On the hypercomplex numbers and normed division algebra of all dimensions: A unified multiplication

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December 7, 2023

Abstract

Mathematics is the mother of all the sciences, engineering and technology, and a normed division algebra of all finite dimensions is the mathematical holy grail. In search of a real three-dimensional, normed, associative, division algebra, Hamilton discovered quaternions that form a non-commutative division algebra of quadruples. Later works showed that there are only four real division algebras with 1, 2, 4, and 8 dimensions. This work overcomes this limitation and introduces generalized hypercomplex numbers of all dimensions that are extensions of the traditional complex numbers. The space of these numbers forms non-distributive normed division algebra that is extendable to all finite dimensions. To obtain these extensions, we defined a unified multiplication, designated as scaling and rotative multiplication, fully compatible with the existing multiplication. Therefore, these numbers and the corresponding algebras reduce to distributive normed algebras for dimensions 1 and 2. Thus, this work presents a generalization of $\mathbb{C}$ in higher dimensions along with interesting insights into the geometry of the vectors in the corresponding spaces.

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Index Terms

[1] Generalized hypercomplex numbers; Normed division algebra; Real numbers; Complex numbers; Quaternions; octonions; Scaling and rotative multiplication (SRM); Non-distributive Field.

I. Introduction

The real numbers ($\mathbb{R}$) form the complete ordered field, wherein addition, subtraction, multiplication, and division are well defined. The imaginary numbers emerged in the quest of finding the solution of the polynomial equation $x^2 + 1 = 0$. The real and imaginary numbers form the complex numbers ($\mathbb{C}$) that are algebraically complete but not ordered. While $\mathbb{R}$ is also a vector space of dimension 'one' over itself

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[1]2020 Mathematics Subject Classification: 13A18, 13A99, 13F07, 13M05, 14A05, 14A20

(i.e., over the field of real numbers), $\mathbb{C}$ is a vector space of dimension 'two' defined over the field $\mathbb{R}$. In particular, $\mathbb{C}$ is an interesting space where one can deal with elements as complex or imaginary numbers (of the form of $a+ib$) or work with them as in abstract algebra, and at the same time can also visualize the elements in the 2-dimensional (2D) space as in traditional geometry with the notion of the length of the vectors, the distance between vectors, and the angle between vectors. This beautiful connection of complex numbers and 2D geometry inspired William Rowan Hamilton to look for a solution of a 3D algebra with a similarly associated 3D geometry. In modern mathematical language, Hamilton was trying for a 3D normed division algebra. In October 1843, Hamilton discovered quaternions ($\mathbb{H}$), and in a very famous act of scientific vandalism, he instantly carved the fundamental equations of quaternions into the stone of the Brougham Bridge as $i^2 = j^2 = k^2 = ijk = -1$. The quaternions are noncommutative because $ij = -ji$.

By now, it is well-established that a 3D normed division algebra does not exist. Frobenius [1] in 1878 obtained the classification of associative normed division algebras and proved that there are only three such algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Hurwitz in 1898 [2] proved that there are only four normed division algebras, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (octonions which are noncommutative and nonassociative, and also known as Cayley numbers) with a natural embedding as $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, where multiplication by a unit vector is distance-preserving. Likewise, Zorn in 1930 [3] had shown that if associativity condition is relaxed with alternativity, then there are only four normed division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The theorems of Adams (1958, 1960) [4], [5], Kervaire (1958) [6], and Bott–Milnor (1958) [7] reveal that the finite-dimensional normed division algebra can have only 1, 2, 4, or 8 dimensions.

The work of Hamilton was seminal because quaternions find applications in various areas such as astronautics, robotics, computer graphics, and animation. They are found to be useful in modern physics, particularly in the general theory of relativity, because they can express the Lorentz transform [8]. The quaternion calculus is useful in crystallography, the kinematics of rigid body motion, classical electromagnetism and quantum mechanics [8]. This is to note that Hamilton was searching for a real, normed, three-dimensional, associative, division algebra that did not exist. In order to equate the Euclidean length of the product of a pair of triples to the product of their lengths, he dropped the property of commutative multiplication and also added a fourth dimension defined by k. Hence, he moved to a 4D hypercomplex number system while trying to define a space of a 3D hypercomplex number system. Similarly, the space of octonions, i.e., the 8D hypercomplex number system, drops not only the property of commutativity but also the additional property of associativity in multiplication. For example, for quaternions (4D), a polynomial of degree n can have infinitely many roots, unlike the result of the fundamental theorem of algebra that guarantees that a polynomial of degree n with complex coefficients has precisely *n* complex roots (counting multiplicity) for 2D complex number system. Furthermore, only four real division algebras with 1, 2, 4, and 8 dimensions can exist, where this existing framework cannot be extended to the other finite dimensions [2], [3], [4], [5], [6], [7].

Intrigued by the above limitation and inspired by the works of Hamilton, where he thought of an entirely different out-of-the-box solution of those times, this work is an attempt to look for a different solution that can work for all finite dimensions. Similar to the works on quaternions and octonions, we have also dropped a property. In addition, we have defined a new multiplication operator. We have proposed a solution with non-distributive normed division algebra along with the definition of a new multiplication operation. The theory turns out to be interesting that is generalizable to all finite higher dimensions. In sum, this work makes the following significant contributions:

1) This work introduces generalized hypercomplex numbers of all dimensions ($\mathbb{S}^M$) that are extensions of the traditional complex numbers with a natural nesting as $\mathbb{S} \subset \mathbb{S}^2 \subset \mathbb{S}^3 \cdots \subset \mathbb{S}^{M-1} \subset \mathbb{S}^M$, where $\mathbb{S} = \mathbb{R}$, $\mathbb{S}^2 = \mathbb{C}$ and $M \in \mathbb{Z}^+$.
2) The space of the defined hypercomplex numbers forms *non-distributive normed division algebra* that holds applicability and generalizability to all finite higher dimensions for $M \geq 3$ and is distributive for $M = 1, 2$.
3) In order to be consistent with the traditional theory of the $\mathbb{R}$ and $\mathbb{C}$ spaces along with the geometry of the vectors in the corresponding spaces, we introduced a new multiplication operation called scaling and rotative (SR) multiplication that is a natural inhabitant of the Spherical Coordinate System (SCS). Unlike the traditional multiplication over the Cartesian space that appears to be derived from addition, the introduced SR multiplication is completely different from addition. As a consequence, it does not follow the distributive property leading to non-distributive normed division algebra.
4) Unlike the quaternions and octonions, these generalized hypercomplex number systems have finite numbers of roots for polynomials of degree n.
5) These hypercomplex numbers and the corresponding algebras reduce to distributive normed algebras for dimensions 1 and 2. This shows backward compatibility. In other words, the introduced concept appears to be a true generalization of $\mathbb{C}$ in higher dimensions.

