

# Representations of Noisy N-Ports

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April 18, 2024

# Representations of Noisy $N$ -Ports

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April 12, 2024

**Abstract**—Much has been written about the representation of noisy linear 2-ports. Here we present a theory of noisy  $N$ -ports. We show how in the general case there are  $(2N)!/(N!)^2$  equivalent representations and give the transformations relating them. We also discuss singular cases in which some of the transformations are not possible as well as how to measure the noise properties of an  $N$ -port.

**Index Terms**—noise, circuit analysis, network theory, circuit noise

## I. INTRODUCTION

There is an extensive literature on noisy linear 2-ports. It is for example well known that a noisy amplifier can be represented by an equivalent circuit consisting of a noiseless amplifier with a voltage and current source placed on the input side as sketched in Fig. 1. (See for example [1], [2].) As sketched, the circuit looks like an inhomogeneous 2-port, but here the complex source amplitudes  $v_n$  and  $i_n$  (at a particular angular frequency  $\omega$ ) are taken to be determined stochastically according to a Gaussian random ensemble. The correlation functions  $\langle v_n^* v_n \rangle$ ,  $\langle i_n^* i_n \rangle$ , and  $\langle v_n^* i_n \rangle$ , which are described by four real parameters, completely suffice to determine the noise properties of the amplifier, or noisy 2-port.<sup>1</sup> The two auto-correlations are real and positive, and the cross-correlation may be expressed as

$$\langle v_n^* i_n \rangle = (\alpha_n + j\beta_n) \sqrt{\langle v_n^* v_n \rangle \langle i_n^* i_n \rangle}$$

where  $\alpha_n^2 + \beta_n^2 \leq 1$ .

Noisy  $N$ -ports (where  $N > 2$ ) have been previously studied, in particular in the work of Haus and Adler [3]–[5] and of Bosma [6]. There the emphasis, generalizing on earlier work by Mason [7] on 2-ports, was on studying the invariant properties of  $N$ -ports independent of their embedding using lossless components to map the  $N$  ports into  $N$  outer ports of the embedding network. Such an embedding network among other things can transform impedances of ports and introduce feedback, both positive and negative. Mason introduced an invariant known as the ‘unilateral power gain’ that does not change as a result of the choice lossless embedding network. This line of investigation clarified the relation between feedback and noise (or noise figure).

Here our motivation is slightly different. We investigate how noise sources can be moved around to give a multitude of

equivalent representations of a noisy  $N$ -port. Our underlying application is the modelling of global 21 cm experiments in radio astronomy, where the object is to make an absolute noise measurement of the sky signal at an accuracy to our knowledge higher than that required for other applications (see [8], [9] and references therein). The techniques presented here allow circuits with noisy  $N$ -ports and loops to be simplified by moving noise sources around. Related work includes work by Twiss generalizing Nyquist’s theorem to lossy multipoint devices at a common temperature [12] and work on noise matching for active phased array antennas [13]–[15].

For the setup in Fig. 1, when the amplifier is not unilateral, we can displace the noise sources to the output, and in the general case there are six equivalent representations for the noisy 2-port, as sketched in Fig. 2.

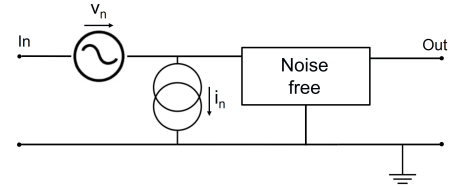


Fig. 1. Noisy Amplifier or 2-Port.

We now proceed to generalize by describing a noisy  $N$ -port. Let  $\bar{v}_I, \bar{i}_I$  ( $I = 1..N$ ) be the voltages and currents as added to the embedded homogeneous  $N$ -port, as illustrated in Fig. 3. Similarly  $v_I, i_I$  are the voltages and currents as seen at the ports of the inhomogeneous  $N$ -port and  $v'_I, i'_I$  are the voltages and currents at the terminals of the embedded homogeneous  $N$ -port, which may for example be related by the impedance matrix  $Z_{IJ}$ , so that

$$v'_I = Z_{IJ} i'_J \quad (1)$$

when this representation is not singular. When this homogeneous  $N$ -port is dressed with voltage and current sources as illustrated in Fig. 3, eqn. (1) is modified to become

$$(v_I - \bar{v}_I) = Z_{IJ} (i_J - \bar{i}_J). \quad (2)$$

Here there are  $2N$  voltage and current sources, but in the general case the number of sources can be reduced to  $N$ . For example, when the  $Z$  representation of the embedded homogeneous  $N$ -port exists, by setting

$$\bar{\bar{v}}_I = \bar{v}_I - Z_{IJ} \bar{i}_J \quad (3)$$

we may eliminate all the  $N$  current sources and obtain an equivalent representation with  $N$  voltage sources, as sketched in Fig. 4. From the point of view of the external ports, from which only  $v_I, i_I$  are visible, the representations in Figs. 3 and 4 are indistinguishable.

