A Theorem on Power Superposition in DC Networks

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October 30, 2023

Abstract

The superposition theorem, a particular case of the superposition principle, states that in a linear circuit with several voltage and current sources, the current and voltage for any element of the circuit is the algebraic sum of the currents and voltages produced by each source acting independently. The superposition theorem is not applicable to power, because it is a non-linear quantity. Therefore, the total power dissipated in a resistor must be calculated using the total current through (or the total voltage across) it. The theorem proposed and proved in this paper states that in a linear DC network consisting of resistors and independent voltage and current sources, the total power dissipated in the resistors of the network is the sum of the power supplied simultaneously by the voltage sources with the current sources replaced by open circuit, and the power supplied simultaneously by the current sources when the voltage sources are replaced by short-circuit. This means that the power is superimposed. The theorem can be used to simplify the power analysis of DC networks. The analysis results are validated via numerical examples.

A Theorem on Power Superposition in DC Networks

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***Abstract*—The superposition theorem, a particular case of the superposition principle, states that in a linear circuit with several voltage and current sources, the current and voltage for any element of the circuit is the algebraic sum of the currents and voltages produced by each source acting independently. The superposition theorem is not applicable to power, because it is a non-linear quantity. Therefore, the total power dissipated in a resistor must be calculated using the total current through (or the total voltage across) it. The theorem proposed and proved in this paper states that in a linear DC network consisting of resistors and independent voltage and current sources, the total power dissipated in the resistors of the network is the sum of the power supplied simultaneously by the voltage sources with the current sources replaced by open circuit, and the power supplied simultaneously by the current sources when the voltage sources are replaced by short-circuit. This means that the power is superimposed. The theorem can be used to simplify the power analysis of DC networks. The analysis results are validated via numerical examples.**

***Index Terms*—DC networks, power calculation, power superposition in DC networks, DC network theorem.**

I. Introduction

THE superposition theorem states that for time-invariant linear DC networks, having more than one independent source, the current or voltage in any branch of the circuit equals the algebraic sum of the response caused by each independent source acting alone, while all the other individual voltage sources are replaced by short circuit and the current sources are replaced by open circuit.

This theorem is very important in circuit theory and finds many practical applications in network analysis. However, it is well established that it works for voltage and current but does not work for power. Usually, the sum of the powers of each current and voltage source acting individually, with the other voltage and current sources set equal to zero, is not equal to the total consumed power [1].

This paper demonstrates that there is a category of linear DC networks, consisting of resistors, independent voltage sources and independent current sources, in which the total power dissipated in the resistors is the algebraic sum of the total power supplied simultaneously by all voltage sources while the current sources are replaced by open circuit, and the total power supplied simultaneously by the current sources while the voltage sources as replaced by short circuit.

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Manuscript received September 14, 2020; revised August 26, 2015.

II. Analysis of a Generic DC Network

Fig. 1 shows a system formed by two independent ideal voltage sources V_1 and V_2, two independent ideal current sources I_1 and I_2, and a network N consisting of resistors of constant, linear and bilateral resistances. The system forms a four-port DC network.

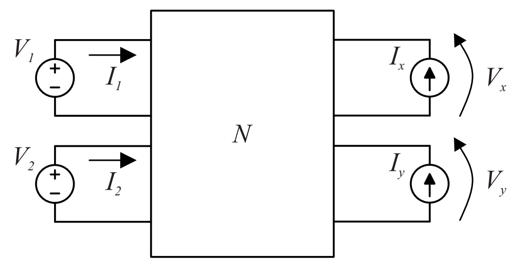

Fig. 1. Four port DC network.

The equations that describe the four port network in terms of hybrid parameters (or g-parameters) are [2]

$$I_1 = g_{11}V_1 + g_{12}V_2 + g_{1x}I_x + g_{1y}I_y \tag{1}$$

$$I_2 = g_{21}V_1 + g_{22}V_2 + g_{2x}I_x + g_{2y}I_y \tag{2}$$

$$V_x = g_{x1}V_1 + g_{x2}V_2 + g_{xx}I_x + g_{xy}I_y \tag{3}$$

$$V_y = g_{y1}V_1 + g_{y2}V_2 + g_{yx}I_x + g_{yy}I_y \tag{4}$$

The power dissipated by the network resistors is

$$P = P_1 + P_2 + P_x + P_y \tag{5}$$

where

$$P_1 = V_1 I_1 \tag{6}$$

$$P_2 = V_2 I_2 \tag{7}$$

$$P_x = V_x I_x \tag{8}$$

$$P_y = V_y I_y \tag{9}$$

Substituting (1), (2), (3) and (4) in (6), (7), (8) and (9) respectively, we find

