Nonlocal Antenna Theory - Part I: Foundations

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Abstract

We propose and develop comprehensive foundations for the analysis of nonlocal radiating systems using a special momentum-space approach. Part I focuses on the rigorous mathematical basis of the theory and their conceptual implications.

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Nonlocal Antenna Theory – Part I

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Abstract—Nonlocal radiating systems are functional structures comprised of externally applied currents radiating in nonlocal domains, for example hot plasma, optically active media, or nano-engineered spatially dispersive metamaterials. We develop here the mathematical foundations of the subject needed for investigating how such new generation of radiating system can be analyzed at a very general level. A key feature in our approach is the adoption of a fully-fledged momentum space perspective, where the spacetime Fourier transform method is exploited to derive, analyze, and understand how externally-controlled currents embedded into nonlocal media radiate. In particular, we avoid working in the spatio-temporal domain as is typical in conventional local radiation theory. Instead, we focus on the basic but nontrivial problem of infinite generic (anisotropic or isotropic) homogeneous nonlocal domain excited by an external source and investigate this structure in depth by deriving the dyadic Green's functions of nonlocal media in the momentum space. Afterwords, the radiated energy in the far-zone is estimated directly in the spectral domain using a generalized momentum space energy density concept and coupled with the power theorem. The resulting derived expression of the radiation pattern power of the source can be computed analytically provided the medium dielectric functions and the dispersion relations of the nonlocal metamaterial are available.

1. INTRODUCTION

The main objective of this paper is to formulate the main themes of electromagnetic radiation theory in a language conductive to research on novel and future types of radiating systems, in particular those operating in complex non-classical environments best described by a *nonlocal* electromagnetic material response function. *Nonlocality* includes most prominently *spatial dispersion*, i.e., the dependence of the material response function on the wavevector $\mathbf{k}$ besides the classical (temporal) dispersion characterized by the appearance of another dependence on ω, the circular frequency [1–3]. Inspired by the earliest formulation of the problem of electromagnetic wave propagation in spatially-dispersive media, we adopt the Fourier space approach to solving and studying the less-known problem of antenna analysis and design in such media. The Fourier space approach replaces the frequency domain formulation where the fields are considered in the frequency-space domain, i.e., functions in the form $\mathbf{F}(\mathbf{r},\omega)$, by moving to a fully-fledged 4-dimensional Fourier space where all fields (electromagnetic fields and their current sources) take the form $\mathbf{F}(\mathbf{k},\omega)$. Following the common convention in physics, we capture the dependence on $\mathbf{k}$ by the term *momentum space* since momentum $\mathbf{p}$ and the wavevector $\mathbf{k}$ are related to each other in quantum physics by mere constant (the de Broglie relation $\mathbf{p}=\hbar\mathbf{k}$). The momentum space formulation of electromagnetic theory is extensively used in diverse disciplines, including condensed-matter physics [4], plasma physics [3], quantum field theory [5], quantum optics [6]. However, momentum space does not seem to have been widely used in classical antenna theory where most treatments tend to favor the frequency-space formulation, with some exceptions like [7], [8–10]. For example, the plane-wave spectrum, a momentum representation of EM fields, was deployed for applications to near-field measurement [11], computation of Green's functions in inhomogeneous media [12, 13], subwavelength imaging [14], and characterizing mutual coupling and interactions [7, 9, 15–17]. Periodic structures are examples of systems in which wave propagation analysis is fundamentally conducted in the spatial Fourier space, although in that case it is usually referred to as *reciprocal space* [18].

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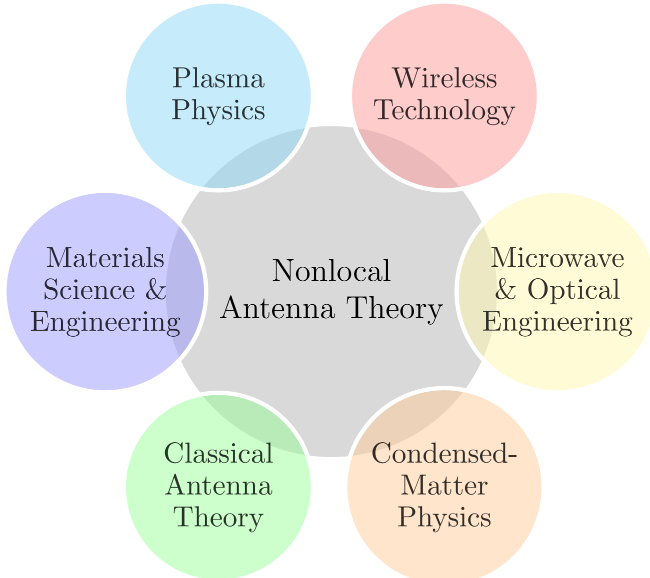

Figure 1: Nonlocal antenna theory is a new emerging cross-disciplinary research area involving several existing disciplines.

Some previous work on controlling the radiation emitted by sources embedded into metamaterials (MTMs) include [19–24], where most of the focus is on conventional metamaterials used to modify the emission characteristics o optical sources in metamaterials. However, all conventional metamaterials exhibit spatial dispersion so the subject of spatial dispersion has been taken up more explicitly in more recent works such as [25]. In this paper, we propose a momentum space formalism for antenna theory using techniques that had originated in some earlier applications in physics but here adapted and extended for the needs of antenna theory in engineering and applied physics. Our goal is to sketch out in broad manner the general ideas, basically how to define radiation patterns and array theory in momentum space instead of time-space or frequency-space when the radiation domain is filled up with infinite homogeneous generic nonlocal metamaterial (NL-MTM). A good theory of electromagnetic radiation in nonlocal domains should also provide a framework for understanding how the surrounding metamaterial domain itself should be designed such that the combined current source/MTM can deliver new functional performance. The main application of the theory is for future antennas utilizing engineered metamaterials exhibiting carefully-tailored nonlocal behaviour, where in that particular case we argue that the Fourier space approach adopted here provides the best means to tackle the subject [10, 26]. One of the key advantages of the proposed theory is that only dispersion relations are needed to construct the radiation pattern (far-zone energy/power density) using an analytical procedure that can be easily automated and canonized. The proposed theory is valid for arbitrary isotropic and anisotropic homogeneous domains surrounding a fully arbitrary radiating (external) current. The only restriction imposed on our metamaterial in this theory is that it must be homogeneous. Aside from this, the entire theory is developed using exact analysis and no approximations are made. The proposed theory is cross-disciplinary and is expected to involve multiple different fields of research, including both fundamental theory and applications, see Fig. 1.

This theory has its background in previous research conducted by the author and collaborators during the last 15 years in which the focus has been on looking for new fundamental ideas and perspectives relevant to both theory and applications of electromagnetic waves and materials [15, 27–33, 33–39], where the emphasis has been from the start on the fundamental *spatial* structures hidden in fields and space and during their interaction with complex material domains [26, 40–46], especially in connection with near fields [8, 47–50], energy [9, 51–54], and nanoelectromagnetics [55–62]. The purpose is to systematically and comprehensibly explicate hpw adding additional spatial degrees of freedom within the context of field-matter correlation/interaction can enhance our understanding of electromagnetic radiation and possible suggest new applications to antenna technology.

The theory of nonlocal antenna systems is expounded in two parts. Part I (the present paper) focus is on the generic and fundamental theoretical aspects of the problem developed rigorously but also provides some additional philosophical and methodological guidelines and remarks on the emerging research field of nonlocal antenna systems. Part II [63] is concerned with applying the general theory developed in Part I to concrete settings (specific metamaterials and radiating sources) and will be reviewed in its own introduction. For the remainder of the current introduction, we give an overall view on the various sections to follow. Sec. 2 provides an overall philosophical and conceptual take on the emerging research area whose main object of study is nonlocal antenna systems. We propose a precise definition of such systems and explain their physical nature and explain why it is important to investigate their behaviour and explore methods to design and build them. The theory proper starts in Sec. 3 where the presentation commences from Maxwell's equations in space-time and gradually introduces all tools needed, like the 4-dimensional Fourier transforms, the proper vector potential formalism, the generalized material (dielectric) response method, and tensor Green's functions. The main goal is to set up the problem entirely in momentum space with all main results created from first principles to facilitate understanding with maximum possible clarity and self-completeness. The Green's functions of nonlocal domains thus derived is exploited in Sec. 4 to recruit an idea originally due to Brillouin [64] in which the far field radiation energy density is estimated directly from the source without the need to invert the spectral Green's functions to obtain the spatio-temporal fields first. Therefore, we can obtain the angular frequency-dependent radiation energy density in the far field, but not the fields themselves in space-time. Since the initial efforts of researchers in the emerging domain of nonlocal antenna systems are now focused on the far zone radiation pattern, these results obtained analytically here using momentum space methods should suffice for the near future. Finally, in Sec. 5 we summarize the overall analytical and computational aspects of the derived radiation pattern expressions and also point out the pros and cons of the proposed theory in addition to providing some remarks on future antennas Finally, we end with the conclusion. At the end, Appendices are given to collect technical results used throughout the present paper in addition to and various mathematical properties and theorems that will be also used in Part II to perform calculations.

2. AN OVERALL VIEW ON NONLOCAL RADIATING SYSTEM: WHAT THEY ARE AND WHY DO WE NEED TO STUDY THEM

Why do we need to study nonlocal antenna systems? First of all, the theory of how spatial dispersion modifies radiation by external sources is interesting in itself and hence has aroused the curiosity of some past researchers several decades ago, with deep roots going back into the earliest days of plasma and condensed-matter physics. Indeed, like any new research area, the subject had not suddenly erupted into the scene without any precursor, but is the cumulative outcome of a long process in waiting. The study of spatially dispersive problems goes back to the 1940s and 1950s when people were investigating propagation of electromagnetic waves in crystals and plasma. The subject till very recently was treated as part of optics and plasma physics where spatial dispersion is key to explaining the generalized response function of magnetic materials (spatial dispersion can be induced by such magnetic response) and optical activity. However, nowadays a resurgence in nonlocal metamaterials is due to the general desire to enlarge the concept of materials from "natural" to "artifical" or metamaterials. Since classical antennas have been investigated very extensively either in free space or temporally dispersive media, nonlocal antenna theory, or the analysis of radiators in nonlocal domains, is a natural enlargement of radiation and antenna theory that is now very timely. A second advantage of studying nonlocal radiation theory is the fact that nonlocal wave propagation may lead to completely different physical behaviour absent in even the "wildest" types of temporally dispersive metamaterials. Some of these new phenomena will be briefly treated in Part II, including virtual arrays, longitudinal waves, and the remarkable ability to design perfectly isotropic radiating structures using dipole antennas as external source. We also add some other potentials, like negative group velocity and dispersion management [65], energy storage and recovery [10, 45, 46], directive emission control using MTMs [19, 25]. These applications and numerous others strongly suggest the need for systematic research programs focusing on exploring new applications with this novel spatially responsive generation of engineered material domains. However, to understand

and motivate some or all of these applications, a strong theoretical and conceptual foundations for the topic, best given in the shape of a comprehensive rigorously developed mathematical theory of radiation by external sources in nonlocal MTM domains, is needed. This paper attempts to provide an initial general theory in this direction.