<sup>1</sup>Here we are assuming that the noise is completely characterized by its second-order correlation functions, which is the case for Gaussian noise, for which there are no phase correlations between different Fourier coefficients. For non-Gaussian noise, of which burst noise is one example, higher-order correlation functions as well as are required to characterize the noise. However even when the noise is non-Gaussian, the two-point correlation functions considered here provide a partial characterization and for many applications suffice.

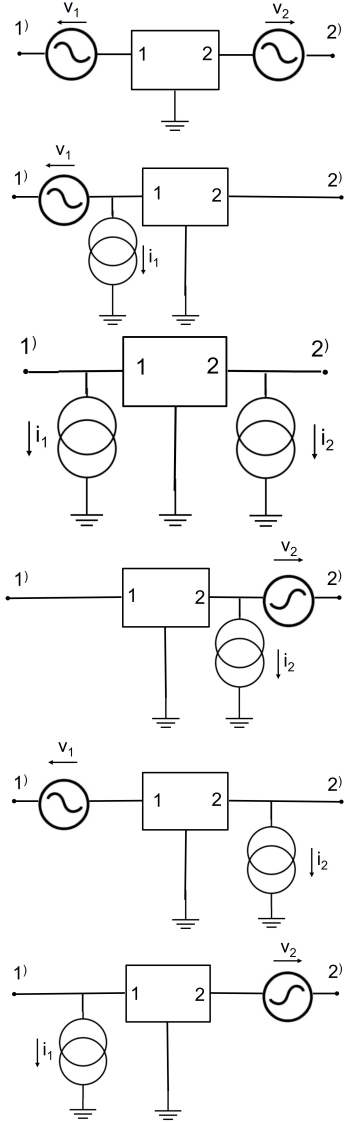


Fig. 2. Six Equivalent Noisy 2-Port Representations.

When the admittance matrix  $\mathbf{Y}$  is defined, where  $\mathbf{Y} = \mathbf{Z}^{-1}$  when  $\mathbf{Z}$  is non-singular and invertible, we may similarly turn all voltage sources into current sources, so that

$$\bar{\bar{i}}_I = \bar{i} - I_J \bar{v}_J \quad (4)$$

and the equivalent representation in Fig. 5 results. This representation is likewise completely equivalent to those in Figs. 3 and 4.

We may also use an S-matrix travelling wave representation, where

$$\begin{aligned} v_I^{(in)} &= v_I^{(right)} = v_I + Z_c i_I \\ v_I^{(out)} &= v_I^{(left)} = v_I - Z_c i_I \end{aligned} \quad (5)$$

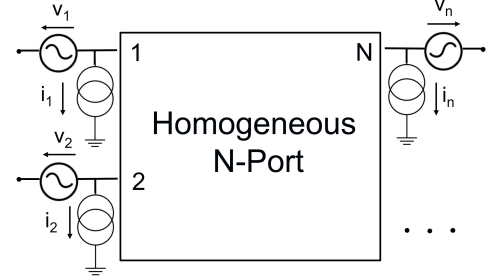
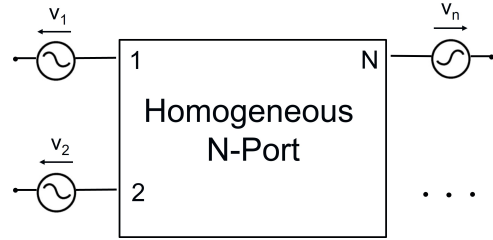
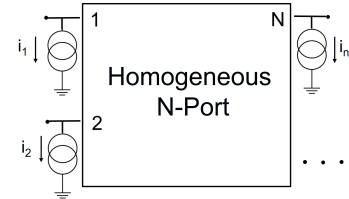
and

$$\mathbf{S} = \frac{\mathbf{Z} - Z_c \mathbf{I}}{\mathbf{Z} + Z_c \mathbf{I}}. \quad (6)$$

Here the characteristic impedance  $Z_c$  is generally taken to be positive and real, corresponding to some lossless transmission

line, although a complex characteristic impedance is possible as well, and the transmission line can be fictitious.