$$P_1 = g_{11}{V_1}^2 + g_{12}V_2V_1 + g_{1x}I_xV_1 + g_{1y}I_yV_1 \tag{10}$$

$$P_2 = g_{21}V_1V_2 + g_{22}{V_2}^2 + g_{2x}I_xV_2 + g_{2y}I_yV_2 \tag{11}$$

$$P_x = g_{x1}V_1I_x + g_{x2}V_2I_x + g_{xx}{I_x}^2 + g_{xy}I_yI_x \tag{12}$$

$$P_y = g_{y1}V_1I_y + g_{y2}V_2I_y + g_{yx}I_xI_y + g_{yy}{I_y}^2 \tag{13}$$

In a reciprocal linear network

$$g_{1x} = -g_{x1} \tag{14}$$

$$g_{2y} = -g_{y2} \tag{15}$$

$$g_{x2} = -g_{2x} \tag{16}$$

$$g_{y1} = -g_{1y} \tag{17}$$

$$g_{12} = g_{21} \tag{18}$$

$$g_{xy} = g_{yx} \tag{19}$$

Substitution of (14), (15), (16), (17), (18) and (19) in (10), (11), (12), (13) and (5) gives

$$\begin{aligned} P =& g_{11}{V_1}^2 + g_{12}{V_2}^2 + 2g_{12}V_1V_2 + g_{xx}{I_x}^2 \\ &+ g_{yy}{I_y}^2 + 2g_{xy}I_xI_y \end{aligned} \tag{20}$$

Let us define P_x and P_y as

$$P_x = g_{11}{V_1}^2 + g_{12}{V_2}^2 + 2g_{12}V_1V_2 \tag{21}$$

$$P_y = g_{xx}{I_x}^2 + g_{yy}{I_y}^2 + 2g_{xy}I_xI_y \tag{22}$$

Thus,

$$P = P_x + P_y \tag{23}$$

Fig. 3 shows the network for the condition where $V_1 \neq 0$, $V_2 \neq 0$, $I_x = 0$ and $I_y = 0$.

Let P_α be the power dissipated by the resistors when $I_x = 0$ and $I_y = 0$. For this condition $P_x = P_y = 0$. Substituting the values of $I_x = 0$ and $I_y = 0$ in (10) and (11) we find

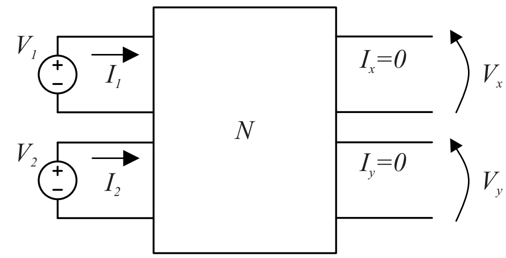

Fig. 2. The four-port network when $I_x = I_y = 0$.

$$P_\alpha = g_{11}{V_1}^2 + g_{12}{V_2}^2 + 2g_{12}V_1V_2 \tag{24}$$

Fig. 3 shows the network for the condition where $V_1 = 0$, $V_2 = 0$, $I_x \neq 0$ and $I_y \neq 0$.

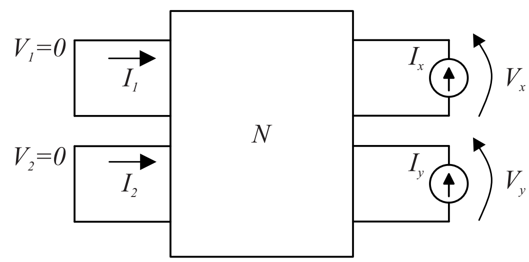

Fig. 3. The four-port network when $V_1 = V_2 = 0$.

Let P_β be the power dissipated by the resistors when $V_1 = 0$ and $V_2 = 0$. For this condition $P_1 = P_2 = 0$. Substitution of $V_1 = 0$ and $V_2 = 0$ in (12) and (13) yields

$$P_\beta = g_{xx}{I_x}^2 + g_{yy}{I_y}^2 + 2g_{xy}I_xI_y \tag{25}$$

Thus, we verify that

$$P_\alpha = P_v \tag{26}$$

$$P_\beta = P_I \tag{27}$$

We can then conclude that the power dissipated in the resistors is the sum of the power P_α supplied simultaneously by the voltage sources when the current sources are replaced by an open circuit, and the power P_β supplied simultaneously by the current sources when the voltage sources are replaced by a short circuit. Therefore, the total power delivered by the sources to the resistors is the result of the superposition of the two powers P_α and P_β.