We next define these nonlocal radiating systems and also provide some additional remarks about the general scope of the present two-part paper. A nonlocal antenna system (NL-AS) is defined as an engineered structure composed of two major components:

(i) An *externally*-controlled current distribution $\mathbf{J}_{\rm ant}(\mathbf{r},t)$.

(ii) A surrounding *nonlocal* metamaterial domain into which the current $\mathbf{J}_{\rm ant}(\mathbf{r},t)$ is embedded.

An overall sketch of this system is given in Fig. 2. The most immediate observation about the definition given above is that the subject of nonlocal radiation theory is inherently multidisciplinary since it involves interaction between classical antenna (radiation) theory, the science of understanding radiation into free space and designing efficient radiators, and the physics of electromagnetic materials, the latter in itself large and cross-disciplinary involving several subdomains like condensed-matter physics, plasma physics, optics, electromagnetic engineering, and so on. A second observation about the definition of nonlocal antenna systems is that it is crucial to maintain independence of the radiating current $\mathbf{J}_{\rm ant}$ from the surrounding metamaterial domain in the sense that the value of the radiating current is not effected by the ongoing radiation processes. This is the fundamental idea behind any externally-controlled radiator. Indeed, if back-reaction of the radiated fields (now existing in the MTM domain) can change the current supplied by the user, then use of the system for applications like wireless communications will be severely limited. For example, if sending a pulse encoding the digital symbol **1** may lead to radiation back-reaction distorting the pulse to the degree it now more resembles the the signal representation of the digital symbol **0**, then the probability of correct detection at the receiver will be degraded even with high signal-to-noise-ratio (SNR). In all traditional antenna theory and applications the assumption that the externally-supplied current is indeed external is taken for granted. However, in nanoscale radiation problems and other processes depending on quantum effects, the combined system of radiating particles and photons (radiation) are usually treated self-consistently, leading to back-reaction of radiation on the radiating currents. For that reason, true antennas do not exist in the ultimate microscopic realm, but only approximations of them can be maintained if a stabilizing mechanism can be put in use in order to ensure the protection of radiating currents from their own radiated fields. In nonlocal domains, since many nonlocal phenomena are due to quantum effects, stating that the current source in the NL-AS is assumed to be exactly external is then important for the purpose of developing an initial viable theory of nonlocal electromagnetic radiation. Clearly, the most simple and direct such theory would be a nonlocal *antenna* theory where the radiating current enjoys stability and absence of back radiation reaction. Laser sources are famous examples of such radiating systems where the radiating current is shielded from the back reaction of the photons it produces through a self-regulating feedback mechanism. Technical,y throughout this paper we write the current in spacetime domain as an externally determined function in the form $\mathbf{J}_{\rm ant}(\mathbf{r},t)$, while the corresponding momentum-space representation is $\mathbf{J}_{\rm ant}(\mathbf{k},\omega)$. As usual in antenna theory, a current distribution is ultimately produced by an external localized source (electric and/or magnetic fields). For simplicity, we refer for completeness to the voltage source signal $v_s(t)$ in Fig. 2. However, in this paper we do not address how the current is excited by a *given* voltage source. Instead, we focus on understanding how a given current distribution will radiate into the surrounding MTM domain and how the domain itself (and possibly the current) may be modified in principle such that a given radiation characteristics might be obtained (only one example, isotropic radiator design, is given in Part II.)

Immediately surrounding the external current $\mathbf{J}_{\rm ant}$ in Fig. 2 is the MTM domain, which in this work is assumed to satisfy the following fundamental assumptions:

(i) The MTM domain is large enough for the radiated fields to reach its outer edge to be considered far-zone fields, i.e. the characteristic scale of the MTM, say D, satisfies $D \gg \lambda = 2\pi/k$.

(ii) The material is electromagnetically homogeneous, i.e., the microscopic scale of matter a satisfies $a \ll \lambda = 2\pi/k$ for all ranges of k we are interested in.

(iii) The material is time invariant.

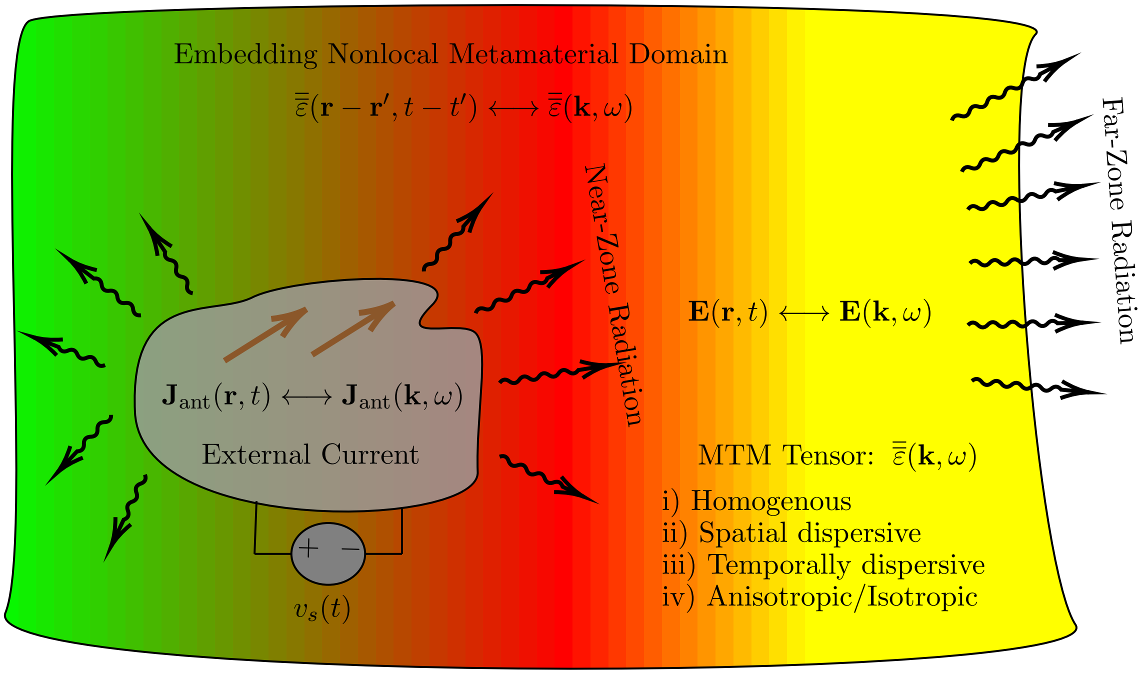

Figure 2: An overall sketch of a generic nonlocal radiating (antenna) system.

Under these conditions, it is well-known that the general dielectric tensor can be written as $\bar{\bar{\varepsilon}}(t-t',\mathbf{r}-\mathbf{r}')$ [1–4, 6, 66, 67]. By performing a Fourier transformation on $\mathbb{R}^4$, we end up with

$$\boxed{\bar{\bar{\varepsilon}}(\mathbf{r}-\mathbf{r}',t-t') \overset{\mathcal{F}}{\longleftrightarrow} \bar{\bar{\varepsilon}}(\mathbf{k},\omega),} \tag{1}$$

which, after with the earlier mentioned assumption on the independence of the radiating current, represent the second key assumption in the momentum space approach of this paper. The dielectric tensor used above will be defined more carefully in Sec. 3. The form $\bar{\bar{\varepsilon}}(\mathbf{k},\omega)$ is the momentum space representation of the space-time response function. The dependence on $\mathbf{k}$ is often referred to in literature as spatial dispersion [1, 2, 68]. However, we should keep in mind that nonlocality is more general than spatial dispersion since inhomogeneous nonlocal domains, which must be modeled by dielectric response function of the form $\bar{\bar{\varepsilon}}(\mathbf{r},\mathbf{r}';t-t')$, can *not* be expressed in Fourier space as $\bar{\bar{\varepsilon}}(\mathbf{k},\omega)$ [4]. Since in this paper our focus is only on homogeneous media, and those *still* fall under the form (1), we need not consider any response function other than the general tensorial form $\bar{\bar{\varepsilon}}(\mathbf{k},\omega)$ of spatially dispersive domains.

An important remark should be added here regarding the scope of the present theory. In both Parts I and II, we completely avoid the issue of how incident waves interact with interfaces separating two different spatially dispersive domains. Such problems clearly involve inhomogeneous dielectric functions and hence are not within the general form (1), the main material tensor model treated in our present work. However, a vast body of literature has been dedicated to the excitation of surface waves in such domains and several solutions to particular problems were proposed in various contexts. A popular approach is the use of additional boundary conditions (ABCs), leading to the analytical derivation and numerical computation of new modes excited in nonlocal domains that would otherwise not show up if the medium is local. Nevertheless, and as was pointed out long time ago in a penetrating analysis of the problem [2], *no completely general description of electromagnetics at the interface between two nonlocal domains is possible using the form* (1), the reason being that the very presence of an interface forces the microscopic nonlocal response to differ near the interface from its behaviour in the deep bulk region. The bulk domain form (1) is indeed valid only away from the edge or surface of the material. It is not clear how a very general boundary between the fuzzy "surface region" and "bulk region" interface can be established without using quantum theory, and that in turns requires working with specialized assumptions about the light-matter interaction Hamiltonian, leading to loss of generality. Since our goal is to develop the theory at the *macroscopic* level of the *phenomenological* bulk model (1), it is assumed throughout that the entire MTM domain is one homogeneous region modeled by otherwise completely unrestricted and general material tensor of the form (1). In particular, the radiating source current is an externally supplied current distribution directly embedded into the surrounding nonlocal MTM domain and no gaps or discontinuities need to be considered since we will use the Green's function (GF) to compute radiation. As is will known, the GF method works best with an external input smoothly embedded into a host domain [12, 13, 69].

We now move on to draw a broad sketch of the subject of future nonlocal antenna systems. The cross-disciplinary research field within which these systems are expected to be investigated is nonlocal antenna theory, see Fig. 1. There are there major components in any viable future nonlocal antenna theory:

(i) Forward radiation theory.

(ii) Reverse design methodology.

(iii) Final physical layout realization.