II. Preliminaries

A field is a triplet $(F, +, \cdot)$ where F is a set, two binary operations on F called addition $(+)$ and multiplication $(\cdot)$ where binary operation on F is a mapping $F \times F \to F$ such that it satisfies the following field axioms for all $g_1, g_2, g_3 \in F$

1) Closure of addition and multiplication: $g_1 + g_2 \in F$ and $g_1 \cdot g_2 \in F$

2) Associativity of addition and multiplication: $g_1+(g_2+g_3) = (g_1+g_2)+g_3,$ and $g_1\cdot(g_2\cdot g_2) = (g_1\cdot g_2)\cdot g_3$
3) Commutativity of addition and multiplication: $g_1 + g_2 = g_2 + g_1$ and $g_1 \cdot g_2 = g_2 \cdot g_1$
4) Additive and multiplicative identities: for every $g \in F$, there exist two elements 0 and 1 in F such that $g + 0 = g$ and $g \cdot 1 = g$
5) Additive and multiplicative inverses: for every $g \in F$, $\exists\, g_{ai} \in F$, called the additive inverse of g, such that $g + g_{ai} = 0$; and for every nonzero $g \in F$, $\exists\, g^{-1}$ or $1/g$ in F, called the multiplicative inverse of g, such that $g \cdot g^{-1} = 1$
6) Distributivity of multiplication over addition:
   $g_1 \cdot (g_2 + g_3) = (g_1 \cdot g_2) + (g_1 \cdot g_3).$

If multiplication is not commutative in a field, it is known as the skew field. Moreover, if multiplication is not distributive over addition, we designate it as a non-distributive field (NDF).

The distributive property of multiplication over addition is a natural consequence of the historical development of multiplication as an operation derived from repeated addition. While multiplication was indeed historically derived from repeated addition, the reverse is not true as addition does not distribute over multiplication. The distributive property is a distinctive feature of multiplication and is not a property shared by addition in the same manner. The distributive property is a fundamental tool in algebra that simplifies expressions, aids in factoring, maintains consistency in operations, facilitates computation, and serves as a building block for further algebraic understanding and manipulation.

III. Proposed Generalized Hypercomplex Number System

In this section, first we define three dimensional (3D) hypercomplex numbers (denoted as the set $\mathbb{S}^3$) as a true extension of existing two dimensional complex numbers ($\mathbb{C}$), which we denote as $\mathbb{S}^2$, i.e., $\mathbb{C} = \mathbb{S}^2$.

A. *Proposed* 3D *Hypercomplex Number System*

We consider a 3D number from the set $\mathbb{S}^3$, in the Cartesian coordinate system (CCS), as

$$g = a + ib + jc, \tag{1}$$

such that the set $\{1,\, i,\, j\}$ is basis where i and j are two imaginary numbers, and $a, b, c \in \mathbb{R}$. This 3D number (1) can also be written using the triplet notation as: $g = \left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$. First, we write (1) in one of the standard spherical coordinate system (SCS) as

$$\begin{aligned} &a = r\cos(\phi)\cos(\theta), \quad b = r\cos(\phi)\sin(\theta), \quad c = r\sin(\phi), \\ &r = \sqrt{a^2 + b^2 + c^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right), \quad \phi = \tan^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right). \end{aligned} \tag{2}$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $\phi \in [-\pi/2, \pi/2]$. Moreover, we can add $2\pi n$ for $n \in \mathbb{Z}$ to both angles θ and ϕ without changing the point. Thus a spherical coordinate triplet (r, θ, ϕ) specifies a single point in three-dimensional space, which has infinitely many equivalent spherical coordinates. Therefore, to remove much of the non-uniqueness in the representation of a point, we fix $n = 0$. There are still (desired) non-uniqueness in the representations (i) if $r = 0$ then θ and ϕ can take any value, which means one can reach to the origin from any direction in 3D, like polar representation in 2D, (ii) if $\phi = \pi/2$ or $\phi = -\pi/2$, then for a fixed r, θ can assume any value, which means one can reach to the z-axis from any angle θ. Thus in these cases, we can change the the values of one or more of the other coordinates without moving the point.

Next, we redefine (2) as

$$\begin{aligned} &a = r \,\mathrm{cos\pi}(\phi)\cos(\theta), \quad b = r\,\mathrm{cos\pi}(\phi)\sin(\theta), \quad c = r\,\mathrm{sin\pi}(\phi), \\ &r = \sqrt{a^2+b^2+c^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \in [0, 2\pi), \quad \phi = \tan^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) \in [-\pi/2, \pi/2]. \end{aligned} \tag{3}$$

where π-periodic functions, as shown in Fig. 1, $\mathrm{cos\pi}(\phi) = \mathrm{cos\pi}(\phi + n\pi)$ and $\mathrm{sin\pi}(\phi) = \mathrm{sin\pi}(\phi + n\pi)$ for all $n \in \mathbb{Z}$, are defined as

$$\mathrm{cos\pi}(\phi) = |\cos(\phi)|, \tag{4}$$

$$\mathrm{sin\pi}(\phi) = \sum_{m=-\infty}^{\infty} \mathrm{sina}(\phi - m\pi), \tag{5}$$

$$\text{where } \mathrm{sina}(\phi) = \begin{cases} \sin(\phi), & \phi \in [-\pi/2, \pi/2], \\ 0, & \phi \notin [-\pi/2, \pi/2]. \end{cases} \tag{6}$$

Thus we can write (1) as

$$g = r\left(\mathrm{cos\pi}(\phi)\cos(\theta) + i\,\mathrm{cos\pi}(\phi)\sin(\theta) + j\,\mathrm{sin\pi}(\phi)\right), \tag{7}$$

where g is $g(r, \theta, \phi)$, θ is an azimuth angle, and ϕ is an elevation angle form XY plane as shown in Fig. 2. This is to note that the conventional notation of spherical coordinate system is not considered in this work[2].

We consider two special cases of (3) as follows:

1) Consider XY plane with $\phi = 0$ which implies

$$\begin{aligned} &a = r\cos(\theta), \quad b = r\sin(\theta), \quad c = 0, \\ &r = \sqrt{a^2+b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \in [0, 2\pi). \end{aligned} \tag{8}$$

[2]One can also use the conventional notation of spherical coordinate system where $a = r\cos(\theta)\sin(\phi)$, $b = r\sin(\theta)\sin(\phi)$, $c = r\cos(\phi)$, $r = \sqrt{a^2+b^2+c^2}$, $\theta = \tan^{-1}(b/a) \in [0, 2\pi)$, $\phi = \tan^{-1}(\sqrt{a^2+b^2}/c) \in [0, \pi]$, which is the angle from the z-axis. This will also lead to a 3D hypercomplex number system.

2) Consider XZ plane with (i) $\theta = 0$ which implies

$$a = r\cos\pi(\phi) \geq 0, \quad b = 0, \quad c = r\sin\pi(\phi), \tag{9}$$
$$r = \sqrt{a^2 + c^2}, \quad \phi = \tan^{-1}\left(\frac{c}{\sqrt{a^2}}\right) \in [-\pi/2, \pi/2],$$

and (ii) $\theta = \pi$ which implies

$$a = -r\cos\pi(\phi) \leq 0, \quad b = 0, \quad c = r\sin\pi(\phi), \tag{10}$$
$$r = \sqrt{a^2 + c^2}, \quad \phi = \tan^{-1}\left(\frac{c}{\sqrt{a^2}}\right) \in [-\pi/2, \pi/2].$$

