Direct Solution of the Stochastic Inverse Eigenvalue Problem for Complex-Valued Eigenspectra

Andre McDonald [1], Anton van Wyk [2], and Guanrong Chen [2]

[1]Defence and Security Cluster
[2]Affiliation not available

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André M. McDonald[a,b,*], Michaël A. van Wyk[b,c], Guanrong Chen[c]

[a] *Defence and Security Cluster, Council for Scientific and Industrial Research, Meiring Naude Rd., Brummeria 0184, Pretoria, South Africa*
[b] *School of Electrical and Information Engineering, University of the Witwatersrand, Braamfontein 2000, Johannesburg, South Africa*
[c] *Department of Electrical Engineering, City University of Hong Kong, Kowloon Tong, Hong Kong SAR, China*

Abstract

We present a direct solution to the problem of constructing a stochastic matrix with prescribed eigenspectrum, widely referred to as the stochastic inverse eigenvalue problem. The solution uses Markov state disaggregation to construct a Markov chain with stochastic transition matrix possessing the required eigenspectrum. Existing solutions that follow the same approach are limited to constructing matrices with real-valued eigenspectra only. The novel solution directly constructs matrices with complex-valued eigenspectra by applying a new disaggregation technique in tandem with a technique from a previous solution. Due to this generalization, the novel solution is able to successfully model physical systems from a larger family. Furthermore, the novel solution constructs the matrix in a finite and predetermined number of iterations, and without numerical approximation. The solution is demonstrated by deriving an expression for a set of 4×4 stochastic matrices sharing the same prescribed complex-valued eigenspectrum and indexed by a real parameter.

Keywords: Inverse eigenvalue problem, inverse spectrum problem, inverse stochastic problem, stochastic matrix, Markov state disaggregation
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1. Introduction

The inverse eigenvalue problem (IEP) requires the construction of a matrix with a prescribed eigenspectrum [1, 2]. Solutions to this problem have important applications in parameter estimation and the modeling of systems on the basis

*Corresponding author
Email addresses: andre.mcdonald1@gmail.com (André M. McDonald), anton.vanwyk@wits.ac.za (Michaël A. van Wyk), eegchen@cityu.edu.hk (Guanrong Chen)

of their observed or expected behavior. In particular, these solutions are utilized when a matrix is to be constructed as a model of a system from information regarding the system's frequency spectrum [3].

In practical applications, constraints regarding the feasibility of a physical system often dictate a certain structure for the matrix to be constructed [2]. Certain systems encountered in practice require the construction of *stochastic matrices* – i.e., matrices with nonnegative entries and with columns[1] that sum to unity. We refer to the IEP with the additional constraint of producing a stochastic matrix as the *stochastic IEP*, or *StIEP*. Solutions to this problem are used to construct Markov chain models when no information is available regarding the mechanism that drives transitions between states [4, 5], as well as models of ergodic dynamical systems with prescribed invariant density and spectral characteristics (this is a more general formulation of the inverse Frobenius-Perron problem [3, 6, 7, 8]).

The StIEP is nonlinear in the sense that the sum of two stochastic matrices is not stochastic. Furthermore, any nonnegative matrix with positive maximal eigenvalue and eigenvector can be mapped to a stochastic matrix using a scalar-weighted similarity transformation [9]. Since similar matrices share the same eigenspectrum, a generic solution to the IEP for *nonnegative matrices*, or the *nonnegative IEP* (*NIEP*), also qualifies as a solution to the StIEP [2].

Iterative methods aimed at solving the StIEP and NIEP numerically have appeared in the literature, with notable examples including [5, 10, 11, 12, 13, 14]. These methods rely on numerical convergence to a matrix with the required eigenspectrum. However, the StIEP is an ill-posed problem and the performance of numerical methods that rely on the minimization of a cost function and numerical convergence depends strongly on the distribution of the eigenvalues prescribed for the matrix [14]. Solutions that do not rely on such methods are of interest from both a practical and theoretical perspective. Ciampolini et al. [4] and Lin [15] proposed solutions to the StIEP that directly construct a stochastic matrix with prescribed real-valued eigenspectrum. Unlike numerical methods that minimize a cost function, the matrix is constructed after a finite and predetermined number of iterations.

The StIEP solution presented in [4] uses *Markov state disaggregation* to construct, over a number of rounds, a discrete-time and homogenous Markov chain with stochastic transition matrix possessing the prescribed real-valued eigenspectrum. During each round, the Markov chain gains an additional state and its transition matrix gains an additional row and column. By using the disaggregation technique specified in [4], one of the prescribed eigenvalues is *inserted* into the eigenspectrum of the transition matrix during each round, while simultaneously leaving its existing eigenvalues unaltered. However, the practical applicability of this solution is restricted by its inability to construct

[1] We follow the column-stochastic convention in our work. The column- and row-stochastic conventions are equivalent, and assuming any particular convention does not affect the integrity of the theory or results presented.

stochastic matrices with *complex-valued* eigenspectra. The same observation holds for the solution proposed by Lin [15].