In (i), the fundamental question is how to compute the radiated fields given a specific current source and embedding (host) nonlocal MTM. This is a forward problem and the usual method to solve it is using the Green's function technique. On the other hand, (ii) asks about how the current source itself and the MTM parameters should be determined such that a desired radiation characteristics may obtain. Finally, in (iii) specific physical domains, processes, manufacturing techniques, etc, are sought in order to realize the model with the optimum design parameters found in (ii). In this paper, Part I will focus mainly on step (i) for the most general level. Part II will continue to address (i) but within a more concrete MTM framework – isotropic nonlocal MTM – but also touches on the design aspects of (ii). The final physical realization requires more concrete focus on specific physical problems like plasma, crystals, manufactured thin films, nanotechnology, quantum optics, and other specialized areas, and hence will be treated elsewhere since the subject falls outside the scope of the present work. However, we would like to mention few more words about steps (ii) and (iii) here before moving forward to the fundamental theory of (i) in the remaining sections of the present paper. Using a proper microscopic theory, ultimately quantum theory, it is possible in general to derive fundamental expressions for the components of the response tensor function $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$ [1–4, 6, 67]. In general, these functions are often expanded using Taylor series summations into few terms in powers of $\mathbf{k}$. For simplicity, let us assume here that only k appears in these expansions (this is the case in isotropic nonlocal MTM treated in more details in Part II. However, there is no loss in generality for the discussion to follow.) Experience accumulated with several types of nonlocal domains since the 1950s suggests that there are two major types of media, *resonant* (R) and *nonresonant* (NR) materials, with domain functions having the forms

$$\varepsilon_{ij}^{\mathrm{R}}(k,\omega)=\frac{\sum_{m=1}^{M}a_{ij}^{m}(\omega)k^{m}}{\sum_{m=1}^{N}b_{ij}^{m}(\omega)k^{m}},\quad \varepsilon_{ij}^{\mathrm{NR}}(k,\omega)=\sum_{m=1}^{N}c_{ij}^{m}(\omega)k^{m}, \tag{2}$$

respectively, where in general $M<N$. These forms (2) are often obtained in the following way: First, fundamental theory is deployed to derive analytical expressions for $\varepsilon_{ij}^{\mathrm{R}}(k,\omega)$ and $\varepsilon_{ij}^{\mathrm{NR}}(k,\omega)$. Afterwards, depending on the concrete values of the various physical parameters that enter into these expressions, e.g., frequency, temperature, molecular charge/mass/spin, density, etc, the obtained analytical expressions are expanded in Taylor series with the proper number of terms. Since the wavenumber is inversely proportional to the field physical spatial scale via $k=2\pi/\lambda$, expanding in terms of powers of k is equivalent to estimating the relative size of the nonlocal domain radius as measured from $\mathbf{r}'$ [1, 68]. In this paper, we define the process of *designing a nonlocal metamaterial* as simply finding the data $M, N, a_{ij}^{m}, b_{ij}^{m}, c_{ij}^{m}$, in multi-scale expressions like (2). In general, we will not assume any form like (2) in Part I, where the discussion is intentionally kept at a very general level conforming only to the homogeneous domain criterion of nonlocality given in (1). In Part II, the nonresonant nonlocal metamaterial case (NR-NL-MTM) is singled out and studied in details for the case of only quadratic dependence on k. We expect in future applications more complicated types of MTMs will need to be considered, for example, those involving resonant excitation of surface waves, but

their investigations is outside the scope of the present paper, which is mainly developing the rudiments of radiation theory in nonlocal antenna system analysis and design.

3. THE DYADIC GREEN'S FUNCTION OF NONLOCAL DOMAINS: THE VIEW FROM MOMENTUM SPACE

The key mathematical apparatus to be deployed throughout this work is the spatio-temporal Fourier transform. In fact, this is precisely the very idea of doing antenna theory in momentum space: expressing the radiated fields, radiating currents, and the material constitutive response functions, in terms of both frequency ω and wavevector $\mathbf{k}$. Both are *spectral* variables, and hence they naturally arise from taking the 4-dimensional Fourier transforms of all relevant quantities. The 4-dimensional Fourier transform of a generic vector field $\mathbf{F}(\mathbf{r},t)$ in space-time is defined by [70]

$$\mathbf{F}(\mathbf{k},\omega) := \int_{\mathbb{R}^4} \mathrm{d}^3 r \mathrm{d}t \; \mathbf{F}(\mathbf{r},t) e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}+\mathrm{i}\omega t}. \tag{3}$$

If the field $\mathbf{F}(\mathbf{r},t)$ is well-behaved in $\mathbb{R}^4$, then the inverse Fourier integral exists and is giving by [71]

$$\mathbf{F}(\mathbf{r},t) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^3 k \mathrm{d}\omega}{(2\pi)^4} \mathbf{F}(\mathbf{k},\omega) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega t}. \tag{4}$$

Throughout this paper, we assume that all relevant electromagnetic fields and currents in nonlocal material domains possess Fourier transforms in the sense that the pair (3) and (4) exist. It is enough for example to merely assume that the fields and currents are smooth (have continuous derivatives of all orders.) However, this might be too restrictive in some applications, especially when it comes to the behaviour of current and charge source distributions, which are usually required only to be Holder continuous (for a condition of slightly stronger continuity, e.g., see [72].) Nevertheless, in what follows, we will not worry much about how exactly the fields and sources mathematical functions are behaving but assume they are "sufficiently well- behaved" such that the integrals (3) and (4) exist.

In the spatio-temporal domain, Maxwell's equations can be expressed as a system of partial differential equations in the form:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{5}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \mu_0 \mathbf{J}(\mathbf{r},t), \tag{6}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{7}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{1}{\varepsilon_0} \rho(\mathbf{r},t). \tag{8}$$

Here, ε_0 and μ_0 are the electric permitivity and magnetic permeability of free space, respectively.† In order to move from the spatio-temporal domain to the momentum-frequency space (hereafter, *momentum space* for brevity), i.e., the main configuration space on which electromagnetic fields live in the present work, we simply apply the 4-dimensional Fourier transformation (3) to the space-time form of Maxwell's equations. In fact, in the Fourier domain, Maxwell's equations (5)-(8) become

$$\mathbf{k} \times \mathbf{E}(\mathbf{k},\omega) = \omega \mathbf{B}(\mathbf{k},\omega), \tag{9}$$

$$\mathrm{i}\mathbf{k} \times \mathbf{B}(\mathbf{k},\omega) = \mu_0 \mathbf{J}(\mathbf{k},\omega) - \frac{\mathrm{i}\omega}{c^2} \mathbf{E}(\mathbf{k},\omega), \tag{10}$$

For further details about the precise mathematical conditions, see [5, 70, 71].

Sometimes even more restrictions might be needed – such as uniform convergence or stronger conditions – in order to ensure that integrals and other limiting operators, e.g., integro-differential operators, can be interchanged. However, we will not address this here at a very general level due to space limitations but instead leave such issues to be handled on case-by-case whenever they arise throughout the development of the theory.

† The electric field vector $\mathbf{E}$ is measured in V/m, while the magnetic flux density has the units of Tesla. The source distributions are two types, volume current density $\mathbf{J}$ measured in A/m^2, and volume charge density ρ in C/m^3.

$$\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)=\frac{-\mathrm{i}}{\varepsilon_0}\rho(\mathbf{k},\omega), \tag{11}$$

$$\mathbf{k}\cdot\mathbf{B}(\mathbf{k},\omega)=0, \tag{12}$$

where $\mathbf{E}(\mathbf{k},\omega)$, $\mathbf{B}(\mathbf{k},\omega)$, and $\mathbf{J}(\mathbf{k},\omega)$ are the space-time Fourier transforms of the electric field, magnetic flux intensity, and the source current distribution, respectively. We immediately note how Maxwell's laws acquire in momentum space a considerably simpler and more manageable *algebraic* form, in direct contrast to the complicated *differential* form possessed by the original set (5)-(8). This is not accidental but has been one of the key motivation to the now universal use of Fourier transform methods in physics and engineering. More remarkable still, the relations (9)-(12) are *still* valid in arbitrary material domain with both temporal and spatial dispersion. This, in fact, provides another fundamental motivation for rebuilding antenna theory in momentum space. Indeed, as will be shown below, it is often very difficult – if not nearly impossible – to fully characterize and understand how antennas embedded into nonlocal media operate if one restricts the formalism to space-time. This is because in generic spatially dispersive domains, the space-time forms of the constitutive relations become 4-dimensional convolution-type integrals, transforming Maxwell's equations from partial differential equations to complicated and unfamiliar integro-differential equations. On the other hand, by working in the momentum-frequency space representation, most of the calculations can be done first algebraically, then only one inverse 4-dimensional Fourier transformation in the form (4) is needed to go back to space-time (if needed.) At this point it is interesting to anticipate what will be rigoursly proved later in this paper, namely that *if we are only interested in the far-field of the antenna, then even the last mentioned inverse Fourier integration becomes unnecessary*, which is one of the key advantages of the choice of the momentum space approach to construct a radiation theory of nonlocal antenna systems.

However, in spite of such considerable reduction in complexity, the purely algebraic equations (9)-(12) cannot be solved till we provide a description of the material response function, most preferably in the form of *constitutive material relations* dictating how matter responds to external fields at the macroscopic level [1, 2, 68]. In this paper, we follow the Fourier transform approach – adopted from some formulations commonly used in plasma and condensed-matter physics – for our quest to construct the constitutive relation.[‡] Our key starting point will be the setup of an unambiguous and direct demarcation between the *antenna* current distribution on one hand, and the current *induced* in the nonlocal medium as a result of interaction between direct fields originally produced by the antenna current source and matter. Since this separation process possesses a microscopic origin, it should be ultimately grounded on solving a set of appropriate coupled quantum-Maxwell's equations, e.g., within a semiclassical approach or fully-fledged quantum-field theoretic framework [4]. However, we will work at the macroscopic level, allowing only *averaged* currents and fields as the permissible type of the unknown dynamic variables to be determined by solving the Maxwell's equation (9)-(12). In particular, within such phenomenological framework the total current $\mathbf{J}(\mathbf{k},\omega)$ may be decomposed into two parts [68, 73]

$$\mathbf{J}(\mathbf{k},\omega)=\mathbf{J}_{\mathrm{ant}}(\mathbf{k},\omega)+\mathbf{J}_{\mathrm{ind}}(\mathbf{k},\omega), \tag{13}$$

where $\mathbf{J}_{\mathrm{ant}}(\mathbf{k},\omega)$ is the *externally* supplied antenna current distribution while $\mathbf{J}_{\mathrm{ind}}(\mathbf{k},\omega)$ is the current *induced* in the medium as a response to the excitation electric field. The ultimate proof of (13) rests on the linearity of the coupled matter-Maxwell's equations since one may envision the current production processes as the quantum-mechanical response of charged microscopic material particles to the presence of electromagnetic fields. Since in microscopic scattering formulations the total field is the sum of the direct and scattered field, one can deduce immediately from this that the various current distribution components, which are here essentially manifestation of a linear processes responding to applied fields, will also add up to give the form (13) above. Alternatively, one may consider (13) a fundamental *axiom* in the antenna theory model developed in this paper. Now, within the regime of linear response theory, the response of matter to applied fields should be a linear operator [1, 4, 6]. The simplest form of such an operator is a dyadic tensor transformation, allowing us to write

$$\mathbf{J}_{\mathrm{ind}}(\mathbf{k},\omega)=\overline{\overline{\sigma}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega), \tag{14}$$