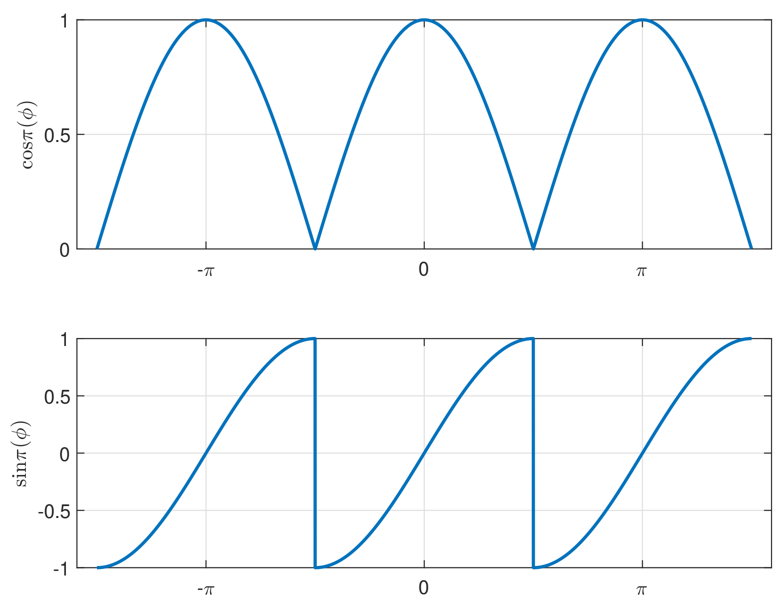

Fig. 1. The π-periodic functions $\cos\pi(\phi)$ (top) and $\sin\pi(\phi)$ (bottom) plotted for $[-3\pi/2, 3\pi/2]$.

To obtain the generalized multiplication of these numbers, we write (1) using SCS (2)–(7) in the triplet notations as

$$g = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}, \quad g_1 = \begin{bmatrix} r_1 \\ \theta_1 \\ \phi_1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix}, \quad \text{and} \quad g_3 = \begin{bmatrix} r_3 \\ \theta_3 \\ \phi_3 \end{bmatrix}. \tag{11}$$

Further, we define a new multiplication operation, named hereby the scaling and rotative (SR) multiplication (SRM), as

$$g_1 g_2 = \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix}, \quad g_1 g_3 = \begin{bmatrix} r_1 r_3 \\ \theta_1+\theta_3 \\ \phi_1+\phi_3 \end{bmatrix}, \quad g_2 g_3 = \begin{bmatrix} r_2 r_3 \\ \theta_2+\theta_3 \\ \phi_2+\phi_3 \end{bmatrix}, \tag{12}$$

and likewise, SR division (SRD) as

$$g_1/g_2 = \begin{bmatrix} r_1/r_2 \\ \theta_1-\theta_2 \\ \phi_1-\phi_2 \end{bmatrix}, g_1/g_3 = \begin{bmatrix} r_1/r_3 \\ \theta_1-\theta_3 \\ \phi_1-\phi_3 \end{bmatrix}, g_2/g_3 = \begin{bmatrix} r_2/r_3 \\ \theta_2-\theta_3 \\ \phi_2-\phi_3 \end{bmatrix}, \tag{13}$$

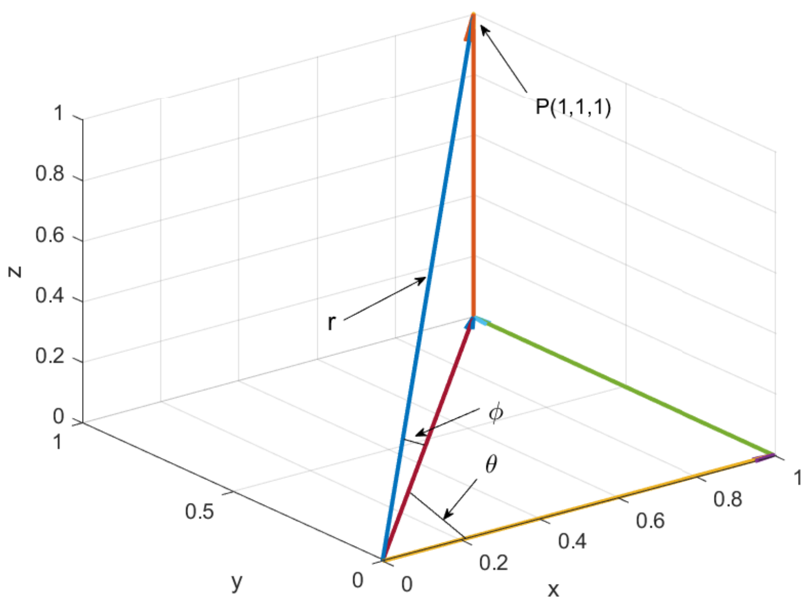

Fig. 2. A point $P = 1 + i + j$ in the considered spherical co-ordinate system, where radius ($r = \sqrt{3}$), azimuth angle ($\theta = \pi/4$ rad), and elevation angle ($\phi = \arctan\left(1/\sqrt{2}\right) = 0.615479709$ rad) are shown.

where we have assumed that $r_2, r_3 \neq 0$ in (13). The multiplication operation defined in (12) consists of scaling and rotation operations such that $\|g_1 g_2\| = \|g_1\|\|g_2\|$. The defined SRM operation reduces to the traditional multiplication when we move from 3D to 2D by considering $c = 0$ in (1).

The complex conjugate of (1) is defined as $\bar{g} = a - ib - jc$. This can be written in the triplet notation as

$$\bar{g} = \begin{bmatrix} r \\ -\theta \\ -\phi \end{bmatrix}, \tag{14}$$

such that $\|g\bar{g}\| = \|g\|^2 = r^2 \implies \|g\| = r$. The conjugation with respect to i and j can be defined as

$$\bar{g_i} = a - ib + jc = \begin{bmatrix} r \\ -\theta \\ \phi \end{bmatrix} \quad \text{and} \quad \bar{g_j} = a + ib - jc = \begin{bmatrix} r \\ \theta \\ -\phi \end{bmatrix}, \tag{15}$$

respectively. Similarly, the multiplicative inverse of (1) is defined as $g^{-1} = \frac{\bar{g}}{g\bar{g}} = \frac{a-ib-jc}{a^2+b^2+c^2}$ for every $g \neq 0$, which is same as inverse of a quaternion. This can be written in the triplet notation as

$$g^{-1} = \begin{bmatrix} 1/r \\ -\theta \\ -\phi \end{bmatrix}, \tag{16}$$

where this result can also be obtained from (13). Further, we can compute power of g^ℓ as

$$g^\ell = \begin{bmatrix} r^\ell \\ \ell\theta \\ \ell\phi \end{bmatrix}, \; \forall \ell \in \mathbb{Z}, \tag{17}$$

and (17) becomes (16) when $\ell = -1$. Addition of two complex numbers (e.g., $g_2 + g_3$) can be written as

$$g_2 + g_3 = (a_2 + a_3) + i(b_2 + b_3) + j(c_2 + c_3) = \begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} r_3 \\ \theta_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} r_{23} \\ \theta_{23} \\ \phi_{23} \end{bmatrix}, \tag{18}$$

and thus

$$g_1(g_2 + g_3) = \begin{bmatrix} r_1 \\ \theta_1 \\ \phi_1 \end{bmatrix} \begin{bmatrix} r_{23} \\ \theta_{23} \\ \phi_{23} \end{bmatrix} = \begin{bmatrix} r_1 r_{23} \\ \theta_1 + \theta_{23} \\ \phi_1 + \phi_{23} \end{bmatrix}. \tag{19}$$