In this travelling wave picture [10], [11], the voltage current source combination may be replaced by ‘in’ and ‘out’ travelling wave sources, as sketched in Fig. 6. Whereas in Fig. 6a the voltage and current sources impose jump conditions between  $v^{ext}$  and  $v^{int}$ , and between  $i^{ext}$  and  $i^{int}$ , respectively, in Fig. 6b the travelling wave sources impose jump conditions between  $v^{in,ext}$  and  $v^{in,int}$ , and between  $v^{out,ext}$  and  $v^{out,int}$ .

Fig. 3. General Noisy  $N$ -Port Representations.Fig. 4. Noisy  $N$ -Port Representation With Only Voltage Sources.Fig. 5. Noisy  $N$ -Port Representation With Only Current Sources.

## II. TRANSFER MATRIX REPRESENTATION AND ITS GENERALIZATIONS

Here we generalize the transfer matrix (or ABCD-matrix) representation

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \mathbf{T}_{21} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A_{21} & B_{21} \\ C_{21} & D_{21} \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}, \quad (7)$$

which is so useful for analyzing a series of 2-ports chained together. For the transfer matrix  $\mathbf{T}_{21}$  the sign conventions are typically modified a bit, so that a current following into port 1 is considered positive whereas a current flowing out of port 2 is regarded positive. For the inhomogeneous case,

$$\begin{pmatrix} v_2 - \bar{v}_2 \\ i_2 - \bar{i}_2 \end{pmatrix} = \mathbf{T}_{21} \begin{pmatrix} v_1 - \bar{v}_1 \\ i_1 - \bar{i}_1 \end{pmatrix}, \quad (8)$$

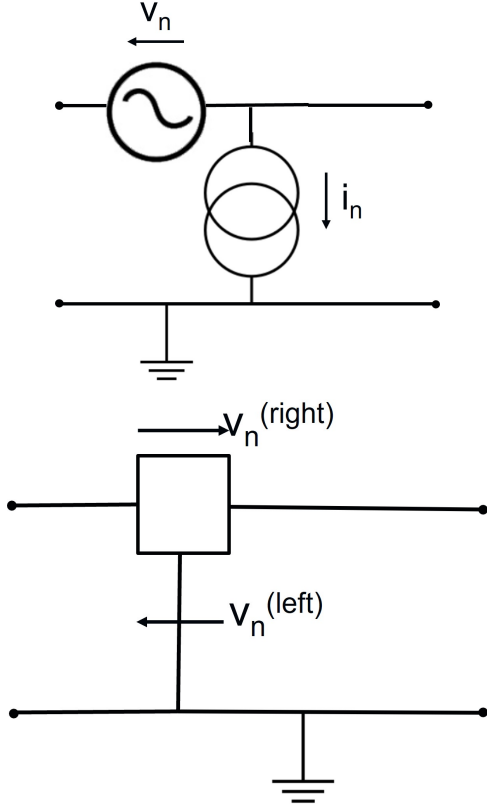


Fig. 6. Travelling Wave Sources.

and we may find the alternative representation

$$\begin{pmatrix} v_2 - \hat{v}_2 \\ i_2 - \hat{i}_2 \end{pmatrix} = \mathbf{T}_{21} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}, \quad (9)$$

where

$$\begin{pmatrix} \hat{v}_2 \\ \hat{i}_2 \end{pmatrix} = \begin{pmatrix} \bar{v}_2 \\ \bar{i}_2 \end{pmatrix} - \mathbf{T}_{21} \begin{pmatrix} \bar{v}_1 \\ \bar{i}_1 \end{pmatrix}. \quad (10)$$

Here both the voltage and current sources on side 1 have been moved to side 2. Since  $\mathbf{T}_{12} = (\mathbf{T}_{21})^{-1}$ , the opposite transformation is possible whenever  $\mathbf{T}_{12}$  is defined.