‡ Therefore, in what follows the much more familiar *multipole expansion* approach, which is very often used in the engineering literature, will *not* be deployed. Monographs on the electromagnetics of continuous media often either use the Fourier transform approach or the multiple expansion, but rarely both. Even rarer is a comparison between the two, but see [10, 26, 66] for more extensive discussion of the background to the use of alternative formulations to describe matter-field interaction.

where $\overline{\overline{\sigma}}(\mathbf{k},\omega)$ is called the *material conductivity tensor* [4]. Note that *spatial dispersion* (nonlocality) is captured by the dependence of conductivity on $\mathbf{k}$, while the appearance of ω reflects normal or temporal dispersion [1, 2]. Spatial and temporal dispersion can be physically interpreted as indication of the presence in the material/metamaterial of key phenomena expressive of *nonlocality* and *memory*, respectively [68, 73]. Let the electric displacement vector in space-time be $\mathbf{D}(\mathbf{r},t)$ with corresponding space-time Fourier transform denoted by $\mathbf{D}(\mathbf{k},\omega)$. In momentum-frequency space, the relation between the electric displacement vector $\mathbf{D}(\mathbf{k},\omega)$ and electric field $\mathbf{E}(\mathbf{k},\omega)$ can now be written as

$$\mathbf{D}(\mathbf{k},\omega) = \varepsilon_0 \overline{\overline{\varepsilon}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega), \tag{15}$$

where the dimensionless dyadic tensorial quantity $\overline{\overline{\varepsilon}}$ is defined by

$$\overline{\overline{\varepsilon}}(\mathbf{k},\omega) := \overline{\mathbf{I}} + \frac{\mathrm{i}}{\omega\varepsilon_0}\overline{\overline{\sigma}}(\mathbf{k},\omega). \tag{16}$$

Here, $\overline{\mathbf{I}}$ is the unit dyad operator, capturing the *direct* field, while the tensor $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$ in (16) is called *the equivalent dielectric function* of the medium in frequency-momentum space [2, 26]. The additive structure in (16) parallels that of (13). The dielectric tensor $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$ supplies the most general description of the nonlocal medium in the frequency-momentum space [4]. Note that in contrast to the traditional multipole formalism, the Fourier space approach to the electromagnetic response of material domains include all electric *and* magnetic responses in one response tensor, namely the tensor $\overline{\overline{\varepsilon}}$, while enjoying several properties that are based on universal principles such as energy conservation, causality, reciprocity, etc, hence valid irrespective to the actual microscopic details of the medium [2,4,10,68,73,74]. We mention here only those generic properties of the material tensor related to dissipation and non-dissipation because they will pop out frequently in the radiation theory to be developed below. Namely, the material response tensor in general can be expanded as

$$\overline{\overline{\varepsilon}}(\mathbf{k},\omega) = \overline{\overline{\varepsilon}}^{\mathrm{H}}(\mathbf{k},\omega) + \overline{\overline{\varepsilon}}^{\mathrm{A}}(\mathbf{k},\omega), \tag{17}$$

where here we define the *hermitian* and *antihermitian* components by

$$\overline{\overline{\varepsilon}}^{\mathrm{H}}(\mathbf{k},\omega) := \frac{1}{2}\left[\overline{\overline{\varepsilon}}(\mathbf{k},\omega) + \overline{\overline{\varepsilon}}^{*}(\mathbf{k},\omega)\right], \quad \overline{\overline{\varepsilon}}^{\mathrm{A}}(\mathbf{k},\omega) := \frac{1}{2}\left[\overline{\overline{\varepsilon}}(\mathbf{k},\omega) - \overline{\overline{\varepsilon}}^{*}(\mathbf{k},\omega)\right]. \tag{18}$$

respectively. Here, $*$ is the complex conjugation operation. In component form, it is clear that the hermitian and antihermitian functions satisfy the symmetry properties

$$\overline{\overline{\varepsilon}}^{\mathrm{H}}_{ij} = \overline{\overline{\varepsilon}}^{\mathrm{H}*}_{ij}, \quad \overline{\overline{\varepsilon}}^{\mathrm{A}}_{ij} = -\overline{\overline{\varepsilon}}^{\mathrm{A}*}_{ij}. \tag{19}$$

It can be shown that only the *antihermitian* part of the response functions $\overline{\overline{\sigma}}$ and $\overline{\overline{\varepsilon}}$ actually contributes to dissipative processes such as wave growth or decay inside the medium [2, 4, 6]. Throughout this paper, we assume as typical in literature that dissipation is either small or negligible. In this case, the relevant dispersion relations of propagating modes will be determined solely by the *hermitian* part of the material response function.

As mentioned in Sec. 2, a key assumption posed in the momentum space radiation theory presented here was that the antenna current is an *independent function* externally imposed from the "outside" of the material. That is, in contrast to $\mathbf{J}_{\mathrm{ind}}$, the antenna current $\mathbf{J}_{\mathrm{ant}}$ is not determined by microscopic processes immanent to the nonlocal material system itself; instead, the induced current $\mathbf{J}_{\mathrm{ind}}$ collects all individual subprocesses in the material system produced in response to the applied external source, e.g., polarization current, conductive current, magnetization, etc. Now, in order to find the electromagnetic fields produced by the antenna current source $\mathbf{J}_{\mathrm{ant}}$, the vector magnetic potential $\mathbf{A}(\mathbf{r},t)$ and the scalar electric potential $\phi(\mathbf{r},t)$ are often introduced where

$$\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}, \quad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t). \tag{20}$$

There is then the well-known freedom of choosing a suitable gauge condition (relation between $\mathbf{A}$ and ϕ) since Maxwell's equations in themselves are compatible with an infinite number of valid choices of these potential functions (gauge freedom, see [5, 69, 70].) It turns out that for the development of antenna theory in nonlocal material domains, a very convenient gauge condition to utilize is the *Fourier gauge*

$$\boxed{\text{The Fourier Gauge :} \quad \phi(\mathbf{r},t)=0.} \tag{21}$$

Consequently, in this case from (20) the electric field in space-time and momentum space can be expressed as

$$\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}, \qquad \mathbf{E}(\mathbf{k},\omega)=\mathrm{i}\omega\mathbf{A}(\mathbf{k},\omega). \tag{22}$$

The gauge condition (21) will be imposed throughout this work. In particular, let us write the wave equation in the Fourier (momentum) space. From (9)-(11), we easily deduce

$$\frac{\omega^2}{c^2}\mathbf{E}(\mathbf{k},\omega)+\mathbf{k}\times[\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega)]=-\mathrm{i}\omega\mu_0\mathbf{J}(\mathbf{k},\omega). \tag{23}$$

With the help of (22), this leads to

$$\frac{\omega^2}{c^2}\mathbf{A}(\mathbf{k},\omega)+\mathbf{k}\times[\mathbf{k}\times\mathbf{A}(\mathbf{k},\omega)]=-\mu_0\mathbf{J}(\mathbf{k},\omega). \tag{24}$$

Note that the remaining field and source components, namely $\mathbf{B}$ and ρ, can be determined from other equations like (9) and the equation of continuity, giving rise to

$$\mathbf{B}(\mathbf{k},\omega)=\frac{1}{\omega}\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega), \qquad \rho(\mathbf{k},\omega)=\frac{\mathbf{1}}{\omega}\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega). \tag{25}$$

In other words, in momentum space, *the only effective unknown is the frequency-momentum space electric field* $\mathbf{E}(\mathbf{k},\omega)$ *while all other quantities can be determined based on this field variable via simple algebraic calculations*. Moreover, using the Fourier gauge, only the momentum space vector potential $A(\mathbf{k},\omega)$ needs to be found by solving (24). Both (23) and (24) are in fact *algebraic* equations, allowing us to derive exact analytical expressions for the antenna radiation in nonlocal media as will be demonstrated later.[§]

Using (13), (14), and (22), we can introduce a slightly different material tensor $\overline{\overline{\zeta}}(\mathbf{k},\omega)$ defined by

$$\mathbf{J}_{\text{ind}}(\mathbf{k},\omega)=\overline{\overline{\zeta}}(\mathbf{k},\omega)\cdot\mathbf{A}(\mathbf{k},\omega), \tag{26}$$

where

$$\overline{\overline{\zeta}}(\mathbf{k},\omega):=\mathrm{i}\omega\overline{\overline{\sigma}}(\mathbf{k},\omega). \tag{27}$$

In terms of this tensor, the effective dielectric tensor can be re-expressed as

$$\overline{\overline{\varepsilon}}(\mathbf{k},\omega)=\bar{\mathbf{I}}+\frac{1}{\omega^2\varepsilon_0}\overline{\overline{\zeta}}(\mathbf{k},\omega). \tag{28}$$

With the help of (27) and (28), we restate the wave equation (24) in the following more compact operator form

$$\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega)\cdot\mathbf{A}(\mathbf{k},\omega)=-\frac{\mu_0c^2}{\omega^2}\mathbf{J}_{\text{ant}}(\mathbf{k},\omega), \tag{29}$$

where the dyadic tensor

$$\boxed{\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega):=-\frac{k^2c^2}{\omega^2}\left(\bar{\mathbf{I}}-\hat{k}\hat{k}\right)+\ \overline{\overline{\varepsilon}}(\mathbf{k},\omega)} \tag{30}$$

[§] In the Fourier gauge, the only restriction is that $\omega\neq0$, an assumption made here. However, for applications to antennas, this is already the case since radiation does not occur in the static regime $\omega=0$.

is the *inverse* of the momentum-space radiation operator (spectral dyadic Green's function) $\overline{\mathbf{G}}(\mathbf{k},\omega)$ defined through the relation

$$\boxed{\mathbf{A}(\mathbf{k},\omega) = -\frac{\mu_0 c^2}{\omega^2}\overline{\mathbf{G}}(\mathbf{k},\omega)\cdot\mathbf{J}_{\mathrm{ant}}(\mathbf{k},\omega).} \tag{31}$$