We can also compute $q_1 g_2 + g_1 g_3$ as

$$q_1 g_2 + g_1 g_3 = \begin{bmatrix} r_1 r_2 \\ \theta_1 + \theta_2 \\ \phi_1 + \phi_2 \end{bmatrix} + \begin{bmatrix} r_1 r_3 \\ \theta_1 + \theta_3 \\ \phi_1 + \phi_3 \end{bmatrix} = \begin{bmatrix} r_{123} \\ \theta_{123} \\ \phi_{123} \end{bmatrix}. \tag{20}$$

From (19) and (20), it is easy to verify that, in general, $g_1(g_2+g_3) \neq g_1g_2+g_1g_3$. Moreover, $g_1(g_2+g_3) = g_1g_2 + g_1g_3$ if $\phi_1 = 0$ and $\theta_1 = 0$, i.e., $g_1 \in \mathbb{R}_+$ Since, we write $g = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in CCS, and $g = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$ in SCS, therefore $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \simeq \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$ where this equality is in the sense that both represent the same point, and like $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$, we can write $\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi \end{bmatrix}$. Thus (7) can be written as

$$r\left(\cos\pi(\phi)\cos(\theta) + i\cos\pi(\phi)\sin(\theta) + j\sin\pi(\phi)\right) = r(\cos(\theta) + i\sin(\theta))(\cos\pi(\phi) + j\sin\pi(\phi)). \tag{21}$$

Using the defined expressions (1)–(11), we can map CCS basis into SCS as:

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix};\ j = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix}, \tag{22}$$

where $\theta \in [0, 2\pi)$, and to make it unique we can consider a particular value say 0 or $\pi/2$. Further we can write

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};\ -1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix};\ 1/i = -i = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ -\pi/2 \\ 0 \end{bmatrix};\ 1/j = -j = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ -\theta \\ -\pi/2 \end{bmatrix}, \tag{23}$$

such that $1(-1) = -1$, $i(-i) = 1$, $j(-j) = 1$, and $0(g) = 0$. The additive identity 0 and multiplicative identity 1 are distinct and unique in both CCS and SCS. It is interesting to observe that, $i^2 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix} \implies i^2 = -1$. This behaves like 2D negative unit multiplication factor because for any $\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$, $\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ \theta + \pi \\ \phi \end{bmatrix} = -a - ib + jc$, i.e., it changes the angle θ by π rad and does not affect ϕ. Moreover, $j^2 = \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2\theta \\ 0 \end{bmatrix} = -1$ if $\theta = \pi/2$, and 1 if $\theta = 0$. Similarly, $ij = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi/2 + \theta \\ \pi/2 \end{bmatrix} \implies (ij)^2 = \begin{bmatrix} 1 \\ \pi + 2\theta \\ 0 \end{bmatrix} = -1$ for $\theta = 0$, and $(ij)^2 = 1$ for $\theta = \pi/2$.

Further, if $g_1 = \begin{bmatrix} 1 \\ 0 \\ \phi_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} 1 \\ 0 \\ \phi_2 \end{bmatrix}$, then $g_1 g_2 = \begin{bmatrix} 1 \\ 0 \\ \phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \phi_1 + \phi_2 \end{bmatrix}$, and, in general, $g_1^m g_2^n = \begin{bmatrix} 1 \\ 0 \\ m\phi_1 + n\phi_2 \end{bmatrix}$. In fact, the new imaginary number j has one degree of freedom because it can be written as $j = \begin{bmatrix} 1 \\ \theta \\ \pi/2 \end{bmatrix}$ for any $\theta \in [0, 2\pi)$ which implies $j^2 = \begin{bmatrix} 1 \\ 2\theta \\ 0 \end{bmatrix}$, and thus, it can have infinite number of representations. For examples (i) $j^2 = 1$ when $\theta = 0$, (ii) $j^2 = -1$ when $\theta = \pi/2$, (iii) $j^2 = i$ when $\theta = \pi/4$, (iv) $j^2 = -i$ when $\theta = 3\pi/4$, and (v) in general $j^2 = \cos(2\theta) + i\sin(2\theta)$ for $\theta \in [0, 2\pi)$.

Intrestingly, the multiplication of a complex number, e.g., $\cos(\theta)+i\sin(\theta)=\begin{bmatrix}1\\ \theta\\ 0\end{bmatrix}$ with specific $j=\begin{bmatrix}1\\ 0\\ \pi/2\end{bmatrix}$ is another representation of $j=\begin{bmatrix}1\\ \theta\\ \pi/2\end{bmatrix}$.

The additive inverse of an element $g=a+ib+jc$ is given by

$$g_{ai}=-a-ib-jc=\begin{bmatrix}r\\ \theta+\pi\\ -\phi\end{bmatrix}=\begin{bmatrix}1\\ \pi\\ 0\end{bmatrix}\begin{bmatrix}r\\ \theta\\ -\phi\end{bmatrix}, \tag{24}$$

and thus $g+g_{ai}=0$. It is important to note that $(-1)g\neq g_{ai}$ because multiplication by -1 changes the angle θ by π rad but does not affect ϕ, whereas we obtain g_{ai} from g by mapping $\theta\mapsto\theta+\pi$ and $\phi\mapsto-\phi$ as defined in (24), which is equivalent to multiplication by -1 and conjugating with respect to j as defined in (15), i.e., $g_{ai}=(-1)\bar{g}_j$. Therefore $g\,g_{ai}=\begin{bmatrix}r\\ \theta\\ \phi\end{bmatrix}\begin{bmatrix}r\\ \theta+\pi\\ -\phi\end{bmatrix}=\begin{bmatrix}r^2\\ 2\theta+\pi\\ 0\end{bmatrix}\neq(-1)g^2$; $(g_{ai})(g_{ai})=\begin{bmatrix}r\\ \theta+\pi\\ -\phi\end{bmatrix}\begin{bmatrix}r\\ \theta+\pi\\ -\phi\end{bmatrix}=\begin{bmatrix}r^2\\ 2\theta+2\pi\\ -2\phi\end{bmatrix}=\begin{bmatrix}r^2\\ 2\theta\\ -2\phi\end{bmatrix}=(g_{ai})^2\neq g^2$, and $(g_{ai})\bar{g}=\begin{bmatrix}r\\ \theta+\pi\\ -\phi\end{bmatrix}\begin{bmatrix}r\\ -\theta\\ -\phi\end{bmatrix}=\begin{bmatrix}r^2\\ \pi\\ -2\phi\end{bmatrix}$. One can notice that the angle θ is being wrapped in multiple of 2π and ϕ in multiple of π.

Because the above defined SR multiplication is not derived from addition, it does not follow the distributive property, and thus, in general, $g_1(g_2+g_3)\neq g_1g_2+g_1g_3$, likewise addition does not distribute over multiplication, i.e., $g_1+(g_2g_3)\neq(g_1+g_2)(g_1+g_3)$. This is a desired property of the defined SRM because, geometrically, the operation on the left side is different from that on the right side.