We begin by describing the plethora of equivalent representations of the homogeneous  $N$ -port before describing the inhomogeneous case and then the case where the inhomogeneous sources are described by a Gaussian stochastic process, otherwise known as Gaussian noise. While the  $\mathbf{Z}$ ,  $\mathbf{Y}$ , and  $\mathbf{S}$  matrix representations straightforwardly generalize to the  $N$  port case, the mixed representations, of which the 2-port transfer matrix described above is an example, generalize into a large number of choices, which we now describe in a perhaps less familiar, more mathematical manner, which facilitates transforming between the various representations, and moreover avoids having to deal separately with special singular cases.

Mathematically, a homogeneous  $N$ -port may be regarded as an  $N$ -dimensional linear subspace of the vector space  $\mathbb{C}^{2N}$  whose coordinates may be taken to be  $v_1, \dots, v_n, i_1, \dots, i_N$ . Said another way, there are  $N$  linearly independent linear equations, or constraints, reducing  $\mathbb{C}^{2N}$  to a subspace having

the structure of  $\mathbb{C}^N$ . The choice of these  $N$  equations, which may be represented as an  $N \times 2N$  rectangular matrix  $\mathbf{M}$ , is not unique: one could have just as well chosen another set of equivalent equations represented by  $\mathbf{M}' = \mathbf{N}\mathbf{M}$  where  $\mathbf{N}$  is any invertible  $N \times N$  square matrix.

Generally, rather than the above representation, we want to partition the  $2N$  variables into two sets such the values of the  $N$  variables of the first set are determined by the values of the  $N$  variables of the second set by means of a linear transformation. These transformations will serve as a tool to eliminate sources on certain ports by moving them to ports on the left-hand side.

We start by counting the number of distinct ways to partition  $2N$  variables into two sets containing  $N$  variables each where order does not matter. There are

$$\binom{2N}{N} = \frac{(2N)!}{(N!)^2} \quad (11)$$

ways to partition the  $2N$  variables into two sets of equal size, which we denote as  $L = (a_1, \dots, a_N)$  and  $R = (b_1, \dots, b_N)$ . The subspace  $S$  defined by the  $N$  linearly independent conditions may be expressed in block form, so that

$$\mathbf{M} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = (\mathbf{A} \ \mathbf{B}) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = 0 \quad (12)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are  $N \times N$  matrices and eqn. (12) becomes

$$\mathbf{A}\mathbf{a} + \mathbf{B}\mathbf{b} = 0. \quad (13)$$

When  $|\mathbf{A}| \neq 0$ ,

$$\mathbf{a} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{b}. \quad (14)$$

In this representation the  $\mathbf{b}$  components are considered the independent variables and the  $\mathbf{a}$  components the dependent variables. When  $|\mathbf{A}| = 0$ , a singular case arises and no such representation is possible.

We now generalize to the inhomogeneous case via the transformations

$$\begin{aligned} v_I &\rightarrow (v_I - \bar{v}_I), \\ i_I &\rightarrow (i_I - \bar{i}_I) \end{aligned} \quad (15)$$

where  $\bar{v}_I$  and  $\bar{i}_I$  are the voltage and current sources at port  $I$ , respectively.

With the above substitutions, eqn. (12) may be generalized to become

$$\mathbf{M} \begin{pmatrix} v_1 \\ \vdots \\ v_N \\ i_1 \\ \vdots \\ i_N \end{pmatrix} = \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix}, \quad (16)$$

and eqn. (14) becomes

$$\mathbf{a} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{b} + \mathbf{A}^{-1}\mathbf{s}, \quad (17)$$

or

$$(\mathbf{a} - \hat{\mathbf{a}}) = -\mathbf{A}^{-1}\mathbf{B}\mathbf{b}. \quad (18)$$

We see that all the sources associated with the  $\mathbf{b}$  components have been eliminated in favor of sources associated with only the  $\mathbf{a}$  components. Such a transformation is possible whenever  $\mathbf{A}$  is non-singular (i.e., an invertible  $N \times N$  matrix).

If we have  $N$  sources with amplitudes  $\alpha_1, \dots, \alpha_N$ , the Gaussian ensemble is described by the probability distribution

$$p(\alpha) = (2\pi)^{-N/2} \det^{-1/2}(\mathbf{C}) \exp \left[ -\frac{1}{2} \alpha^\dagger \mathbf{C}^{-1} \alpha \right] \quad (19)$$

where

$$C_{ij} = \langle \alpha_i^\dagger \alpha_j \rangle \quad (20)$$

is the correlation matrix, which is Hermitian and positive definite (i.e., all its eigenvalues are strictly positive). The matrix  $\mathbf{C}$  has  $N^2$  independent real components.