Here, $k := |\mathbf{k}|$ is the magnitude of the wavevector $\mathbf{k}$ and $\hat{k} := \mathbf{k}/k$ is the unit vector pointing in the direction of $\mathbf{k}$. The tensor $\overline{\mathbf{G}}$ is often called in the physics literature the *photon propagator* [5,6]; here, we just refer to it as the nonlocal medium dyadic Green's function (GF). The relations (30) and (31) are essential for the entire momentum-space radiation theory to follow in this paper. In particular, (31) provides a very convenient method to compute and understand the radiation field $\mathbf{A}(\mathbf{k},\omega)$ in momentum space when the space-time Fourier components of the antenna current $\mathbf{J}(\mathbf{k},\omega)$ are available. In fact, *the momentum-space Green's function* $\overline{\mathbf{G}}(\mathbf{k},\omega)$ *turns out to be much easier to work with in nonlocal domains than the conventional dyadic GFs often used in local electromagnetics such as in* [12,13]. Note that the nonlocal medium Green's function (30) can be expressed analytically by the following *closed-form* expression:

$$G_{ij}(\mathbf{k},\omega) = \frac{C_{ij}(\mathbf{k},\omega)}{G^{-1}(\mathbf{k},\omega)}, \tag{32}$$

where C_{ij} are the co-factors of the matrix representation of the tensor $\overline{\mathbf{G}}^{-1}$ satisfying

$$\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega)\cdot\overline{\mathbf{C}}(\mathbf{k},\omega) = \overline{\mathbf{I}}G^{-1}(\mathbf{k},\omega), \tag{33}$$

where

$$G^{-1}(\mathbf{k},\omega) := \det\left[\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega)\right]. \tag{34}$$

Here, 'det' is the determinant operator.‖ The detailed expressions of the co-factor matrix are lengthy and will not be given here but can be found in comprehensive textbooks on matrix theory, e.g., see [75]. However, in Appendix C we give a series of general expressions suitable for electromagnetic theory applications derived using tensor algebra methods. What is important for us here is that aside from the the general functional dependence of $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$, the nonlocal medium Green's function is essentially a polynomial rational function in both $\mathbf{k}$ and ω. If the dielectric function itself is now expanded in power series of both $\mathbf{k}$ and ω as in (2) – and also as will be done in Part II – then the GF effectively becomes a rational polynomial in the spectral variables $\mathbf{k}$ and ω.

When there is no source ($\mathbf{J}_{\mathrm{ant}} = 0$), the relation (29) reduces to

$$\overline{\mathbf{G}}^{-1,\mathrm{H}}(\mathbf{k},\omega)\cdot\mathbf{A}(\mathbf{k},\omega) = 0, \tag{35}$$

where $\overline{\mathbf{G}}^{-1,\mathrm{H}}(\mathbf{k},\omega)$ is the hermitian part of the tensor $\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega)$. Here, we adopted the general approach in plasma and condensed-matter physics where losses – introduced in our case by dissipation in $\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega)$, ultimately caused by the antihermitian part of $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$ via (30) – is treated as small perturbation added to the main component of the Green's function tensor, which is hermitian [1,2,68]. For that reason, only the hermitian part is relevant to the determination of radiation modes.¶

From the homogeneous wave equation in momentum space (35), the existence of nonzero solutions representing source-free wave fields propagating in the nonlocal domain with positive group velocity is guaranteed only at those special combinations of ω and $\mathbf{k}$ at which the operator $\overline{\mathbf{G}}^{-1,\mathrm{H}}(\mathbf{k},\omega)$ becomes singular (non-invertible). Since the latter operator is dyadic, this case occurs when the following equation holds

$$\boxed{G^{-1,\mathrm{H}}(\mathbf{k},\omega) = 0,} \tag{36}$$

‖ It can be shown that the determinant of $\overline{\mathbf{C}}$ is equal to G^{-2}. Consequently, from the determinant product rule and (33), it follows that $\det(\overline{\mathbf{G}}) = G$, which explains our notation.

¶ We will however drop the superscript H in the future whenever that does not cause confusion in order to simplify the notation. Note that dissipation is a more general concept than losses. For example, Landau damping in plasma domains is a collision-free form of dissipation, hence cannot be understood as losses. The mathematical treatment of all dissipation processes, however, invariably involves the decomposition of some suitable quantities into hermitian and antihermitian components, see [2,4,68,74] for more details.

which is referred to universally as the *dispersion relation* of the domain who dielectric function is $\overline{\overline{\varepsilon}}(\mathbf{k},\omega)$. In general, there exists multiple solutions to (36), each labeled by an integer l captured by the lth mode dispersion relation, which is usually put either in the form $\omega = \omega_l(\mathbf{k})$ or $\mathbf{k} = \mathbf{k}_l(\omega)$. Each such solution of the dispersion relation will give rise to a *field* distribution $\mathbf{A}_l(\mathbf{k},\omega)$ obtained by solving the corresponding wave equation (35), where this field can be thought of as a vector element belonging to the null space of the Green's dyad operator $\overline{\mathbf{G}}^{-1,\mathrm{H}}(\mathbf{k},\omega)$. Note that since the wave equation is homogeneous, an arbitrary complex normalization factor is present and hence it is best to work with normalized modal fields $\hat{e}_l(\mathbf{k},\omega)$ instead of $\mathbf{A}_l(\mathbf{k},\omega)$, which we define by:

$$\hat{e}_l(\mathbf{k}) := \frac{\mathbf{A}(\mathbf{k},\omega_l(\mathbf{k}))}{|\mathbf{A}(\mathbf{k},\omega_l(\mathbf{k}))|}. \tag{37}$$

Furthermore, since a polarization vector is defined only with respect to a given modal dispersion relation $\omega = \omega_l(\mathbf{k})$, the modal field is really a function of $\mathbf{k}$ only and hence it will always be written as $\hat{e}_l(\mathbf{k})$ instead of $\hat{e}_l(\mathbf{k},\omega)$. In addition, to normalize the modal fields, the following standard orthonormality condition is imposed:

$$\hat{e}_l(\mathbf{k}) \cdot \hat{e}_l^*(\mathbf{k}) = 1. \tag{38}$$

Clearly, the modal fields are complex. Additionally, a further classification of modes is also possible depending on how the vector $\hat{e}_l(\mathbf{k})$ is oriented with respect to $\hat{k}$. Whenever $\hat{e}_l(\mathbf{k})$ is parallel to $\mathbf{k}$, we say that the corresponding mode is *longitudinal* (L). On the other hand, when $\hat{e}_l(\mathbf{k})$ is orthogonal to $\mathbf{k}$, i.e., $\hat{e}_l(\mathbf{k}) \cdot \hat{k} = 0$, the mode is called *transverse* (T). However, note that these L and T modal concepts acquire clear meaning only within momentum space.

As will be illustrated next, *it turns out that the antenna radiation pattern (evaluated here in the momentum space), is completely determined by the propagating modes arising from the solution of the dispersion relation* (36). *For that reason, the art and science of designing antennas with desired far-field radiation patterns in nonlocal metamaterials requires careful proper engineering of these radiation modes.*+ For these reasons, the dispersion relation (36) will play a crucial role in the remaining fundamental and applied sections of this paper, especially in Part II where concrete solutions of specific dispersion equations of L and T modes will be computed. The overall body of theory and experiments devoted to investigating the different types of modes arising from solving dispersion relations in various local and nonlocal materials is enormous, extending over a bewilderingly large range of applications. To probe deeper into dispersion relations and their solutions, the reader may consult several references on plasma physics [1, 3, 74], optics of solids [2, 76], condensed-matter physics [4, 77], magnetohydrodynamics [68, 78], and many other applications in astrophysics [3, 79] and cosmology [80, 81].

4. THE NONLOCAL ANTENNA SYSTEM RADIATION PATTERN IN MOMENTUM SPACE

In mainstream antenna theory and the treatment of other emission processes, the conventional approach to estimating far-field radiation consists of solving Maxwell's equations (often in vacuum) to find the electric and magnetic fields in spacetime, forming the Poynting vector, then computing the radiated power by integrating the latter in space and time [82–84]. This approach, however, is extremely difficult to apply in generic anisotropic media, and in the case when the material tensor is also nonlocal, it is probably not possible at all to work exclusively in spacetime. In what follows, we propose an alternative pathway toward building the essential components of a computationally viable theory of antennas radiating in nonlocal domains. The key idea is that in the momentum space of Fourier transformed fields it is much easier to work with spectral components since in the latter case they acquire their distinctive pure tensor-algebraic form developed above; simultaneously, using Parseval (power) theorems, one may relate the physical meaning of some (squared) quantities in one domain to the other. Our goal then will be to evaluate power delivered to the far zone by estimating (in momentum space) the energy leaving the current source in the near zone and use the Green's function to relate the two. This general idea appears to be due to Brillouin [64].

+ This conclusion was also reached – using different route – in the near-field engineering theory proposed in [10, 45]. Also, see [44, 65] for similar conclusions in applications to dispersion management.

We start by rewriting The nonlocal medium Green's function (32) in the slightly different form

$$G_{ij}(\mathbf{k},\omega) = \frac{C_{ij}^{\mathrm{H}}(\mathbf{k},\omega)}{G^{-1,\mathrm{H}}(\mathbf{k},\omega)}. \tag{39}$$

That is, only the *hermitian* part is included in the momentum space Green's function. The motivation is what we have already alluded to earlier: dissipation in nonlocal media is often treated as *perturbation* added to the pure propagation scenario described by hermitian response functions [3, 68, 85]. It is remarkable, though, that in spite of the fact that only the hermitian part was originally taken into the Green's function (39) as can be seen there in the medium's dyadic GF in co-factor form, *there still exists an antihermitian component in this Green's function that must be added in order to enforce causality* [3,68]. This surprising observation, which is not new but often overlooked in many accounts on electromagnetic radiation, deserves special attention because – as will become clearer below – it lies at the heart of the momentum space antenna theory being presented in this paper. For that reason, we will revisit the proof in a slightly detailed form in order to better appreciate in depth the connection with the fundamental theory of antennas embedded into nonlocal metamaterials. Let us begin by noting that the Green's function (39) possesses "temporal poles," i.e., poles in the complex ω-plane, determined by the solutions of the dispersion equation (36). This implies that $G_{ij}(\mathbf{k},\omega)$ would become singular at those values of ω and $\mathbf{k}$ satisfying the dispersion relation $\omega = \omega_l(\mathbf{k})$ by definition we have $G^{-1,\mathrm{H}}(\mathbf{k},\omega_l(\mathbf{k})) = 0$. This presents a serious problem because usually in order to find the spatio-temporal fields, one needs to invert the Fourier transform by computing (4). This then will lead to divergent integrals when the dummy variables ω and $\mathbf{k}$ hit the special values satisfying the dispersion relation $\omega = \omega_l(\mathbf{k})$. Unless a small perturbation in the pole location is introduced, which is usually attained by replacing ω by $\omega + \mathrm{i}\epsilon$, where ϵ is a very small positive real number, no actual solution of the electromagnetic problem is possible [5, 68, 70, 86]. More explicitly, we often enact the following formal transformation

$$\omega \to \omega + \mathrm{i}0, \tag{40}$$

a notation adopted hereafter. The symbolic expression i0 indicates the presence of a formal perturbation in the medium, i.e., small dissipation inserted by hand in order to push the pole slightly beyond the real ω-axis [12, 13, 68]. It is clear then that around the lth mode, the determinant G^{-1} appearing in the denominator of (39) may be approximated by

$$G^{-1}(\mathbf{k},\omega) \approx \frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial \omega}[\omega - \omega_l + \mathrm{i}0]. \tag{41}$$