In this paper we generalize the StIEP solution proposed by Ciampolini et al. in [4]. We propose a novel disaggregation technique that causes the transition matrix to gain *two* rows and columns during each round. This technique permits the insertion of a complex conjugate pair of eigenvalues from the set $\mathbb{C} \setminus \mathbb{R}$ into the eigenspectrum of the transition matrix during each round. By using this disaggregation technique in tandem with the technique specified in [4], a solution to the StIEP for *complex*-valued eigenspectra is realized.

The proposed solution to the StIEP has broader applicability than previous solutions limited to constructing matrices with real-valued eigenspectra, and may be used to successfully model physical systems from a larger family. In addition, the proposed solution constructs the stochastic matrix directly and in an explicit fashion. This property accommodates the construction of sets containing stochastic matrices that share the same eigenspectrum; we demonstrate this in section 5.1 by deriving an expression for a set of 4×4 stochastic matrices indexed by a real parameter. The solution also does not suffer from the slow (and in some cases, problematic) convergence of the numerical methods described earlier. Instead, a stochastic matrix is constructed after a finite and predetermined number of iterations, and without numerical approximation.

The remainder of this paper is set out as follows. A brief literature review is provided in section 2. In section 3, the Markov state disaggregation process is described, and both the original and novel disaggregation techniques are defined. Theorems regarding properties of the disaggregation techniques that are relevant towards solving the StIEP are stated in the same section. An algorithm for solving the StIEP using the two disaggregation techniques is presented in section 4. The algorithm is demonstrated by solving two examples of the StIEP in section 5. Several conclusions are drawn in section 6.

2. Literature review

The StIEP and NIEP are associated with work aimed at deriving necessary and sufficient conditions for a specified list of numbers to be realizable as the eigenvalues of a stochastic or nonnegative matrix. Theorems regarding necessary and sufficient conditions have appeared in the literature [16, 17], but the corresponding proofs are generally not constructive in nature [18]. Hence, these theorems cannot be used to solve the IEP in a direct manner. When used to solve the IEP, those theorems that were proved in a constructive manner are generally limited to producing matrices of small dimensions. The work of Suleimanova [19] and Perfect [20] are exceptions in this regard; however, the theory presented in these papers is restricted to real-valued eigenspectra.

An alternative approach to solving the StIEP involves the use of numerical methods to convergence to a matrix with the required eigenspectrum. One class of method aims to minimize a predefined cost function by using optimization methods to perform a descent on the underlying matrix manifolds (refer to [14]). The first method of this class was developed by Chu and Driessel [10] to solve

the NIEP for symmetric matrices. This method was later generalized by Chu and Guo [11] to solve the NIEP for matrices that are not necessarily symmetric. Whereas the authors reported promising numerical results, problems with regards to the methods' convergence were observed.

More recently, Zhao et al. [5] proposed a numerical method from the same class as [10] and [11]. This method solves the StIEP using a conjugate gradient descent algorithm, where minimization is carried out over the product manifold of the orthogonal matrices and matrices with rows on the unit sphere. Whereas numerical results presented by the authors demonstrate the method's effectiveness, the solution depends on an existence theorem for isospectral stochastic matrices that has not been proved thus far (as reported in [14]). Steidl and Winkler [14] extended the method proposed in [5], thereby addressing this shortcoming. Additional methods for solving the NIEP and StIEP and that belong to the same class appeared in [12] and [13], respectively.

Whereas considerable progress has been made regarding numerical methods for solving the StIEP, we observe that methods relying on the minimization of a cost function and numerical convergence depend strongly on the distribution of the eigenvalues prescribed for the matrix [14].

3. Markov state disaggregation

We refer to the process whereby one or more states of a discrete-time and homogenous Markov chain are disaggregated (or divided) into multiple new states as *Markov state disaggregation* [4]. This process generates a longer Markov chain with larger transition matrix. The probabilities of transitions to and from the new states of the Markov chain are selected in a systematic manner, thereby selecting certain characteristics of the larger transition matrix.

A stochastic matrix may be constructed by disaggregating the states of a Markov chain over several rounds, starting with the elementary single-state Markov chain. During each round, a particular *disaggregation technique* is applied to divide one of the Markov chain's states; we use the term disaggregation technique when referring to a particular method for dividing any *single* state of a Markov chain, including the manner in which the relevant transition probabilities are selected. After completing a number of rounds, the transition matrix of the Markov chain (i.e., a stochastic matrix) is produced as output.

Within the context of the StIEP, we are interested in the family of disaggregation techniques that cause the transition matrix of the Markov chain obtained after one round of disaggregation to (i) inherit the eigenvalues of the transition matrix belonging to the Markov chain at the start of the round, and (ii) gain one or more additional eigenvalues that may be prescribed by appropriately selecting the transition probabilities associated with the new states. These disaggregation techniques may be mathematically formulated as mappings between the transition matrices at the start and end of each round. These mappings have parameters that are selected to specify the eigenvalue(s) that are inserted into the eigenspectrum of the transition matrix. Disaggregation techniques from

this family are used to solve the StIEP by iteratively selecting and inserting the prescribed eigenvalues into the eigenspectrum of the initial transition matrix.

