocal metamaterials. Moreover, the fiber bundle theory opens the door for several new applications, including the ability to understand the deep connection between topology and electromagnetics in engineered media. Overall, the author proposes that future research in metamaterials will gradually require more extensive collaboration between engineers and mathematicians to explore the full consequences of this organic topology/electromagnetics relation.

Appendix: A Guide to the Mathematical Background

We provide an informal overview regarding how to read the mathematical portions of this paper and where to find detailed references that might be needed in order to expand some of the technical proof sketches provided in the main text. We emphasize that in this paper only the *elementary* definitions of 1) differential manifolds, 2) Banach and Sobolev spaces, 3) vector bundles, and 4) partition of unity are needed to understand the mathematical development.

Differential manifolds. A differential manifold is a collection of fundamental “topological atoms” each composed of an open set U_i and a chart $\phi_i(x)$, which serves as a coordinate system, basically an invertible smooth map to the Euclidean space

$\mathbb{R}^n$. That is, locally, every manifold looks like a Euclidean space with dimension n . The collection of open sets $U_i, i \in I$, where I is an index set, covers this n -dimensional manifold. Since some of these open sets are allowed to overlap, the key idea of the differential manifold is that on the overlap region $U_i \cap U_j$, there exist a smooth reversible coordinate transformation function connecting the coordinates of the same point when expressed in the two (different) languages of the topological atoms U_i and U_j . The key concept of topology is how to propagate information from the local to the global. In this sense, differential manifolds present elementary structure allowing us to model this process using the efficient technology of the differential calculus. Only the basic definition of smooth manifolds is required in this paper, which can be found in virtually any book on differential or Riemannian geometry, e.g., see [5, 55, 57–59, 86–88].

Banach and Sobolev spaces. In a nutshell, a Banach space is a vector space equipped with a norm satisfying the familiar properties (being positive, being zero only for the null vector, scale linearity, and the triangle inequality.) Most importantly, Banach spaces are also required to be *topologically complete* in the sense that every Cauchy sequence converges to an element in the space itself. In this way, no “holes” are left in the space and one may deploy them to perform analysis like solving differential equations. A Hilbert space is a Banach space equipped with an inner product. An important thing to note about Banach and Hilbert spaces is that when they are used to model *function* spaces (as in this paper), they lead to *infinite* dimensional vector spaces [86]. Sobolev spaces are Hilbert spaces consisting of (Lebesgue) square integrable functions that possess “generalized derivative,” a concept in itself technical but straightforward, e.g., see [57] for a very readable account. In Sec. 4, we introduced Sobolev space over the open domain D instead of simply Banach space. However, that was done mainly to simplify the technical development and in anticipation of future work. Indeed, in this paper, the fiber bundle $\mathcal{M}$ is referred to just as Banach bundle, not Sobolev bundle for the reason that all our essential results and insights apply to the more general concept of Banach space than Sobolev space. The latter however, is easier to implement and we only invoked here the key definition of the space itself. In particular, none of the other technical properties of Sobolev spaces are needed in the paper. However, since in the future the material bundle space $\mathcal{M}$ is expected to be used to construct solution of Maxwell’s equations in new form, Sobolev spaces are projected to play the most important role since they have proved very efficient in analysis. For the basic definition of Sobolev spaces and their applications to partial differential equations in mathematical physics and finite-element method in engineering, we recommend [57]. The subject of Banach manifolds is less commonly treated in literature than finite-dimensional manifolds, but good concise treatments of the topic include [5, 59, 87].

Vector bundles. Fiber bundles, of which vector bundles are special case, are now standard topics in both mathematics (topology, geometry, differential equations), theoretical physics (quantum field theory, cosmology, quantum gravity), and applied physics (condensed-matter physics, many-body problems). For the major importance of vector and fiber bundles within the overall structure of modern fundamental physics, see [86, 88, 89]. In quantum field theory, gauge field theories use vector bundles as essential ingredients in the standard model of particle physics [86]. In condensed-matter physics, the increasing role played by quantum field theory in applications to condensed-matter physics has made knowledge of fiber bundles useful and more widespread, e.g., see the Berry phase and the associated gauge connection [31, 33].

The key idea of a vector bundle is to attach an entire vector space to every point on a base manifold. To be more specific, consider a differential manifold $\mathcal{D}$. Each such vector space will be called the *fiber* at that point. The tangent space of the manifold is the most obvious example of such vector bundles. However, more complicated structures than finite-dimensional tangent spaces can also be encoded by the vector bundle concept. In this paper, we have shown that EM nonlocality can be modeled naturally by considering the Banach space of all fields on the microdomains based at a point in the material configuration space. Fiber bundles then can be seen as highly efficient and economic ways to encapsulate large amount of topological and geometrical data and they lend themselves easily to complex calculations. Very readable technical descriptions of vector bundles can be found in [56, 58, 59, 86].

Partition of unity techniques. These are somehow technical tools used by topologists to propagate information from the local to the global and are quite handy and easy to apply. The main theorems allow moving from one topological atom to another by “gluing” them together using smooth standard domain-division functions. The technique was stated and used only toward the end of Sec. 5 to write the expansion (48) and can be skipped in first reading of the paper. Partition of unity is usually taught in all topology and some geometry textbooks, e.g., see [5, 56, 58, 59].

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