Using the Plemelj formula [70, 87]

$$\frac{1}{\omega + \mathrm{i}0} = \mathcal{P}\frac{1}{\omega + \mathrm{i}0} - \mathrm{i}\pi\delta(\omega), \tag{42}$$

where $\mathcal{P}$ is the principal Cauchy value operator and δ the Dirac delta function, the relations (39) and (41) when summed over all radiation modes jointly imply the existence of the following *antihermitian* component

$$G_{ij}^{\mathrm{A}}(\mathbf{k},\omega) = -\mathrm{i}\pi \sum\nolimits_l \omega_l(\mathbf{k}) R_{ij}^l(\mathbf{k}) \delta(\omega - \omega_l(\mathbf{k})), \tag{43}$$

where

$$\boxed{R_{ij}^l(\mathbf{k}) := \left. \frac{C_{ij}^{\mathrm{H}}(\mathbf{k},\omega)}{\omega \partial G^{-1}(\mathbf{k},\omega)/\partial \omega} \right|_{\omega=\omega_l(\mathbf{k})}} \tag{44}$$

is what we term *the momentum-space radiation mode Green's function.* It captures the lth mode contribution to the ijth component ($i, j = 1, 2, 3$) of the nonlocal medium Green's function tensor $\overline{\mathbf{G}}(\mathbf{k},\omega)$. It turns out that *only the antihermitian part of this medium Green's function as determined by* (43) *actually contributes to the real radiated power of any antenna.* On the other hand, the hermitian part of $\overline{\mathbf{G}}(\mathbf{k},\omega)$ contributes only to the antenna near field.

We next explicitly compute this radiation power pattern in momentum space. Unfortunately, the available method commonly applied to antennas radiating in free space or nondispersive media depends on the Poynting theorem interpreted as energy conservation relation [82, 83]. It is well known that this direct view cannot be extended without further assumptions to generally temporally dispersive media [69]. Worse still, in nonlocal (spatially dispersive) domains, the standard interpretation of the Poynting theorem itself is not valid since power will flow along new directions emerging from higher-order corrections [1, 2, 4, 44, 65]. Instead, we adopt here an alternative method due to Brillouin [64] and often adopted in various settings [88]. The key idea is to estimate the energy transferred from the source to the near field right at the source and equate this with the net (real) power delivered to the medium. Since in low-loss media most of the delivered power (energy) will escape to the environment's far zone, we may then use this energy "balance" to estimate the antenna radiation pattern. To achieve this in momentum space, we introduce a new radiation pattern intensity $U_l(\mathbf{k})$, which is formally defined as

$$\boxed{U_l(\mathbf{k}) := \begin{array}{l}\text{Density of energy transferred from the source current } \mathbf{J}_{\rm ant}(\mathbf{r},t) \text{ into the } l\text{th radiation}\\ \text{mode's field per the momentum-space differential volume element } \mathrm{d}^3\mathbf{k}/(2\pi)^3.\end{array}} \tag{45}$$

Clearly, the units of this quantity will be $\mathrm{J}\cdot\mathrm{m}^3$. Let the antenna current source $\mathbf{J}_{\rm ant}(\mathbf{r},t)$ be examined within a standard time interval $[-T/2, T/2]$. Since radiation modes do not exchange energy with each other, we can sum over all radiation intensity functions $U_l(\mathbf{k})$ defined by (45) to obtain

$$-\int_{-T/2}^{T/2}\mathrm{d}t\int_{V_{\rm ant}}\mathrm{d}^3r\mathbf{J}_{\rm ant}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)=\sum_l\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3}U_l(\mathbf{k}). \tag{46}$$

Here, $V_{\rm ant}$ is the antenna (source) region (see Fig. 2). The equality (46) is a more general statement of energy conservation since it does not require using the Poynting vector, the latter being inadequate when nonlocality is present. Moreover, the same relation may serve as an implicit formal definition of the modal momentum-space density function $U_l(\mathbf{k})$. With the help of the standard Parseval (power) theorem, the total radiated energy can be re-expressed in momentum space as

$$\mathcal{E}_{\rm rad} := -\int_{-T/2}^{T/2}\mathrm{d}t\int_{V_{\rm ant}}\mathrm{d}^3r\ \mathbf{J}_{\rm ant}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = -\int_{\mathbb{R}^4}\frac{\mathrm{d}\omega\mathrm{d}^3k}{(2\pi)^4}\ \mathbf{J}^*_{\rm ant}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega). \tag{47}$$

The momentum space integration in the RHS of (47) is generally performed over the entire ω-$\mathbf{k}$-four-dimensional space $\mathbb{R}^4$, but in practice it has to be terminated by an upper cuttoff frequency. Using (22) and (31), (47) becomes

$$-\int_{-T/2}^{T/2}\mathrm{d}t\int_{V_{\rm ant}}\mathrm{d}^3r\ \mathbf{J}_{\rm ant}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)=\int_{\mathbb{R}^4}\frac{\mathrm{d}\omega\mathrm{d}^3k}{(2\pi)^4}\frac{\mathrm{i}\mu_0c^2}{\omega}\mathbf{J}^*_{\rm ant}(\mathbf{k},\omega)\cdot\overline{\mathbf{G}}(\mathbf{k},\omega)\cdot\ \mathbf{J}_{\rm ant}(\mathbf{k},\omega). \tag{48}$$

The integral appearing at the RHS of (48) is real (because energy in the LHS is real), so it can be written as half its sum with the complex conjugate, which implies that only the *antihermitian* part of $\overline{\mathbf{G}}(\mathbf{k},\omega)$ will contribute to the total integral. The latter, however, has already been computed and its expression is given by (43), which upon substitution into (48), evaluating the trivial ω-integral involving the delta function, and noticing that negative frequencies have identical contribution to positive frequencies, will yield the following result

$$\mathcal{E}_{\rm rad}=\frac{1}{\varepsilon_0}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_l\mathbf{J}^*_{\rm ant}[\mathbf{k},\omega_l(\mathbf{k})]\cdot\overline{\mathbf{R}}_l(\mathbf{k})\cdot\mathbf{J}_{\rm ant}[\mathbf{k},\omega_l(\mathbf{k})]. \tag{49}$$

The dyadic function $\overline{\mathbf{R}}_l(\mathbf{k})$ is defined as the 3-dimensional dyad with Cartesian components given by $R^l_{ij}(\mathbf{k})$ as per (44), i.e., we have $\overline{\mathbf{R}}_l(\mathbf{k}) := \sum_{i,j=1}^3 R^l_{ij}(\mathbf{k})\hat{x}_i\hat{x}_j$, where $\hat{x}_i, i=1,2,3$, are three Cartesian unit bases. This crucial tensor can be further expanded with the help of (B3) and (44), leading to

$$\overline{\mathbf{R}}_l(\mathbf{k})=R_l(\mathbf{k})\ \hat{e}_l(\mathbf{k})\ \hat{e}^*_l(\mathbf{k}), \tag{50}$$

with

$$\boxed{R_l(\mathbf{k}) := \left. \frac{\gamma_l(\mathbf{k})}{\omega \partial G^{-1}(\mathbf{k},\omega)/\partial\omega} \right|_{\omega=\omega_l(\mathbf{k})}.} \tag{51}$$

For the definition of $\gamma_l(\mathbf{k})$, see (B4). The momentum-space function $R_l(\mathbf{k})$ is the most important quantity in the theory of nonlocal antennas proposed in this paper. As will be seen later, all calculations distinctive of this type of antennas rely on accurate estimation of $R_l(\mathbf{k})$ for various material domains. Example calculations will be given in Part II.

Finally, by comparing (49) with (46), the following expression for the lth mode radiation intensity is derived:

$$\boxed{U_l(\mathbf{k}) = \frac{1}{\varepsilon_0} \mathbf{J}^*_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})] \cdot \overline{\mathbf{R}}_l(\mathbf{k}) \cdot \mathbf{J}[\mathbf{k},\omega_l(\mathbf{k})].} \tag{52}$$

Using (50), the radiation mode intensity formula (52) can be further simplified into

$$U_l(\mathbf{k}) = \frac{1}{\varepsilon_0} R_l(\mathbf{k}) \left| \hat{e}^*_l(\mathbf{k}) \cdot \mathbf{J}_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})] \right|^2. \tag{53}$$

The relation (52) is very general and fundamental. It expresses the amount of radiated energy within a unit volume in momentum-space in terms of the momentum-space radiation mode (spectral) Green's function $\overline{\mathbf{R}}_l(\mathbf{k})$. However, this formula is directly applicable to the case of nondegenerate waves like longitudinal waves where for each l only one modal field $\hat{e}_l(\mathbf{k})$ exists. In radiation theory in general, and antenna theory in particular, *transverse* waves are ubiquitous. These waves are degenerate since for each mode index (order) l, there exist *two* modal fields $\hat{e}^1_l(\mathbf{k})$ and $\hat{e}^2_l(\mathbf{k})$ that must be taken into account satisfying $\hat{e}^1_l(\mathbf{k}) \cdot \hat{e}^2_l(\mathbf{k}) = 0$, while both vectors are orthogonal to $\hat{k}$. The total energy radiated by such degenerate modes must then include an additional summation over the multiplicity (polarization) index $s = 1, 2$. The expanded tensor (50) can then be modified to become

$$\overline{\mathbf{R}}_{ls}(\mathbf{k}) = R_l(\mathbf{k}) \ \hat{e}_{ls}(\mathbf{k}) \ \hat{e}^*_{ls}(\mathbf{k}), \tag{54}$$

while the radiation energy formula (49) acquires the form

$$\mathcal{E}_{\text{rad}} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_l \underbrace{\sum_s U_{ls}(\mathbf{k})}_{U_l(\mathbf{k})}. \tag{55}$$

Here, the modified momentum-space modal radiation density function $U_{ls}(\mathbf{k})$ is defined by

$$U_{ls}(\mathbf{k}) := \mathbf{J}^*_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})] \cdot \overline{\mathbf{R}}_{ls}(\mathbf{k}) \cdot \mathbf{J}_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})]. \tag{56}$$

It is apparent from (56) that the following summation needs to be evaluated

$$\sum_s \overline{\mathbf{R}}_{ls}(\mathbf{k}) = R_l(\mathbf{k}) \sum_s \hat{e}_{ls}(\mathbf{k}) \ \hat{e}^*_{ls}(\mathbf{k}). \tag{57}$$

Using (B10), this can be readily computed as follows: If we define $U_l(\mathbf{k}) := \sum_s U_{ls}(\mathbf{k})$, then

$$\boxed{U_l(\mathbf{k}) = \frac{1}{\varepsilon_0} R_l(\mathbf{k}) \ \mathbf{J}^*_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})] \cdot (\overline{\mathbf{I}} - \hat{k}\hat{k}) \cdot \mathbf{J}_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})].} \tag{58}$$

Noting that the transverse component of the current has an amplitude $\hat{k} \times \mathbf{J}$, the relation (58) can be put in the alternative form

$$U_l(\mathbf{k}) = \frac{1}{\varepsilon_0} R_l(\mathbf{k}) \ |\hat{k} \times \mathbf{J}_{\text{ant}}[\mathbf{k},\omega_l(\mathbf{k})]|^2. \tag{59}$$

The two main formulas (52) and (58) will be used in Part II in order to estimate the radiation pattern of antennas embedded into example isotropic nonlocal metamaterials. The idea is to re-write the wavevector $\mathbf{k}$ as $\mathbf{k} = k\hat{k}$ then make use of the dispersion relation $\omega = \omega(\mathbf{k})$ in order to replace k by ω and then evaluate the radiation energy density as function of $\hat{k}$. Since the latter is an angular function of θ and φ, we may then plot $U_l(k, \hat{k})$ as function of angles around the source for a given k, where k is obtained from both the desired frequency ω and the specific values of $\hat{k}$ by solving the dispersion relation $\omega = \omega_l(k, \hat{k})$ for k and then using the fundamental formulas (52) and (58).