Ciampolini et al. [4] proposed a disaggregation technique that permits the insertion of a single real eigenvalue into the eigenspectrum of the transition matrix during each round. We present this technique in the following subsection. Thereafter, we present a novel disaggregation technique that permits the insertion of a complex conjugate pair of eigenvalues from the set $\mathbb{C} \setminus \mathbb{R}$ (i.e., the *strictly complex* eigenvalues) into the eigenspectrum of the transition matrix during each round. This novel technique complements the technique proposed in [4]; by using both disaggregation techniques in tandem, the StIEP may be solved for prescribed *complex*-valued eigenspectra[2].

As a point of departure for defining the disaggregation techniques, let the sequence $X_1, X_2, \ldots$ of random variables denote a discrete-time and homogenous Markov chain, referred to as the *original* Markov chain. Let this Markov chain have states $s_1, s_2, \ldots, s_N$ and transition probabilities defined by

$$p_{i,j} \triangleq \Pr(X_{t+1} = s_i | X_t = s_j). \tag{1}$$

We define the transition matrix of the original Markov chain as $\mathbf{P} \triangleq [p_{i,j}]_{i,j=1}^{N}$; this matrix may be expressed as

$$\mathbf{P} = \left(\begin{array}{c|c|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \mathbf{w}_{k,1}^T & p_{k,k} & \mathbf{w}_{k,2}^T \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array}\right). \tag{2}$$

Since the transition matrix has at least one eigenvalue of unity, there exists at least one *stationary probability vector* $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_N]^T$ satisfying $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$; we refer to the elements of any such vector as *stationary state probabilities*.

3.1. Original disaggregation technique

Ciampolini et al. [4] proposed a disaggregation technique that divides a particular state of a Markov chain into two states. We refer to this as the *original disaggregation technique* and to its application as *original disaggregation*. Suppose that original disaggregation of state s_k in the original Markov chain is to be performed, with the requirement of inserting a real eigenvalue λ into the eigenspectrum of the transition matrix. This produces a longer Markov chain with states $s'_1, s'_2, \ldots, s'_{N+1}$, where s'_k and s'_{k+1} denote the new states obtained by dividing state s_k, and

$$s'_i = \begin{cases} s_i, & \text{if } i < k, \\ s_{i-1}, & \text{if } i > k+1. \end{cases} \tag{3}$$

[2]This statement follows from the observation that the eigenvalues of any matrix with real elements are the roots of a characteristic polynomial in one variable and with *real* coefficients; thus, the strictly complex eigenvalues of these matrices appear in complex conjugate pairs.

Original disaggregation is performed by replacing element $p_{k,k} > 0$ of the original Markov chain's transition matrix in (2) by a matrix $\mathbf{T}$ and selecting the remaining transition matrix elements in a systematic manner, thus obtaining a larger transition matrix $\mathbf{P}'$. Let $\mathbb{P}_N$ denote the set of $N \times N$ stochastic matrices. The larger transition matrix is obtained by applying the map $\mathcal{T} : \mathbb{P}_N \to \mathbb{P}_{N+1}$ to the original transition matrix, such that

$$\mathbf{P}' = \mathcal{T}(\mathbf{P}, \lambda, \alpha, k) \tag{4}$$

and

$$\mathbf{P}' = \left(\begin{array}{c|cc|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \alpha\mathbf{w}_{k,1}^T & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{T}$}} & \alpha\mathbf{w}_{k,2}^T \\ (1-\alpha)\mathbf{w}_{k,1}^T & & & (1-\alpha)\mathbf{w}_{k,2}^T \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array}\right). \tag{5}$$

In the latter expression, $\mathbf{T} \triangleq T(p_{k,k}, \lambda, \alpha)$ is defined according to three real-valued scalar parameters. Parameter λ denotes the eigenvalue to be inserted into the eigenspectrum of the transition matrix and α is selected in order to choose the stationary probabilities of the new states obtained by disaggregating state s_k. The operator $T(p_{k,k}, \lambda, \alpha)$ is defined as

$$T(p_{k,k}, \lambda, \alpha) \triangleq \frac{p_{k,k}}{h_{\max}} \begin{pmatrix} \alpha & \alpha - h_{\min} \\ h_{\max} - \alpha & 1 - \alpha \end{pmatrix}, \tag{6}$$

where

$$h_{\max} \triangleq \frac{p_{k,k}}{p_{k,k} + \lambda} \tag{7}$$

and

$$h_{\min} \triangleq \frac{\lambda}{p_{k,k} + \lambda}. \tag{8}$$

To maintain a valid (i.e., stochastic) transition matrix, the selected parameters λ and α must satisfy

$$|\lambda| \leq p_{k,k} \tag{9}$$

and

$$\max\{h_{\min}, 0\} \leq \alpha \leq \min\{h_{\max}, 1\}. \tag{10}$$

The original disaggregation technique has the following properties, as proved[3] in [4]. We assume in each case that original disaggregation of state s_k is carried out in the original Markov chain, thereby producing the transition matrix $\mathbf{P}'$.

Theorem 3.1. *The transition matrix $\mathbf{P}'$ inherits the eigenvalues of $\mathbf{P}$, and gains the additional real eigenvalue λ.*

[3]Reference [4] follows the row-stochastic convention. By transposing the relevant matrices in each proof of [4], the corresponding proof is obtained for the column-stochastic convention followed in this paper.