### III. MEASUREMENT

We shall assume that the homogeneous  $N$ -port parameters have already been characterized, for example using a VNA, and are known. The conceptually most straightforward way of characterizing the inhomogeneous parameters would be for example one of the following: (i) each of the outputs of the  $N$ -port could be shorted to ground with port  $I$  terminal connected to an ideal ammeter (of vanishing internal impedance), so that  $i_I$  at each port could be measured. Then from these measurements the correlation matrix  $C_{IJ}^i(\omega) = \langle i_I^\dagger(\omega) i_J(\omega) \rangle$  could be calculated. (ii) The same could be done with open circuit boundary conditions at each port, with an idealized infinite internal impedance voltmeter attached between each port and ground so that  $v_I(t)$  is monitored over a sufficiently long time, so that  $C_{IJ}^v(\omega) = \langle v_I^\dagger(\omega) v_J(\omega) \rangle$  can be evaluated. (iii) A hybrid method where some ports are shorted and others left open, with the current measured on the former and the voltage on the latter. In each of these cases, the inhomogeneous  $N$ -port can be cast into the representation eqn. (16), which can be manipulated to any of the equivalent representations described in the preceding sections. An alternative (iv) exists where each port is shunted to ground with a nonzero impedance  $Z_I$  and the voltage at each port is monitored.

The above methods involve either measuring  $N$  voltages and currents simultaneously, or equivalently making pairwise measurements between each pair of ports so that all the necessary cross-correlations can be measured. But in some instances, for example in characterizing the noise properties of an amplifier, this would require measuring minute voltages or currents at the input port. Much more accurate measurements can be obtained by placing different shunt impedances across the input ports and measuring the total power at the output port.

Let us for the moment assume that a voltages-only representation of the  $N$ -port is possible—in other words, that the  $\mathbf{Z}$ -matrix description of the port is non-singular. (When this condition is not satisfied, the argument given here may be straightforwardly modified.) Let us designate one of the ports, which we shall take to be the output port and label as port  $N$ . We shall deduce the noise properties, more specifically the correlation function  $C_{IJ}^{v^*v} = \langle v_I^* v_J \rangle$ , based on total power measurements at port  $N$  for a set of impedances at the ports  $Z_I^{(a)}$ ,  $[(a) = 1 \dots \mathcal{N}]$ . The data consists of the  $\mathcal{N}$  real quantities  $d^{(a)} = \langle v_N^* v_N \rangle (Z_1^{(a)}, \dots, Z_N^{(a)})$ . No cross correlations are measured directly. Rather these are inferred from the impedance dependence of the total power measured

at port  $N$ . The correlation matrix  $C_{IJ}^{v^*v}$  has  $N^2$  independent real parameters, so one would naively expect  $C_{IJ}^{v^*v}$  to be determined whenever  $\mathcal{N} \geq N^2$  assuming that none of the linear algebra is singular. Mathematically, port  $N$  could be any port, but as a practical matter we would want to choose the measurement port for maximal noise output power, in order to maximize the accuracy of the measurement and the minimize the requirements on the measurement apparatus. Once  $\langle v_I^* v_J \rangle$  has been measured, as was described above, the noise properties can be transformed to any other representation for which the transformation is non-singular.

### IV. CONCLUSION

We have explored the large number of equivalent representations of an inhomogeneous  $N$ -port and the transformations relating them. In a nutshell, when there are no obstructions arising from singularities in the linear algebra, it suffices to place  $N$  voltage or current sources at the output ports of the embedded homogeneous  $N$ -port so that each port has at most one voltage source and one current source. An alternative description in terms of travelling wave sources is also described. The most general Gaussian noise of a noisy  $N$ -port is described by an  $N \times N$  complex Hermitian correlation matrix for these complex sources, corresponding to  $N^2$  real degrees of freedom. When a number of such multi-ports are connected together to form a new port, it is possible by means of the transformations described in this paper to move the noise sources so that they are all situated on the external legs of the new port—that is, so that one has an embedded homogeneous port with noises source as described above. We also described how to measure this correlation matrix.

### ACKNOWLEDGMENTS

MB acknowledges a SKA-LOFAR travel grant from the Observatoire de Paris for a trip to Cambridge where part of this work was done. MB and DM thank Eloy de Lera Acedo, Christophe Craeye, and Dirk de Villiers for useful discussions and comments. DM also acknowledges Murray Edwards College of the University of Cambridge.

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