III. Analysis of a Parcitular Circuit

Consider the two-port network shown in Fig. 4, which consists of a voltage source V_1, a current source I_x and three resistors R_1, R_2 and R_3.

The describing equations are

$$I_1 = g_{11}V_1 + g_{1x}I_x \tag{28}$$

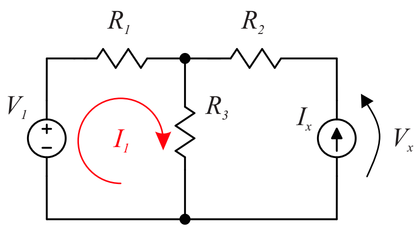

Fig. 4. A particular two-port network.

$$V_x = g_{x1}V_1 + g_{xx}I_x \tag{29}$$

where

$$g_{11} = \frac{1}{R_1 + R_3} \tag{30}$$

$$g_{xx} = R_2 + R_3 \tag{31}$$

$$g_{1x} = \frac{-R_3}{R_1 + R_2} \tag{32}$$

$$g_{x1} = \frac{R_3}{R_1 + R_2} \tag{33}$$

The power transferred from the sources V_1 and I_x to the resistors is

$$P = V_1 I_1 + I_x V_x \tag{34}$$

Substitution of (28) and (29) in (34) yields

$$P = g_{11}{V_1}^2 + g_{1x}I_xV_1 + g_{x1}V_1I_x + g_{xx}{I_x}^2 \tag{35}$$

Making $g_{1x} = -g_{x1}$ in (35) gives

$$P = g_{11}{V_1}^2 + g_{xx}{I_x}^2 \tag{36}$$

Substituting (30) and (31) in (36) we find

$$P = \frac{{V_1}^2}{R_1 + R_2} + (R_2 + R_3){I_x}^2 \tag{37}$$

The first term of (38) is the power transferred to the resistors by the voltage source V_1 when $I_x = 0$ and the equivalent circuit for this condition is shown in Fig. 5(a). The second term of (38) is the power transferred to the resistors by the current source I_x when $V_1 = 0$ and the equivalent circuit for this condition is shown in Fig. 5(b).

IV. Numerical Example

Let us consider the circuit shown in Fig. 6, which is used to illustrate the use of the proposed theorem to calculate the total power dissipated in the resistors R_1, R_2 and R_3.

The parameters of the circuit are $V_1 = 100\,V$, $V_2 = 50\,V$, $I_x = 10\,A$, $R_1 = 5\,\Omega$, $R_2 = 9\,\Omega$ and $R_3 = 15\,\Omega$.

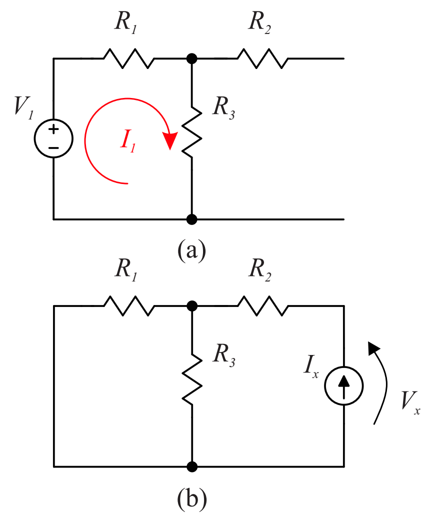

Fig. 5. (a) Equivalent circuit when $I_x = 0$ and (b) equivalent circuit when $V_1 = 0$.

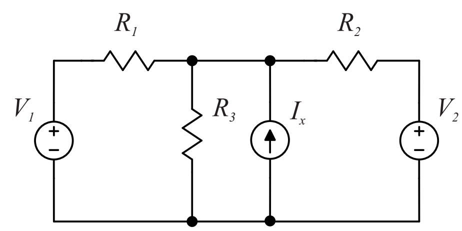

Fig. 6. Electric circuit for the numerical example.