Theorem 3.1 implies that original disaggregation successfully inserts the selected real eigenvalue λ into the eigenspectrum of the Markov chain's transition matrix during each round of disaggregation, while leaving the existing eigenvalues of the matrix unaltered.

Theorem 3.2. *The Markov chain obtained after one round of original disaggregation possesses a stationary probability vector* $\boldsymbol{\pi}' = [\pi'_1, \pi'_2, \ldots, \pi'_{N+1}]^T$ *that is associated with a corresponding stationary probability vector* $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_N]^T$ *of the original Markov chain. The elements of these vectors are related according to*

$$\pi'_i = \begin{cases} \pi_i, & \text{if } i < k, \\ \beta\pi_k, & \text{if } i = k, \\ (1-\beta)\pi_k, & \text{if } i = k+1, \\ \pi_{i-1}, & \text{if } i > k+1, \end{cases} \tag{11}$$

where

$$\beta \triangleq \frac{\alpha(1+\lambda) - \lambda}{1-\lambda}. \tag{12}$$

The property described in Theorem 3.2 is particularly useful when attention is restricted to irreducible and aperiodic Markov chains, which possess a unique stationary probability vector [21]. In this case, there exists some freedom to select the stationary probabilities of the new states obtained by disaggregating state s_k. This is achieved by selecting an appropriate value for α, thereby choosing β according to (12).

The original disaggregation technique only permits insertion of a *real* eigenvalue into the eigenspectrum of the Markov chain's transition matrix during each round. Hence, the practical applicability of solutions to the StIEP that rely solely on this disaggregation technique is limited. A complementary disaggregation technique that permits insertion of *strictly complex* eigenvalues is proposed next.

3.2. Novel disaggregation technique

The new disaggregation technique proposed in this subsection divides a particular state of a Markov chain into three states. We refer to this as the *novel disaggregation technique* and to its application as *novel disaggregation*. Suppose that novel disaggregation of state s_k in the original Markov chain is to be performed, with the requirement of inserting a pair of strictly complex eigenvalues λ and λ^* into the eigenspectrum of the transition matrix. This produces a longer Markov chain with states $s'_1, s'_2, \ldots, s'_{N+2}$, where s'_k, s'_{k+1} and s'_{k+2} denote the new states obtained by dividing state s_k, and

$$s'_i = \begin{cases} s_i, & \text{if } i < k, \\ s_{i-2}, & \text{if } i > k+2. \end{cases} \tag{13}$$

Novel disaggregation is performed by replacing element $p_{k,k} > 0$ of the original Markov chain's transition matrix in (2) by a matrix $\mathbf{U}$ and selecting the remaining transition matrix elements in a systematic manner, thus obtaining a larger

transition matrix $\mathbf{P}'$. The larger transition matrix is obtained by applying the map $\mathcal{U} : \mathbb{P}_N \rightarrow \mathbb{P}_{N+2}$ to the original transition matrix, such that

$$\mathbf{P}' = \mathcal{U}(\mathbf{P}, r, \theta, k) \tag{14}$$

and

$$\mathbf{P}' = \left(\begin{array}{c|ccc|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \mathbf{w}_{k,1}^T/3 & & & & \mathbf{w}_{k,2}^T/3 \\ \mathbf{w}_{k,1}^T/3 & & \mathbf{U} & & \mathbf{w}_{k,2}^T/3 \\ \mathbf{w}_{k,1}^T/3 & & & & \mathbf{w}_{k,2}^T/3 \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array} \right). \tag{15}$$

In the latter expression, $\mathbf{U} \triangleq U(p_{k,k}, r, \theta)$ is defined according to three real-valued scalar parameters, where $\lambda \triangleq re^{i\theta}$ denotes one of the eigenvalues of the complex conjugate pair to be inserted into the eigenspectrum of the transition matrix. The operator $U(p_{k,k}, r, \theta)$ was derived by generalizing the approach followed in [22], and is defined as

$$U(p_{k,k}, r, \theta) \triangleq \left(\begin{array}{ccc} \eta_1 & \eta_2 & \eta_3 \\ \eta_3 & \eta_1 & \eta_2 \\ \eta_2 & \eta_3 & \eta_1 \end{array} \right), \tag{16}$$

where

$$\eta_1 \triangleq \eta_1(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} + \frac{2r}{3}\cos\theta, \tag{17}$$

$$\eta_2 \triangleq \eta_2(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} - \frac{r}{3}(\cos\theta - \sqrt{3}\sin\theta), \tag{18}$$

$$\eta_3 \triangleq \eta_3(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} - \frac{r}{3}(\cos\theta + \sqrt{3}\sin\theta). \tag{19}$$