5. SUMMARY, DISCUSSION OF THE RESULTS, AND TRANSITION TO PART II

The ultimate goal of the the momentum-space theory derived above is to provide a practical path toward the estimation of the antenna current's angular energy radiation pattern without first solving for the spatio-temporal fields themselves by inverting the spectral domain GFs, a difficult and hard to generalize task. The radiation density function $U_l(k, \hat{k})$ discussed at the end of Sec. 4 is the key quantity, but it is not the final form we will eventually work with in Part II. Instead, a more convenient form involves the slightly different function $U_l(\omega, \hat{k})$, where ω is the radiator's frequency. Since the technical details will be provided in details in Part II, we summarize here the fundamental main features in the computational approach developed in this paper as follows:

(i) Everything starts from knowledge of the dispersion relations whether in the form $\omega = \omega_l(\mathbf{k})$ or $\mathbf{k}_l = \mathbf{k}_l(\omega)$.

(ii) From the proper dispersion relations, the calculations of the source radiation density pattern in the momentum space representation rests completely on evaluating the fundamental scalar functions $R_l(\mathbf{k})$.

(iii) The previous procedure is conducted for the lth mode. The total radiation energy density will be the direct sum of the same calculations of all modes (superposition applies to modal energy densities). This is a direct consequence of the way in which the momentum space radiation density function $U_l(\mathbf{k})$ was constructed.

(iv) Using the angular form

$$\hat{k} = \hat{k}(\Omega) = \hat{x}\cos\varphi\sin\theta + \hat{y}\cos\varphi\sin\theta + \hat{z}\cos\theta, \qquad \Omega := (\theta, \varphi). \tag{60}$$

the radiation density function $U_l(\mathbf{k})$ can be expressed as an *angular* radiation density function $U_l(k, \Omega)$.

(v) From the dispersion relation, enact the transformation

$$\boxed{U_l(k, \Omega) \xrightarrow{\mathbf{k}=k\hat{k},\ \omega=\omega_l(\mathbf{k})} U_l(\omega, \Omega),} \tag{61}$$

which is based on the dispersion relations of the lth mode. Here, ω is the frequency of the external source energizing the radiating antenna system.

The transformation (61) will be studied in details in Part II for isotropic nonlocal MTMs but will also be generalized to arbitrary domains in the Appendix of [63].

Like every general theory developed with a specific formalism or mathematical/conceptual apparatus in mind, the proposed momentum space theory offers several advantages but also faces some difficulties. For the sake of completeness, we give a brief summary of these issues here. The advantages of the proposed theory include:

(i) The theory leads to a straightforward computational formalism that requires only knowledge of the dielectric tensor function $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$, the dispersion relations (36), and the modal fields $\hat{e}_l(\mathbf{k})$ (these can be obtained using the recipes of Appendix B.

(ii) Once the above data are found, the radiation pattern can be computed in a quasi-analytical fashion, hence very efficiently.

(iii) There is no need to perform Sommerfeld-type spectral integrals to invert the Green's functions of nonlocal domains.

(iv) The theory is completely general and works with both isotropic and anisotropic media with the same general mathematical expressions.

(v) The theory, being developed in momentum space, makes it quite natural and straightforward to adapt to other settings in contemporary applied physics and condensed-matter physics. For example, it is straightforward to generalize the present theory to account for fluctuation phenomena using the methods of [74]. Moreover, the momentum space formalism provides a direct path toward quantization of the theory [4, 6, 67].

On the other hand, some of the open issues still faced by the proposed theory include the following current and future points of potential further developments, which we provide only in summary form while anticipating forthcoming work:

(i) The theory, while very general, does not yet apply to inhomogeneous media.

(ii) It is tedious to perform all the required analytical computations when the number of modes excited by the current source within the nonlocal MTM is large.

(iii) Building concrete examples based on the general theory of this paper requires extensive analytical calculations. For example, Part II is entirely devoted to few examples with only one or two modes excited simultaneously.

(iv) The theory requires knowledge of the dielectric tensor $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$ and dispersion relations in some analytical form. If these information are only available numerically, for instance via measurement, then some substantial additional work might be needed to evaluate the radiation pattern based on our exact formulas. However, curve fitting and analytical approximation tool-kits can be used to replace the numerical data by a proxy analytical form that can be used to compute the needed fundamental radiation functions $U_l(\omega, \Omega)$.

(v) The theory does not give the fields in the far zone, only the energy/power associated with them, which is often enough for majority of antenna applications.

(vi) The theory cannot give the near field produced by the source, only the net power escaping into the far zone.

(vii) The theory requires an array of mathematical methods not usually available to engineers and applied physicists working on device system developments. Some of these methods include the language of tensor calculus, the 4-dimensional Fourier analysis methods, and dyadic Green's functions of nonlocal domains.

Part II will be focused on explicating the points of strength of the proposed theory, especially in light of concrete examples involving isotropic nonlocal MTMs. There, the derivation of $U_l(\omega, \Omega)$ for specific nonlocal domains will be illustrated with various examples and the potential for engineering applications will be highlighted. For the ongoing need to overcome some of the expected difficulties mentioned above, the author believes such open problem are quite natural and expected in any new and emerging research field and hope they will be tackled soon in future publications.

The theory of electromagnetic radiation in nonlocal domains has been designed and developed with the expectation that it will help stimulate the research & development of futuristic radiating systems exhibiting unusual or nonclassical behaviour. *Future antenna systems* can be defined as novel antenna technologies that are currently actively being developed or predicted to play a major role in the near or far futures. Examples include plasma and quantum antennas, biological and molecular transmitters, and intelligent electromagnetic agents, just to mention few. The key feature in such systems is not necessarily that they serve a specific role in an existing applications, but instead what distinguish such future generation of systems is their ability to exhibit a new and unique radiation properties unseen in conventional antennas. The author believes that nonlocal antenna systems as an example of future antennas present one of the most promising and exciting applications of fundamental theory to technology. In Part II, we continue to explore how these future systems perform in terms of standard measures like bandwidth, directivity, and isotropicity, in addition to new ideas, meanwhile pointing out their potential deployment for further development especially in areas like wireless communications and nanotechnology.

Acronym	Meaning
EM	Electromagnetics/Electromagnetic
MTM	Metamaterial
NL-MTM	Nonlocal metamaterial
NR-NL-MTM	Nonresonant nonlocal metamaterial
L	Longtitdunal
T	Transverse
GF	Green's Function
NL-AS	Nonlocal antenna system

Table A1: List of abbreviations used in this paper (Parts I and II).

6. CONCLUSION

We have provided a complete and rigorous derivation of an equivalent quantity that gives the amount of energy radiated by an antenna embedded into a generic nonlocal metamaterial per unit Hertz per unit solid angle. The method is based on carrying out all calculations in frequency-momentum space instead of the conventional approach in spacetime. Since the Poynting vector in nonlocal media fails to describe the direction of power flow, we computed the energy injected directly from the antenna current into the near field in order to estimate the radiation energy intensity per unit frequency per unit solid angle. It was found that the total radiation pattern is the sum of radiation functions each controlled by the corresponding longitudinal and/or transverse mode that the antenna launch into the nonlocal metamaterial. The derived expression can be evaluated analytically if the dispersion relation and hence the modes of the nonlocal medium are known.

APPENDIX A. LIST OF ABBREVIATIONS

In Table A1, we list all abbreviations used in this paper. In general, we tried to avoid using such acronyms as much as possible.

APPENDIX B. AN ALGORITHM FOR COMPUTING POLARIZATION IN NONLOCAL MEDIA

A rather direct routine exists for the computation of the polarization vectors corresponding to a given mode provided the mode's dispersion relation $\omega_l(\mathbf{k})$ is available. We start from (33), which can be re-adapted into the component form

$$\sum_{n=1}^{3} G^{\mathrm{H}}_{in}(\mathbf{k},\omega) C_{nj}(\mathbf{k},\omega) = \delta_{ij} G^{-1}(\mathbf{k},\omega). \tag{B1}$$

For a given mode, $G^{-1} = 0$ and hence we have $\sum_n G_{in} C_{nj} = 0$. Therefore, every vector of the form $[\ C_{1j} \quad C_{2j} \quad C_{3j}\ ]^T$ satisfies the homogeneous wave equation (35) for each j, implying that C_{ij} is proportional to the mode polarization form e_{li} defined by (37), where $i = 1, 2, 3$, enumerates the three cartesian components of $\hat{e}_l$. However, note that from its defining relation (B1), the co-factor matrix C_{ij} is hermitian because we operate only with the hermitian part of the propagator (Green's function) $\overline{\mathbf{G}}$. Consequently, C_{nj} is *also* proportional to e^*_{lj}, $j = 1, 2, 3$. Combining these two proportionality conditions, we write

$$C_{ij}(\mathbf{k},\omega) = \gamma e_{li}(\mathbf{k}) e^*_{lj}(\mathbf{k}) \tag{B2}$$

for some constant γ. To find this constant, we take the trace of (B2) and use (38), leading to

$$\overline{\mathbf{C}}(\mathbf{k},\omega_l(\mathbf{k})) = \gamma_l(\mathbf{k})\ \hat{e}_l(\mathbf{k})\ \hat{e}^*_l(\mathbf{k}), \tag{B3}$$

where

$$\gamma_l(\mathbf{k}) := \mathrm{tr}[\overline{\mathbf{C}}(\mathbf{k}, \omega_l(\mathbf{k}))]. \tag{B4}$$

This decomposition of the co-factor matrix (dyad) into modal polarization factors is very general and is often useful in calculations. Note further that the following important symmetry relation also applies

$$\omega_l(-\mathbf{k}) = -\omega_l(\mathbf{k}), \tag{B5}$$

which links the forward and backward wave solutions of the wave equation. Using the reality and hermitian conditions, this leads to

$$\hat{e}_l(-\mathbf{k}) = \hat{e}_l^*(\mathbf{k}). \tag{B6}$$

Moreover, the expansion (B3) together with (B6) jointly imply

$$\overline{\mathbf{C}}(-\mathbf{k}, -\omega) = \overline{\mathbf{C}}^T(\mathbf{k}, \omega). \tag{B7}$$

Assuming that the dispersion relation is available, we may now summarize the algorithm needed to compute polarization in nonlocal domains as follow:

(i) Find the co-actor matrix $C_{ij}(\mathbf{k}, \omega)$ using the standard inversion approach in matrix analysis.