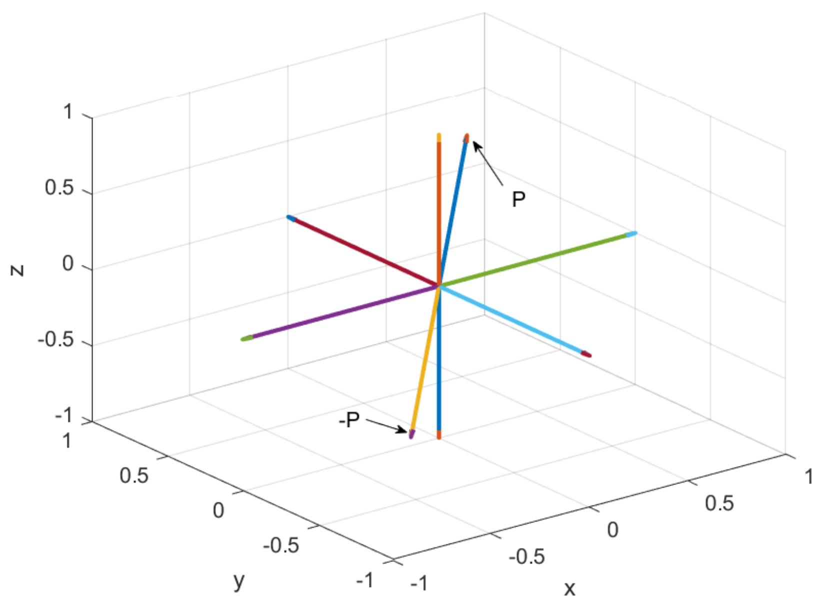

Fig. 3. A point $P=\frac{1}{2\sqrt{2}}(\sqrt{3}+i\sqrt{3}+j\sqrt{2})$ in the considered spherical co-ordinate system where radius $r=1$, azimuth angle $\theta=\pi/4$ and elevation angle $\phi=\pi/6$ are shown, and point $-P$ with $r=1$, $\theta=5\pi/4$ and elevation angle $\phi=7\pi/6=-\pi/6$ are also shown.

Remark 1. *As discussed, if $\phi=\pm\pi/2$, then θ is indeterminate, and thus j has an infinite representations, and that is why there is non-uniqueness in the representations. Therefore, in order to obtain a complete uniqueness, we restrict $\phi\in(-\pi/2,\pi/2)$ which excludes only two end points from $\phi\in[-\pi/2,\pi/2]$. This* *means in* (1) *if* $c \neq 0$, *then either* $a \neq 0$ *or* $b \neq$ *or both are not zero. Thus practically, we can always ensure uniqueness in the representation of numbers in CCS, such that*

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix}, \textit{ and } \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_{12} \\ b_{12} \\ c_{12} \end{bmatrix} = \begin{bmatrix} r\cos\pi(\phi)\cos(\theta) \\ r\cos\pi(\phi)\sin(\theta) \\ r\sin\pi(\phi) \end{bmatrix}, \tag{25}$$

where from (3) *we compute* $r = \sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}$, $\theta = \left(\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right)\right)$ *and* $\phi = \left(\tan^{-1}\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right) + \tan^{-1}\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right)$; $\cos\pi(\phi)$ *and* $\sin\pi(\phi)$ *are defined in* (4) *and* (5), *respectively.*

Result 1. *The distributive property of the defined SR multiplication over addition (i.e.,* $g_1(g_2+g_3) = g_1g_2 + g_1g_3$*) holds if* $g_1 \in \mathbb{R}_{\geq 0}$.

Remark 2. *By not considering the units, the traditional multiplication can be considered as a repetitive addition for rational numbers. In all practical applications, we always consider real numbers up to a finite precision only. Moreover, the rationals are dense in reals, so a real number is either a rational number or it can be approximated by a rational with an arbitrary precision. Thus, in practice, real numbers are used as rational numbers. Therefore, the traditional multiplication is derived from addition, which leads to the distributivity of multiplication over addition. Thus, one can observe that of the two binary operations (i.e.,* $+$ *and* $\cdot$*), one seems redundant for rational numbers.*

It is pertinent to note that the defined SR multiplication is backward compatible with the traditional (existing) multiplication for the complex number system. In fact, it is a generalization of the traditional multiplication to higher dimensional hypercomplex number systems. To demonstrate this, we present the following results.

Theorem 1. *A non-distributive normed division algebra (ND2A), as defined in* (1)–(24)*, is a number system where one can add, subtract, multiply and divide, and satisfy the norm* $\|g_1g_2\| = \|g_1\|\|g_2\|$. *Further, this algebra is of dimension* $M = 3$, *and becomes distributive when* $M \in [1, 2]$.

Proof. To prove the theorem, we have to prove that the 3D numbers given in (1)–(24) satisfy the axioms of *non-distributive field* for all $g_1, g_2, g_3 \in \mathbb{S}^3$:

1) Closure of addition and multiplication: $g_1 + g_2 \in \mathbb{S}^3$ and $g_1 \cdot g_2 \in \mathbb{S}^3$.
2) Associativity of addition and multiplication: $g_1 + (g_2 + g_3) = (g_1 + g_2) + g_3 = (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3) + j(c_1 + c_2 + c_3)$ and $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 = \begin{bmatrix} r_1r_2r_3 \\ \theta_1+\theta_2+\theta_3 \\ \phi_1+\phi_2+\phi_3 \end{bmatrix}$.
3) Commutativity of addition and multiplication: $g_1 + g_2 = g_2 + g_1 = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2)$ and $g_1 \cdot g_2 = g_2 \cdot g_1 = \begin{bmatrix} r_1r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix}$.

4) Additive and multiplicative identities: For every $g \in \mathbb{S}^3$, there exist two different elements 0 and 1 in $\mathbb{S}^3$, as defined in (22), such that $g + 0 = g$ and $g \cdot 1 = g$.
5) Additive and multiplicative inverses: For every $g \in \mathbb{S}^3$, $\exists\, g_{ai} \in \mathbb{S}^3$ called the additive inverse of g such that $g + g_{ai} = 0$ where $g = a + ib + jc$ and $g_{ai} = -a - ib - jc$; and for every nonzero element $g \in \mathbb{S}^3$, $\exists\, g^{-1}$ or $1/g$ in $\mathbb{S}^3$ called the multiplicative inverse of g such that $g \cdot g^{-1} = 1$, where $g^{-1} = \frac{\bar{g}}{g\bar{g}} = \frac{a-ib-jc}{a^2+b^2+c^2} = \begin{bmatrix} 1/r \\ -\theta \\ -\phi \end{bmatrix}$ as defined in (16), which is unique and also coincides with the quaternion inverse.
6) Distributivity of multiplication over addition: In general, this is not true because
$g_1 \cdot (g_2 + g_3) \neq (g_1 \cdot g_2) + (g_1 \cdot g_3)$
$\begin{bmatrix} r_1 \\ \theta_1 \\ \phi_1 \end{bmatrix} \cdot \left(\begin{bmatrix} r_2 \\ \theta_2 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} r_3 \\ \theta_3 \\ \phi_3 \end{bmatrix} \right) \neq \begin{bmatrix} r_1 r_2 \\ \theta_1+\theta_2 \\ \phi_1+\phi_2 \end{bmatrix} + \begin{bmatrix} r_1 r_3 \\ \theta_1+\theta_3 \\ \phi_1+\phi_3 \end{bmatrix}.$

Thus, 3D numbers given in (1)–(24) satisfy the axioms of non-distributive field. Thus we complete the proof. □

Remark 3. *We can observe that the proposed set* $(\mathbb{S}^3, +)$ *is an additive Abelian group, and* $(\mathbb{S}^3 \setminus \{0\}, \cdot)$ *is a multiplicative Abelian group. These sets satisfy the four fundamental group axioms of (i) closure, (ii) associativity, (iii) the identity property, and (iv) the inverse property, along with commutativity that is required only for an Abelian group.*

Proposition 1. *The following elementary consequences of the field axioms are also being satisfied by the proposed number systems,* $\forall\, g, g_1, g_2, g_3 \in \mathbb{S}^3$

1) $(-1)[(-1)g] = g$
2) $(g^{-1})^{-1} = g$
3) $g_1 + g_2 = g_1 + g_3 \implies g_2 = g_3$
4) $g\,0 = 0$
5) $[(-1)g_1]g_2 = (-1)(g_1 g_2)$
6) $[(-1)g_1][(-1)g_2] = g_1 g_2$
7) $g_1 g_2 = g_1 g_3$ *and* $g_1 \neq 0$ *implies* $g_2 = g_3$
8) $g_1 g_2 = 0 \implies g_1 = 0$ *or* $g_2 = 0$.