The region of permissible values for λ is determined by first observing that (15) is required to be a valid transition matrix (i.e., stochastic). Since $\eta_1+\eta_2+\eta_3 = p_{k,k}$ by definition, each column of $\mathbf{P}'$ sums to unity. Thus, to ensure a stochastic matrix $\mathbf{P}'$, the elements of $U(p_{k,k}, r, \theta)$ are required to be nonnegative. This holds if

$$\mathrm{Re}(\lambda) \geq -\frac{p_{k,k}}{2} \tag{20}$$

and

$$\frac{1}{\sqrt{3}}(\mathrm{Re}(\lambda) - p_{k,k}) \leq \mathrm{Im}(\lambda) \leq -\frac{1}{\sqrt{3}}(\mathrm{Re}(\lambda) - p_{k,k}). \tag{21}$$

Second, we require that λ must be strictly complex (i.e., not real-valued); this implies that

$$\theta \neq k\pi, \quad k \in \mathbb{Z}. \tag{22}$$

The preceding expressions define the region of permissible values for $\lambda = re^{i\theta}$ when applying the novel disaggregation technique. This region coincides with the shaded triangle in the complex plane of Fig. 1, with the real line excluded.

The following theorems describe two properties of the novel disaggregation technique. We assume in each case that novel disaggregation of state s_k is carried out in the original Markov chain, thereby producing the transition matrix $\mathbf{P}'$.

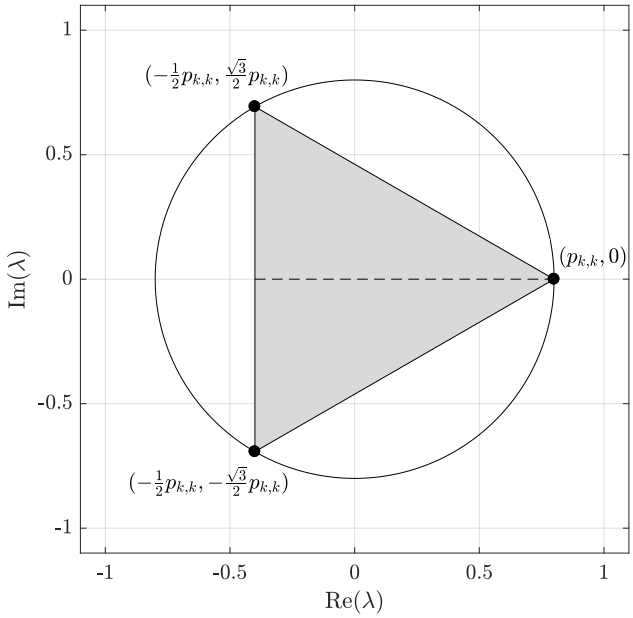

Figure 1: Complex plane with shaded region representing the permissible values for eigenvalue $\lambda = re^{i\theta}$ when applying the novel disaggregation technique.

Theorem 3.3. *The transition matrix* $\mathbf{P}'$ *inherits the eigenvalues of* $\mathbf{P}$*, and gains the complex conjugate pair of eigenvalues* $\lambda \triangleq re^{i\theta}$ *and* $\lambda^* \triangleq re^{-i\theta}$*.*

Proof. The theorem is proved in Appendix A. □

Theorem 3.3 implies that novel disaggregation successfully inserts a selected pair of strictly complex eigenvalues (i.e., a complex conjugate pair) into the eigenspectrum of the Markov chain's transition matrix during each round of disaggregation, while leaving the existing eigenvalues of the matrix unaltered. In particular, by selecting parameters r and θ, both $\lambda = re^{i\theta}$ and its complex conjugate $\lambda^* = re^{-i\theta}$ are inserted as eigenvalues (provided λ is within the region of permissible values illustrated in Fig. 1).

Theorem 3.4. *The Markov chain obtained after one round of novel disaggregation possesses a stationary probability vector* $\boldsymbol{\pi}' = [\pi'_1, \pi'_2, \ldots, \pi'_{N+2}]^T$ *that is associated with a corresponding stationary probability vector* $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_N]^T$ *of the original Markov chain. The elements of these vectors are related according to*

$$\pi'_i = \begin{cases} \pi_i, & \text{if } i < k, \\ \pi_k/3, & \text{if } i = k, k+1, k+2, \\ \pi_{i-2}, & \text{if } i > k+2. \end{cases} \tag{23}$$

Proof. The theorem is proved in Appendix B. □

Whereas original disaggregation affords a measure of freedom to select the stationary probabilities of the new states obtained by disaggregating state s_k, Theorem 3.4 implies that novel disaggregation affords no such freedom. The new states are always equiprobable.

4. Solution to the stochastic inverse eigenvalue problem

The disaggregation techniques defined in the previous section may be applied in tandem to solve the StIEP. Starting with the elementary single-state Markov chain[4], the techniques are used to insert the prescribed eigenvalues into the eigenspectrum of the Markov chain's transition matrix over several rounds. We proceed by developing an algorithm for solving the StIEP in this fashion.

There exist multiple degrees of freedom when applying the two disaggregation techniques to solve the StIEP. These are (i) the order in which the prescribed eigenvalues are inserted into the eigenspectrum of the transition matrix, (ii) the state selected for disaggregation during each round, and (iii) during those rounds where original disaggregation is performed, a measure of freedom to select the stationary probabilities of the two new states obtained through disaggregation. Simultaneously, there exist multiple constraints with regards to the parameter values that may be selected during each round of disaggregation. The parameters of the original disaggregation technique are constrained according to (9) and (10), and the parameters of the novel disaggregation technique are constrained according to (20), (21) and (22) (the region of permissible values is plotted in Fig. 1).