A. Power dissipated in the resistors when $V_1 = 0$ and $V_2 = 0$

When $V_1 = V_2 = 0$, the equivalent circuit is that shown in Fig. 7, where

$$R_x = \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3} = 2.647\,\Omega$$

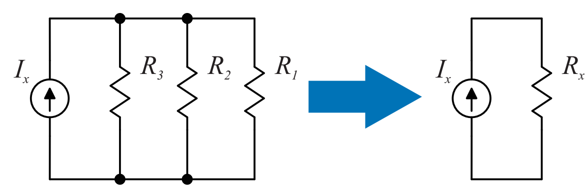

Fig. 7. Equivalent circuit when $V_1 = V_2 = 0$.

The power supplied by the current source I_x is

$$P = R_x.I_x^2 = 264.71\,W$$

B. Power dissipated in the resistors when $I_x = 0$

When $I_x = 0$ the equivalent circuit is that shown in Fig. 8. According to Millman's Theorem [3],

$$V_x = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = 67.65\,V$$

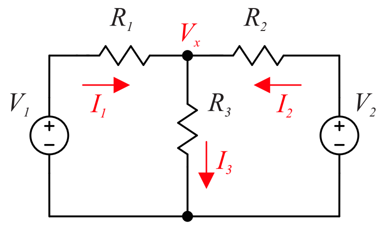

Fig. 8. Equivalent circuit when $I_x = 0$.

The power dissipated in the resistors is

$$P_v = \frac{(V_x - V_1)^2}{R_1} + \frac{{V_x}^2}{R_3} + \frac{(V_x - V_2)^2}{R_2} = 549.00\,W$$

C. Total power dissipated in the resistors

The total power dissipated in the resistors, according to the proposed theorem is

$$P = P_v + P_I = 549.00\,W + 264.71\,W = 813.73\,W$$

D. Total power calculation using conventional approach

The operation of the network shown in Fig. 6, according to Kirchhoff's voltage and current laws (KVL and KCL), is described by the system of three equations with three unknowns

$$V_1 = R_1 I_1 + V_x \tag{38}$$

$$V_2 = R_2 I_2 + V_x \tag{39}$$

$$I_x = -I_1 - I_2 + \frac{V_x}{R_3} \tag{40}$$

The solution of this system of equations is

$$\begin{aligned} \begin{bmatrix} I_1 \\ I_2 \\ V_x \end{bmatrix} &= \begin{bmatrix} R_1 & 0 & 1 \\ 0 & R_2 & 1 \\ -1 & -1 & \frac{1}{R_3} \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_1 \\ V_2 \\ I_x \end{bmatrix} \\ &= \begin{bmatrix} 1.176A \\ -4.902A \\ 94.118V \end{bmatrix} \end{aligned} \tag{41}$$

The power dissipated in the resistors is

$$\begin{aligned} P &= \begin{bmatrix} V_1 & V_2 & I_x \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V_x \end{bmatrix} \\ &= \begin{bmatrix} 100 & 50 & 10 \end{bmatrix} \begin{bmatrix} 1.176A \\ -4.902A \\ 94.118V \end{bmatrix} = 813.72\,W \end{aligned}$$

This result, obtained by the conventional analysis method, using KVL and KCL, is identical to the previous one using the power superposition approach.

V. A DC Network Theorem

In the previous sections, the proof of the theorem formulated below was presented, which provides simplification in the power calculation for linear and bilateral DC networks, consisting of resistors and independent voltage and current sources.

For a linear DC network formed by resistors and independent voltage and current sources, let P_v be the sum of the powers supplied simultaneously to the resistors of the network by the voltage sources when all current sources are replaced by an open circuit. Let P_I be the sum of the powers supplied simultaneously to the resistors of the network by the current sources when all voltage sources are replaced by a short circuit. Hence, the total power dissipated in the resistors when all voltage and current sources are simultaneously present is equal to $P_v + P_I$.

VI. Conclusion

This paper introduces a new DC network theorem that can be used to simplify the determination of the total power dissipated in the resistors of a linear DC network formed by resistors and independent voltage and current sources. It is demonstrated theoretically that the total power is the sum of the power supplied simultaneously by the voltage sources with all current sources replaced by open circuit, and the power supplied by the current sources with all voltage sources replaced by short circuit.

Acknowledgment

The work of Victor Luiz F. Borges, Engineer with the Brazilian Institute of Power Electronics and Renewable Energy (IBEPE), who collaborated in the manuscript editing, is gratefully acknowledged.

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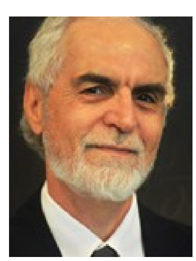

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