Result 2. *Now, we present two important results of using* $j^2 = \pm 1 \Leftrightarrow j^3 = \pm j$ *and* $j^2 = \pm 1 \implies j^4 = 1$ *as follows:*

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \cdots + \frac{(j\phi)^n}{n!} + \cdots \tag{26}$$

On using $j^2=-1$, $j^3=-j$ and $j^4=1$, one can easily obtain Euler identity as

$$e^{j\phi}=\left[1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}-\frac{\phi^6}{6!}+\cdots\right]+j\left[\frac{\phi}{1!}-\frac{\phi^3}{3!}+\frac{\phi^5}{5!}-\frac{\phi^7}{7!}+\cdots\right], \tag{27}$$

$$=\cos(\phi)+j\sin(\phi). \tag{28}$$

Interestingly, on using $j^2=1$, $j^3=j$ and $j^4=1$, we obtain hyperbolic Euler type identity as

$$e^{j\phi}=\left[1+\frac{\phi^2}{2!}+\frac{\phi^4}{4!}+\frac{\phi^6}{6!}+\cdots\right]+j\left[\frac{\phi}{1!}+\frac{\phi^3}{3!}+\frac{\phi^5}{5!}+\frac{\phi^7}{7!}+\cdots\right], \tag{29}$$

$$=\cosh(\phi)+j\sinh(\phi). \tag{30}$$

Example 1. *Now, we consider an interesting example that will be useful in obtaining next result as follows: Let $g_1=\left[\begin{smallmatrix}r_1\\ \theta\\ 0\end{smallmatrix}\right]$ and $g_2=\left[\begin{smallmatrix}r_2\\ 0\\ \phi\end{smallmatrix}\right]$, where $r_1,r_2>0$. Using new multiplication defined in* (12) *we obtain $g_1g_2=\left[\begin{smallmatrix}r_1\\ \theta\\ 0\end{smallmatrix}\right]\left[\begin{smallmatrix}r_1\\ 0\\ \phi\end{smallmatrix}\right]=\left[\begin{smallmatrix}r_1r_2\\ \theta\\ \phi\end{smallmatrix}\right]$. Using* (1) *to* (11)*, this can be written as $g_1g_2=r_1r_2[\,\cos\!\pi(\phi)\cos(\theta)+i\cos\!\pi(\phi)\sin(\theta)+j\sin\!\pi(\phi)]$, and thus,*

$$re^{i\theta}e^{j\phi}=r(\,\cos\!\pi(\phi)\cos(\theta)+i\cos\!\pi(\phi)\sin(\theta)+j\sin\!\pi(\phi)), \tag{31}$$

Moreover, $g_1^m=r_1^me^{im\theta}$ and $g_2^m=r_2^me^{jm\phi}$, thus the behaviour of these two complex numbers is very similar.

Result 3. *We observe that if $g=a+ib+jc$, then $e^g=e^ae^{ib}e^{jc}=\left[\begin{smallmatrix}e^a\\ 0\\ 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ b\\ 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ 0\\ c\end{smallmatrix}\right]=\left[\begin{smallmatrix}e^a\\ b\\ c\end{smallmatrix}\right]$. Thus, $\ln(e^g)=g \implies \ln\left(\left[\begin{smallmatrix}e^a\\ b\\ c\end{smallmatrix}\right]\right)=a+ib+jc$. Therefore, if we consider any $3D$ hypercomplex number $g=a+ib+jc \implies g=\left[\begin{smallmatrix}r\\ \theta\\ \phi\end{smallmatrix}\right]$, then $\ln(g)=\ln(r)+i\theta+j\phi \implies g=re^{i\theta}e^{j\phi}$, and it will reduce to the traditional $2D$ complex number system if $c=0$ and hence, $\phi=0$.*

Thus, the proposed $3D$ hypercomplex number system is a true generalization of the existing $2D$ complex number system. To obtain the multiplication of two numbers, we can use *Result 3* as follows: Let $\ln(g_1)=\ln(r_1)+i\theta_1+j\phi_1$ and $\ln(g_2)=\ln(r_2)+i\theta_2+j\phi_2$, and thus $\ln(g_1)+\ln(g_2)=\ln(g_1g_2)=\ln(r_1r_2)+i(\theta_1+\theta_2)+j(\phi_1+\phi_2) \implies g_1g_2=\left[\begin{smallmatrix}r_1r_2\\ \theta_1+\theta_2\\ \phi_1+\phi_2\end{smallmatrix}\right]$. Therefore, we conclude that the addition of hypercomplex numbers is naturally defined in the Cartesian coordinates and *multiplication is naturally defined in the spherical coordinates through the natural logarithmic addition.*

B. Examples of the Proposed 3D *Hypercomplex Numbers*

Example 2. *For example, let us consider the quadratic equation (QE) $x^2+1=0$. If $x\in\mathbb{R}$, then there are no real roots. If $x\in\mathbb{S}^2$ (traditionally, $x\in\mathbb{C}$), then there are two roots $x=\pm i$ where $i^2=-1$. If $x\in\mathbb{S}^3$, then there are four roots as $x^2=e^{i(\pi+2\pi k)}e^{j(\pi l)} \implies x=e^{i(\pi+2\pi k)/2}e^{j(\pi l)/2}$, for $k,l=0,1$. Therefore, four roots are $\left[\begin{smallmatrix}1\\ \pi/2\\ 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\ 3\pi/2\\ 0\end{smallmatrix}\right]$; $\left[\begin{smallmatrix}1\\ \pi/2\\ \pi/2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\ 3\pi/2\\ \pi/2\end{smallmatrix}\right]$ where last two roots represent the same point* *in z-axis. The other two roots are in the same z-axis as* $\begin{bmatrix} 1 \\ \pi/2 \\ -\pi/2 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 3\pi/2 \\ -\pi/2 \end{bmatrix}$ *which represent the same point in z-axis. So there are total 6 roots of unity in 3D complex number system, and out of 6, 4 are distinct roots.*

Example 3. *Here, we consider $x^3 - 1 = 0$, where $x \in \mathbb{S}^3$ and compute its roots as $x^3 = e^{i2\pi k} e^{j\pi l} \implies x = e^{i2\pi k/3} e^{j\pi l/3}$, where $k, l = 0, 1, 2$. Therefore, roots are* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2\pi/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4\pi/3 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ \pi/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2\pi/3 \\ \pi/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4\pi/3 \\ \pi/3 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 2\pi/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2\pi/3 \\ 2\pi/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4\pi/3 \\ 2\pi/3 \end{bmatrix}$ *where $2\pi/3$ is same as $2\pi/3 - \pi = -\pi/3$ for angle ϕ. So there are 9 distinct roots of unity in 3D complex number system.*