Equations (9), (20) and (21) imply that the disaggregation of a state with a larger self-transition probability during any round permits the insertion of a real eigenvalue with larger modulus or a pair of complex conjugate eigenvalues with larger modulus into the eigenspectrum of the transition matrix. This property, together with the observation that the self-transition probabilities of the Markov chain's states become smaller as disaggregation proceeds, motivates the following guideline: *Eigenvalues are to be inserted into the eigenspectrum of the Markov chain's transition matrix in nonincreasing order of their modulus when solving the StIEP.* The intention behind this guideline is to reduce the likelihood of a scenario where the prescribed eigenspectrum cannot be realized due to incompatible values selected for parameter α during earlier rounds where original disaggregation is performed (this parameter determines, in part, the new states' self-transition probabilities).

The guideline presented above does not determine which state to select for disaggregation during each round. Whereas one may select any state with self-transition probability large enough to accommodate the eigenvalue(s) to be inserted during each round, in the proposed algorithm we select the state with the *largest* self-transition probability for disaggregation. In addition, the guideline

[4]This Markov chain has a transition matrix $\mathbf{P} = [1]$ with single eigenvalue equal to unity. Also, any stochastic matrix has at least one unity eigenvalue due to it being stochastic.

does not determine how the stationary probabilities of the new states obtained during each round of original disaggregation are to be selected (i.e., the selection of a value for parameter α). Selecting a value of $\alpha = 1/2$ is permissible in any round, regardless of the eigenvalue λ selected for insertion. However, the values chosen for α during earlier rounds may affect whether certain of the prescribed eigenvalues can be successfully inserted into the eigenspectrum of the transition matrix during future rounds. If required, parameter α may be selected to preserve states with larger self-transition probabilities, thereby accommodating the insertion of eigenvalues with large modulus during subsequent rounds. Alternatively, α may be considered a free parameter and remain unspecified. By following this approach, an expression may be derived for a set of stochastic matrices that share the same eigenspectrum, provided that (10) is satisfied during each round. We illustrate this approach in the example of section 5.1.

We develop Algorithm 1 for solving the StIEP according to the above observations. Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ denote the eigenspectrum prescribed for the stochastic matrix; we assume that one of the eigenvalues equals unity, and that the prescription of a strictly complex eigenvalue implies the prescription of its complex conjugate. To simplify the presentation of the algorithm, let $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{\tilde{M}}$ denote the eigenvalues that remain after removing one of the unity eigenvalues from the prescribed eigenspectrum, and removing one of the strictly complex eigenvalues from each of the complex conjugate eigenvalue pairs prescribed. These remaining eigenvalues are indexed in nonincreasing order of their modulus; i.e., $i < j \implies |\tilde{\lambda}_i| \geq |\tilde{\lambda}_j|$. In addition, the notation $\mathbf{P}_{(m)}$ is used to denote the Markov chain's transition matrix at the end of round m, where $m \geq 0$.

5. Examples

We apply Algorithm 1 to solve two examples of the StIEP. Both examples prescribe complex-valued eigenspectra.

5.1. Example 1

We solve the StIEP of deriving an expression for a set of 4×4 stochastic matrices with eigenspectrum $\lambda_1 = 1$, $\lambda_2 = 0.6$, $\lambda_3 = 0.5e^{i2\pi/3}$ and $\lambda_4 = 0.5e^{-i2\pi/3}$. After removing the unity eigenvalue λ_1 and eigenvalue λ_4 from the prescribed complex conjugate pair (λ_3, λ_4), the eigenvalues $\tilde{\lambda_1} = 0.6$ and $\tilde{\lambda_2} = 0.5e^{i2\pi/3}$ remain.

The elementary single-state Markov chain, which possesses the transition matrix $\mathbf{P}_{(0)} = [1]$ with unity eigenvalue, is used as the starting point. Original disaggregation of state s_1 is performed during the first round to insert the real eigenvalue $\tilde{\lambda}_1 = 0.6$ into the eigenspectrum of the transition matrix. The parameter α is not specified during this round. The transition matrix $\mathbf{P}_{(1)} = \mathcal{T}(\mathbf{P}_{(0)}, \tilde{\lambda}_1, \alpha, k)$ is produced at the end of the first round, where $k = 1$,

$$\mathbf{P}_{(1)} = \frac{1}{10} \begin{pmatrix} 16\alpha & 16\alpha - 6 \\ 10 - 16\alpha & 16 - 16\alpha \end{pmatrix} \tag{24}$$

Algorithm 1 Solution of the StIEP using Markov state disaggregation.