(ii) Substitute the dispersion relation $\omega_l(\mathbf{k})$ into the co-factor matrix elements $C_{ij}(\mathbf{k}, \omega)$ in order to obtain the $\mathbf{k}$-functions $C_{ij}(\mathbf{k}, \omega_l(\mathbf{k}))$.

(iii) Select any column $C_{ij}(\mathbf{k}, \omega_l(\mathbf{k}))$. Normalize this column using the relation (37). This will serve as a possible modal field solution.

In this way, the normalized $\hat{e}_l(\mathbf{k})$ of the mode under consideration can be explicitly computed starting from the material tensor function $\overline{\overline{\varepsilon}}(k, \omega)$, but only if the dispersion relation $\omega = \omega_l(\mathbf{k})$ of the mode under consideration is available. Usually it is this dispersion law that is more expensive to compute in the analysis of nonlocal materials.

It is instructive to add few words about energy exchange between different modes in nonlocal domains since this behaviour is less intuitive than normal dispersion. Consider two different modes l_1 and l_2 with dispersion relations $\omega = \omega_{l_1}(\mathbf{k})$ and $\omega = \omega_{l_2}(\mathbf{k})$, where dissipation is neglected. The corresponding modal field distributions are captured by the vectors $\hat{e}_{l_1}(\mathbf{k})$ and $\hat{e}_{l_2}(\mathbf{k})$. By direct calculation of the energy transferred from one the fields of one mode to another, it can be shown that a nonzero such energy exchange may occur if and only if the following strict condition is satisfied

$$\hat{e}_{l_2}^*(\mathbf{k}) \cdot \left[\overline{\overline{\varepsilon}}^{\mathrm{H}}(k, \omega) - \overline{\mathbf{I}}\right] \cdot \hat{e}_{l_1}(\mathbf{k}) = 0. \tag{B8}$$

In other words, in momentum space, geometrical orthogonality $\hat{e}_{l_2}^*(\mathbf{k}) \cdot \hat{e}_{l_1}(\mathbf{k}) = 0$ is not equivalent to natural mode orthogonality.

We also add a useful identity enjoyed by modal polarization vectors of degenerate waves like transverse modes with degeneracy index $s = 1, 2$. By its construction, the vectors $\hat{e}_{l1}$, $\hat{e}_{l2}$, and $\hat{k}$ form an orthogonal basis set for $\mathbb{R}^3$. Therefore, we have by the resolution of identity (completeness) relation

$$\hat{e}_{l1}(\mathbf{k})\ \hat{e}_{l1}^*(\mathbf{k}) + \hat{e}_{l2}(\mathbf{k})\ \hat{e}_{l2}^*(\mathbf{k}) + \hat{k}\hat{k} = \overline{\mathbf{I}} \tag{B9}$$

which is the completeness relation for the transverse modal fields. Summing over the two degenerate modes, we find

$$\sum_s \hat{e}_{ls}(\mathbf{k})\ \hat{e}_{ls}^*(\mathbf{k}) = \overline{\mathbf{I}} - \hat{k}\hat{k}, \tag{B10}$$

another version of the resolution of identity for modal fields. The relation (B10) is very useful when attempting to estimate the total radiation power/energy without needing to account for the details of polarization, which is often the case in scattering and radiation in random media.

APPENDIX C. EXPLICIT GENERIC SPATIAL DISPERSION DOMAIN TENSOR FORMULAS AND SOME OF THEIR PROPERTIES

(i) The reality condition requires that

$$\overline{\mathbf{G}}^{-1}(\mathbf{k},\omega) = \left[\overline{\mathbf{G}}^{-1}\left(-\mathbf{k}^{*},-\omega^{*}\right)\right]^{*}, \tag{C1}$$

where we consider the most general case of dissipative medium in which both the spatial and temporal frequencies $\mathbf{k}$ and ω are allowed to become complex.

(ii) The fundamental quantity G^{-1}, the converse of the determinant of the forward Green's function dyad, can be explicitly computed in terms of the material tensor $\overline{\overline{\varepsilon}}(k,\omega)$. The result is given by

$$G^{-1} = n^4\ \hat{k}\cdot\overline{\overline{\varepsilon}}(k,\omega)\cdot\hat{k} - n^2\left(\hat{k}\cdot\overline{\overline{\varepsilon}}(k,\omega)\cdot\hat{k}\ \mathrm{tr}\left[\overline{\overline{\varepsilon}}(k,\omega)\right] - \hat{k}\cdot\overline{\overline{\varepsilon}}^{2}(k,\omega)\cdot\hat{k} + |\overline{\overline{\varepsilon}}(k,\omega)|\right), \tag{C2}$$

where $\mathrm{tr}\left[\overline{\overline{\varepsilon}}(k,\omega)\right]$ and $|\overline{\overline{\varepsilon}}(k,\omega)|$ are the matrix trace and determinant operations applied to the material tensor $\overline{\overline{\varepsilon}}(k,\omega)$, while

$$n^2 := \frac{k^2c^2}{\omega^2}. \tag{C3}$$

(iii) The tensor expression for the co-factor matrix is considerably more complicated but can be put in the form

$$\begin{aligned}\overline{\mathbf{C}}(\mathbf{k},\omega) = n^4\ \hat{k}\hat{k} - n^2&\left\{\hat{k}\hat{k}\ \mathrm{tr}\left[\overline{\overline{\varepsilon}}(k,\omega)\right] + \overline{\mathbf{I}}\hat{k}\cdot\overline{\overline{\varepsilon}}(k,\omega)\cdot\hat{k} - \hat{k}\ \hat{k}\cdot\overline{\overline{\varepsilon}}(\mathbf{k},\omega) - \hat{k}\ \overline{\overline{\varepsilon}}(\mathbf{k},\omega)\cdot\hat{k}\right\}\\ &+\frac{1}{2}\overline{\mathbf{I}}\left\{\left(\mathrm{tr}\left[\overline{\overline{\varepsilon}}(\mathbf{k},\omega)\right]\right)^2 - \mathrm{tr}\left[\overline{\overline{\varepsilon}}^{2}(\mathbf{k},\omega)\right]\right\} + \overline{\overline{\varepsilon}}^{2}(\mathbf{k},\omega) - \mathrm{tr}\left[\overline{\overline{\varepsilon}}(\mathbf{k},\omega)\right]\overline{\overline{\varepsilon}}(\mathbf{k},\omega).\end{aligned} \tag{C4}$$

It is worth reminding the reader that when $G^{-1}=0$, the matrix representation of the tensor $\overline{\mathbf{C}}$, i.e, the array C_{ij}, is a rank-one matrix, while G_{ij}^{-1} has rank two. The formal proof of the above relations (C2) and (C4) can be obtained using the Cayley-Hamilton's theorem, which states that a matrix satisfies its own characteristic equation. The details are straightforward but the computations are lengthy and will hence be omitted. Some of the texts that discuss these calculations include [2, 4].

(iv) The determinant of $\overline{\mathbf{C}}$ is clearly given by

$$\det\left[\overline{\mathbf{C}}(\mathbf{k},\omega)\right] = G^{-2}(\mathbf{k},\omega). \tag{C5}$$

(v) A very general formula for evaluating the fundamental $R_l(\mathbf{k})$ alternative to (51) can be derived using (C2) and (C4). The result is

$$R_l(\mathbf{k}) = \left.\frac{\omega}{\frac{\partial}{\partial\omega}\left[\omega^2\ \hat{e}_l^{*}(\mathbf{k})\cdot\overline{\overline{\varepsilon}}^{\mathrm{H}}(\mathbf{k},\omega)\cdot\hat{e}_l(\mathbf{k})\right]}\right|_{\omega=\omega_l(\mathbf{k})} \tag{C6}$$

This alternative form can be very useful to either perform all calculations in practical settings or to double check the correctness of results obtained by other means and will be used in Part II. Its usefulness resides in the fact that only the dielectric function is needed and the latter is often available from previous analysis or measurement. We provide only a brief sketch of the proof. Taking the hermitian of (33) then the differential and using some of the tensorial (algebraic) properties of the co-factor matrices, the following relation can be derived

$$\mathrm{d}\left[G^{-1,\mathrm{H}}(\mathbf{k},\omega)\right] = \sum_{ij=1}^{3} C_{ij}^{\mathrm{H}}(\mathbf{k},\omega)\mathrm{d}\left[G^{-1,\mathrm{H}}(\mathbf{k},\omega)\right] \tag{C7}$$

Combining the last equation with (B3), the main expression of $R_l(\mathbf{k})$ given by (51) can be put in the alternative shape

$$R_l(\mathbf{k}) = \left. \frac{1}{\hat{e}_l^*(\mathbf{k}) \cdot \frac{\partial}{\partial \omega} \left[\omega G^{-1,\mathrm{H}}(\mathbf{k},\omega) \right] \cdot \hat{e}_l(\mathbf{k})} \right|_{\omega=\omega_l(\mathbf{k})} . \tag{C8}$$

The dyadic GF tensorial form in momentum space (30) can then be exploited in order to show that

$$\hat{e}_l^*(\mathbf{k}) \cdot \frac{\partial}{\partial \omega} \left[\omega G^{-1,\mathrm{H}}(\mathbf{k},\omega) \right] \cdot \hat{e}_l(\mathbf{k}) = \frac{1}{\omega} \frac{\partial}{\partial \omega} \left[\omega^2 \ \hat{e}_l^*(\mathbf{k}) \cdot \overline{\overline{\varepsilon}}^{\mathrm{H}}(\mathbf{k},\omega) \cdot \hat{e}_l(\mathbf{k}) \right] , \tag{C9}$$

after which (C6) readily follows.

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