Example 4. *Here, we consider $x^4 - 1 = 0$, where $x \in \mathbb{S}^3$ and compute its roots as $x^4 = e^{i2\pi k} e^{j\pi l} \implies x = e^{i\pi k/2} e^{j\pi l/4}$, where $k, l = 0, 1, 2, 3$. Therefore, roots are* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/2 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi/2 \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/2 \\ \pi/4 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/2 \\ \pi/2 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi/2 \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/2 \\ 3\pi/4 \end{bmatrix}$ *where $3\pi/4$ is same as $3\pi/4 - \pi = -\pi/4$ for angle ϕ. Other four roots are* $\begin{bmatrix} 1 \\ 0 \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi/2 \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/2 \\ -\pi/2 \end{bmatrix}$. *So there are total 20 roots in 3D complex number system, and out of these, there are 14 distinct $4th$ roots of unity in 3D complex number system.*

Example 5. *Here, we consider $x^4 + 1 = 0$, where $x \in \mathbb{S}^3$ and compute its roots as $x^4 = e^{i(\pi + 2\pi k)} e^{j(\pi l)} \implies x = e^{i(\pi + 2\pi k)/4} e^{j(\pi l)/4}$, where $k, l = 0, 1, 2, 3$. Therefore, roots are* $\begin{bmatrix} 1 \\ \pi/4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5\pi/4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7\pi/4 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 \\ \pi/4 \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/4 \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5\pi/4 \\ \pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7\pi/4 \\ \pi/4 \end{bmatrix}$; $\begin{bmatrix} 1 \\ \pi/4 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/4 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5\pi/4 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 7\pi/4 \\ \pi/2 \end{bmatrix}$; $\begin{bmatrix} 1 \\ \pi/4 \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/4 \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5\pi/4 \\ 3\pi/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7\pi/4 \\ 3\pi/4 \end{bmatrix}$. *Other four roots are* $\begin{bmatrix} 1 \\ \pi/4 \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3\pi/4 \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5\pi/4 \\ -\pi/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 7\pi/4 \\ -\pi/2 \end{bmatrix}$, *so there are total 20 roots. There are some roots which represent the same point in z-axis, so there are 14 distinct roots in 3D complex number system in this case as well.*

We can conclude the above observation as follows:

Result 4. *The number of n-th roots of unity in $x \in \mathbb{S}^3$ are (i) n^2 if n is an odd, and (ii) $n^2 + n$ if n is an even number.*

Example 6. *In this example, we demonstrate that the SR multiplication does not distribute over addition in this generalized hypercomplex number system. Let us consider $g_1 = 1$, $g_2 = i$, $g_3 = j$, which can be written using (1)–(11) as*

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix} \implies g_1 + g_2 = \begin{bmatrix} \sqrt{2} \\ \pi/4 \\ 0 \end{bmatrix}, \quad g_1 + g_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ \pi/4 \end{bmatrix}, \quad g_2 + g_3 = \begin{bmatrix} \sqrt{2} \\ \pi/2 \\ \pi/4 \end{bmatrix}.$$

Therefore, using (12), one can compute

$$g_1 g_2 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}, \quad g_1 g_3 = \begin{bmatrix} 1 \\ \pi/2 \\ \pi/2 \end{bmatrix}, \quad g_2 g_3 = \begin{bmatrix} 1 \\ \pi \\ \pi/2 \end{bmatrix}, \quad g_3(g_1 + g_2) = \begin{bmatrix} \sqrt{2} \\ 3\pi/4 \\ \pi/2 \end{bmatrix}, \quad g_1 g_3 + g_2 g_3 = \begin{bmatrix} 2 \\ \pi/2 \\ \pi/2 \end{bmatrix},$$

and thus, $g_3(g_1 + g_2) \neq g_1 g_3 + g_2 g_3$.

C. Geometrical Insights into the Generalized Hypercomplex Number System

We note that algebraically, the additional imaginary axis j considered in $\mathbb{S}^{\mathbb{M}}$ behaves similar to i. For example, $i^2 = -1$ and $j^2 = -1$. Similarly, one can also show that $(1+j)^2 = 2j$ and $(1-j)^2 = -2j$. Similar identities are satisfied by i. Moreover, this j axis geometrically plays interestingly on the hypercomplex numbers. If there is a point $P = a + ib = \begin{bmatrix} r \\ \theta \\ 0 \end{bmatrix}$ in the complex XY plane and if it is multiplied by a unit norm complex number $\begin{bmatrix} 1 \\ \varphi \\ 0 \end{bmatrix}$, then that point will rotate counterclockwise by φ, i.e., new point $Q = re^{i(\theta+\varphi)} = \begin{bmatrix} r \\ \theta+\varphi \\ 0 \end{bmatrix}$. Similarly, if a 3D point $P = a + ib + jc \implies P = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$ is multiplied by a unit norm point $\begin{bmatrix} 1 \\ \varphi \\ \psi \end{bmatrix}$, then it will rotate to new point $Q = \begin{bmatrix} r \\ \theta+\varphi \\ \phi+\psi \end{bmatrix}$. Thus in the proposed 3D hypercomplex number system, one can rotate a point in both θ and ϕ directions with desired azimuth and elevation angles.

D. Generalized (MD) Hypercomplex Number System

The 3D hypercomplex number system can be easily generalized to the MD hypercomplex number $\mathbb{S}^M$ system by using the generalized MD spherical coordinate system. For example, 4D hypercomplex number system can be written, for all $\theta_1 \in [0, 2\pi)$ and $\theta_2, \theta_3 \in [-\pi/2, \pi/2]$, as

$$
\begin{aligned}
&d_0 = r\cos\pi(\theta_3)\ \cos\pi(\theta_2)\cos(\theta_1),\\
&d_1 = r\cos\pi(\theta_3)\ \cos\pi(\theta_2)\sin(\theta_1),\\
&d_2 = r\cos\pi(\theta_3)\sin\pi(\theta_2),\\
&d_3 = r\sin\pi(\theta_3),\\
&\theta_1 = \tan^{-1}\left(\frac{d_1}{d_0}\right), \quad \theta_2 = \tan^{-1}\left(\frac{d_2}{\sqrt{d_0^2 + d_1^2}}\right), \quad \theta_3 = \tan^{-1}\left(\frac{d_3}{\sqrt{d_0^2 + d_1^2 + d_2^2}}\right),\\
&r = \sqrt{d_0^2 + d_1^2 + d_2^2 + d_3^2}, \quad g = d_0 + j_1 d_1 + j_2 d_2 + j_3 d_3,
\end{aligned}
\tag{32}
$$

where j_2 has one degree of freedom (θ_1), and j_3 has two degree of freedom (θ_1 and θ_2), and in general, for all $\theta_1 \in [0, 2\pi)$ and $\theta_2, \theta_3, \cdots, \theta_{M-1} \in [-\pi/2, \pi/2]$,