Input: Remaining eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{\tilde{M}}$ ▷ Refer to section 4.
$\mathbf{P}_{(0)} \leftarrow [1]$; ▷ Transition matrix of single-state chain.
for $m \leftarrow 1, 2, \ldots, \tilde{M}$ **do** ▷ Perform $\tilde{M}$ disaggregation rounds.
    $k \leftarrow \arg\max(\mathrm{diag}(\mathbf{P}_{(m-1)}))$; ▷ Index of state to disaggregate.
    if $\tilde{\lambda}_m \in \mathbb{R}$ **then** ▷ Original disaggregation to be performed?
        if $|\tilde{\lambda}_m| > p_{k,k}$ **then**
            return; ▷ Eigenspectrum cannot be realized.
        end if
        Select value for α satisfying (10). ▷ Refer to section 4.
        $\mathbf{P}_{(m)} \leftarrow \mathcal{T}(\mathbf{P}_{(m-1)}, \tilde{\lambda}_m, \alpha, k)$; ▷ Apply original disaggregation.
    end if
    if $\tilde{\lambda}_m \in \mathbb{C} \setminus \mathbb{R}$ **then** ▷ Novel disaggregation to be performed?
        if $\lambda \triangleq \tilde{\lambda}_m$ does not satisfy (20) and (21) **then**
            return; ▷ Eigenspectrum cannot be realized.
        end if
        $r \leftarrow |\tilde{\lambda}_m|$;
        $\theta \leftarrow \arg(\tilde{\lambda}_m)$;
        $\mathbf{P}_{(m)} \leftarrow \mathcal{U}(\mathbf{P}_{(m-1)}, r, \theta, k)$; ▷ Apply novel disaggregation.
    end if
end for
return $\mathbf{P}_{(\tilde{M})}$ ▷ Column stochastic matrix with prescribed eigenspectrum.

and

$$\frac{6}{16} \leq \alpha \leq \frac{10}{16}. \tag{25}$$

During second round, the eigenvalue $\tilde{\lambda_2} = 0.5e^{i2\pi/3}$ and its complex conjugate are inserted into the eigenspectrum of the transition matrix using the novel disaggregation technique. State s_2' is selected[5] for disaggregation with parameters $r = 0.5$ and $\theta = 2\pi/3$. The transition matrix $\mathbf{P}_{(2)} = \mathcal{U}(\mathbf{P}_{(1)}, r, \theta, k)$ is produced at the end of the second round, where $k = 2$ and

$$\mathbf{P}_{(2)} = \frac{1}{30}\begin{pmatrix} 48\alpha & 48\alpha - 18 & 48\alpha - 18 & 48\alpha - 18 \\ 10 - 16\alpha & 11 - 16\alpha & 26 - 16\alpha & 11 - 16\alpha \\ 10 - 16\alpha & 11 - 16\alpha & 11 - 16\alpha & 26 - 16\alpha \\ 10 - 16\alpha & 26 - 16\alpha & 11 - 16\alpha & 11 - 16\alpha \end{pmatrix}. \tag{26}$$

Equations (25) and (26) define a set of stochastic matrices with the prescribed eigenspectrum.

5.2. Example 2

We solve the StIEP of constructing a 5×5 stochastic matrix with eigenspectrum $\lambda_1 = 1$, $\lambda_2 = 0.8$, $\lambda_3 = 0.3i$, $\lambda_4 = -0.3i$ and $\lambda_5 = 0.1$. After removing the unity eigenvalue λ_1 and eigenvalue λ_4 from the prescribed complex conjugate pair (λ_3, λ_4), the eigenvalues $\tilde{\lambda_1} = 0.8$, $\tilde{\lambda_2} = 0.3i$ and $\tilde{\lambda_3} = 0.1$ remain.

The elementary single-state Markov chain with transition matrix $\mathbf{P}_{(0)} = [1]$ is again used as the starting point. Original disaggregation of state s_1 is performed during the first round to insert the real eigenvalue $\tilde{\lambda}_1 = 0.8$ into the eigenspectrum of the transition matrix. The parameter $\alpha = 0.5$ is selected during this round. The transition matrix $\mathbf{P}_{(1)} = \mathcal{T}(\mathbf{P}_{(0)}, \tilde{\lambda}_1, \alpha, k)$ is produced at the end of the first round, where $k = 1$ and

$$\mathbf{P}_{(1)} = \frac{1}{10}\begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix}. \tag{27}$$

During the second round, the eigenvalue $\tilde{\lambda_2} = 0.3i$ and its complex conjugate are inserted into the eigenspectrum of the transition matrix using the novel disaggregation technique. State s_2' is selected for disaggregation with parameters $r = 0.3$ and $\theta = \pi/2$. The transition matrix $\mathbf{P}_{(2)} = \mathcal{U}(\mathbf{P}_{(1)}, r, \theta, k)$ is produced at the end of the second round, where $k = 2$ and

$$\mathbf{P}_{(2)} = \frac{1}{30}\begin{pmatrix} 27 & 3 & 3 & 3 \\ 1 & 9 & 9 + \sqrt{27} & 9 - \sqrt{27} \\ 1 & 9 - \sqrt{27} & 9 & 9 + \sqrt{27} \\ 1 & 9 + \sqrt{27} & 9 - \sqrt{27} & 9 \end{pmatrix}. \tag{28}$$

[5]Strict application of Algorithm 1 will result in state s_2' being selected for disaggregation only if $6/16 \leq \alpha < 8/16$; however, the final solution remains valid over the original range of parameter values specified in (25).