$$\begin{aligned}
d_0 &= r\cos\pi(\theta_{M-1})\ \cos\pi(\theta_{M-2}) \cdots \cos\pi(\theta_2)\cos(\theta_1), \\
d_1 &= r\cos\pi(\theta_{M-1})\ \cos\pi(\theta_{M-2}) \cdots \cos\pi(\theta_2)\sin(\theta_1), \\
d_2 &= r\cos\pi(\theta_{M-1})\ \cos\pi(\theta_{M-2}) \cdots \cos\pi(\theta_3)\sin\pi(\theta_2), \\
&\vdots \\
d_{M-3} &= r\cos\pi(\theta_{M-1})\ \cos\pi(\theta_{M-2})\sin\pi(\theta_{M-3}), \\
d_{M-2} &= r\cos\pi(\theta_{M-1})\sin\pi(\theta_{M-2}), \\
d_{M-1} &= r\sin\pi(\theta_{M-1}),
\end{aligned} \tag{33}$$

$$\begin{aligned}
\theta_1 &= \tan^{-1}\left(\frac{d_1}{d_0}\right), \quad \theta_2 = \tan^{-1}\left(\frac{d_2}{\sqrt{d_0^2 + d_1^2}}\right), \cdots, \\
\theta_{M-1} &= \tan^{-1}\left(\frac{d_{M-1}}{\sqrt{d_0^2 + d_1^2 + \cdots + d_{M-2}^2}}\right), \\
r &= \sqrt{d_0^2 + d_1^2 + \cdots + d_{M-2}^2 + d_{M-1}^2},
\end{aligned} \tag{34}$$

and thus, we write MD hypercomplex number as

$$g = d_0 + j_1 d_1 + \cdots + j_{M-2} d_{M-2} + j_{M-1} d_{M-1}, \tag{35}$$

where $m = 1, 2, \cdots, M-1$. The M-tuple representations are

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, j_1 = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, j_2 = \begin{bmatrix} 1 \\ \theta_1 \\ \pi/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, j_3 = \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ \pi/2 \\ \vdots \\ 0 \end{bmatrix}, \cdots, j_{M-1} = \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \pi/2 \end{bmatrix}, -1 = \begin{bmatrix} 1 \\ \pi \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{36}$$

where j_{M-1} has $(M-2)$ degree of freedom for $M \geq 2$. To obtain the generalized multiplication of these numbers, we write (34) and (35) using SCS in M-tuple notations as

$$g_1 = \begin{bmatrix} r_1 \\ \theta_{1,1} \\ \theta_{2,1} \\ \vdots \\ \theta_{M-1,1} \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} r_2 \\ \theta_{1,2} \\ \theta_{2,2} \\ \vdots \\ \theta_{M-1,2} \end{bmatrix}. \tag{37}$$

We, hereby, define the scaling and rotative (SR) multiplication (SRM) as

$$g_1 g_2 = \begin{bmatrix} r_1 r_2 \\ \theta_{1,1}+\theta_{1,2} \\ \theta_{2,1}+\theta_{2,2} \\ \vdots \\ \theta_{M-1,1}+\theta_{M-1,2} \end{bmatrix}, \tag{38}$$

and division as

$$g_1/g_2 = \begin{bmatrix} r_1/r_2 \\ \theta_{1,1}-\theta_{1,2} \\ \theta_{2,1}-\theta_{2,2} \\ \vdots \\ \theta_{M-1,1}-\theta_{M-1,2} \end{bmatrix}, \tag{39}$$

where $r_2 > 0$. Similar to 3D, there is complete a uniqueness in the MD numbers representation if we restrict $\theta_2, \theta_3, \cdots, \theta_{M-1} \in (-\pi/2, \pi/2)$, which can be always achieved practically.

IV. Conclusion

The fundamental and most important contributions are the introduction of generalized hypercomplex numbers and the associated algebra in all finite dimensions. Interestingly, this framework reduces to the traditional theory of $\mathbb{R}$ and $\mathbb{C}$ spaces along with the geometry of the vectors in the corresponding spaces. In order to ensure this generalizability, an out-of-the-box solution is proposed with 1) non-distributive normed division algebra, and 2) a new multiplication operation defined in the spherical coordinate system, which is also backward compatible with the multiplication operation of numbers in $\mathbb{C}$. The proposed generalized hypercomplex numbers and the approach to derive it may open the floodgates for higher-dimensional algebra, which may find greater utility in various applications in the near future across science, engineering and technology.

Data accessibility statement

A MATLAB code for the proposed hypercomplex numbers and normed division algebra of all finite dimensions has been made publicly available to download and verify the various properties at http://dx.doi.org/10.13140/RG.2.2.30297.70242

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What was the motivation to explore hyper-complex numbers of all dimensions? In the vibrant town of Hamirpur (HP), where the streets whispered with mathematical secrets and the air buzzed with the electrifying energy of intellectual pursuits, Dr. Singh, a distinguished faculty member at NIT Hamirpur, found himself embarking on a riveting quest to unveil the mysteries of hyper-complex numbers across all dimensions. The journey started in October 2021 when Singh, not just a scholar but also a father eager to ignite the spark of curiosity in his daughter Prisha, a high school student, decided to plunge her into the enchanting realm of complex numbers. As he led her through the intricate journey from natural to real numbers to complex numbers, he explained that like $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, real numbers are also a subset of complex numbers ($\mathbb{R} \subset \mathbb{C}$). Prisha asked a deceptively simple yet profoundly existential question: "Are complex numbers a mere super-set of reals, or do they belong to a grander mathematical family? Where does this numerical odyssey come to an end? And, if it does, how and why?"

Captivated by his daughter's inquiry, Singh embarked on a mathematical odyssey to unlock the answers. He dove deep into the labyrinth of mathematical literature, only to discover that extending the realm of complex numbers was a complex puzzle itself. Undeterred, Singh rallied forces with the brilliant minds of A. Gupta from IIIT Delhi and SD Joshi from IIT Delhi. Together, they embarked on a journey through the hierarchies of hyper-complex numbers ($\mathbb{S} \subset \mathbb{S}^2 \subset \mathbb{S}^3 \subset \cdots \subset \mathbb{S}^M$), where $\mathbb{S} = \mathbb{R}$ and $\mathbb{S}^2 = \mathbb{C}$.

As the months rolled on, Prisha (along with her brother Siddhartha and mother Shraddha) transformed into an avid spectator of her father's intellectual escapade, peppering him with thought-provoking questions that fueled Singh's determination to pursue the enigma. The trio engaged in intense debates, weathering challenges and setbacks that peppered their path. Despite more than two years of unwavering effort, the elusive solution seemed to dance just out of reach, pushing Singh to the brink of abandoning the quest.

Then, on the pivotal day of November 15, 2023, in the throbbing heart of the ECE laboratory at the School of Engineering, JNU Delhi, India, Singh experienced a revelation. The solution to Prisha's seemingly innocuous question materialized before him like a mathematical apparition. The elusive hierarchy of hyper-complex numbers unfurled, revealing new dimensions in the very fabric of number theory.

The ripples of this groundbreaking discovery could reverberate through the hallowed halls of academia, etching a significant milestone in the annals of mathematical exploration. Singh had unearthed a mathematical treasure with the potential to reshape the future of number theory. The fundamental question, born on a lively November day, might ignite a revolution in the world of mathematics, proving that sometimes, the most profound answers lie concealed in the unassuming curiosity of a simple mind.

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