During the third round, the eigenvalue $\tilde{\lambda}_3 = 0.1$ is inserted into the eigenspectrum of the transition matrix using the original disaggregation technique. State s_1'' is selected for disaggregation with parameter $\alpha = 0.6$. The transition matrix $\mathbf{P}_{(3)} = \mathcal{T}(\mathbf{P}_{(2)}, \tilde{\lambda}_3, \alpha, k)$ is produced at the end of the third round, where $k = 1$ and

$$\mathbf{P}_{(3)} = \frac{1}{300} \begin{pmatrix} 180 & 150 & 18 & 18 & 18 \\ 90 & 120 & 12 & 12 & 12 \\ 10 & 10 & 90 & 90 + 10\sqrt{27} & 90 - 10\sqrt{27} \\ 10 & 10 & 90 - 10\sqrt{27} & 90 & 90 + 10\sqrt{27} \\ 10 & 10 & 90 + 10\sqrt{27} & 90 - 10\sqrt{27} & 90 \end{pmatrix}. \quad (29)$$

Matrix $\mathbf{P}_{(3)}$ is stochastic and possesses the prescribed eigenspectrum.

6. Conclusions

The novel disaggregation technique proposed in this paper generalizes the original technique of Ciampolini et al. [4] for inserting real eigenvalues into the eigenspectrum of a Markov chain's transition matrix. Specifically, it permits the insertion of complex conjugate pairs of eigenvalues from the set $\mathbb{C} \setminus \mathbb{R}$ into the eigenspectrum of the transition matrix while leaving its existing eigenvalues unaltered. An algorithm that uses these techniques in tandem to solve the StIEP for complex-valued eigenspectra was proposed. This algorithm was demonstrated by solving two examples of the StIEP that prescribe complex-valued eigenspectra.

By generalizing the StIEP solution of [4] to complex-valued eigenspectra, the solution proposed in this paper achieves broader applicability. The proposed solution may be used to successfully model physical systems from a larger family. An example of such an application is the modeling of ergodic one-dimensional dynamical systems with multimodal power spectra [3, 6, 7, 8]. In this application, the arguments of the eigenvalues belonging to the stochastic matrix coincide with the angular center frequencies (in radians per sample) of the system's spectral modes. Using the proposed solution, a system with one or more spectral modes having center frequencies in the interval $0 < |\omega_c| < \pi$ can be successfully modeled. In contrast, previous StIEP solutions limited to constructing matrices with real-valued eigenspectra can only be used to model those systems having all spectral modes restricted to center frequencies satisfying $\omega_c \in \{0, \pi\}$.

The proposed solution solves the StIEP directly and explicitly. This property accommodates the construction of sets containing stochastic matrices that share the same eigenspectrum. In section 5.1, this was demonstrated by deriving an expression for a set of 4×4 stochastic matrices sharing the same complex-valued eigenspectrum and indexed by a real parameter. Furthermore, the solution does not rely on numerical methods for minimizing a cost function. Due to the ill-posed nature of the StIEP, these numerical methods generally suffer from slow (and in some cases, problematic) convergence; refer to [14]. By constructing the stochastic matrix in a finite and predetermined number of iterations, and

without numerical approximation, the proposed method exhibits none of these deficiencies.

Appendix A. Proof of Theorem 3.3

We first prove the theorem for novel disaggregation of the elementary single-state Markov chain with transition matrix $\mathbf{P} = [1]$. Thereafter, we complete the proof by addressing novel disaggregation of the original Markov chain possessing an $N \times N$ transition matrix, where $N > 1$.

Appendix A.1. Elementary single-state Markov chain

Let $\mathbf{P'}$ denote the transition matrix obtained after novel disaggregation of the single-state Markov chain with parameters r and θ satisfying (20) to (22). We prove that $\mathbf{P'}$ inherits the unity eigenvalue λ_1 from the original transition matrix $\mathbf{P}$ and gains the additional eigenvalues $\lambda_2 = re^{i\theta}$ and $\lambda_3 = re^{-i\theta}$.

Suppose that disaggregation is carried out as stated above. Equations 14 to 16 imply that $\mathbf{P'} = U(1, r, \theta)$, where

$$\mathbf{P'} = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_3 & \eta_1 & \eta_2 \\ \eta_2 & \eta_3 & \eta_1 \end{pmatrix} \tag{A.1}$$

and $\eta_q \triangleq \eta_q(1, r, \theta)$ is defined for $q \in \{1, 2, 3\}$ by (17) to (19). The eigenvalues of $\mathbf{P'}$ are the roots of its characteristic polynomial $\det(\mathbf{P'} - \mu\mathbf{I})$. Let the second and third rows of $\mathbf{P'} - \mu\mathbf{I}$ both be added to its first row, thereby yielding $\mathbf{W}_1 - \mu\mathbf{I}$. Equations (17) to (19) imply that $\eta_1 + \eta_2 + \eta_3 = 1$; hence,

$$\mathbf{W}_1 = \begin{pmatrix} 1 & 1-\mu & 1-\mu \\ \eta_3 & \eta_1 & \eta_2 \\ \eta_2 & \eta_3 & \eta_1 \end{pmatrix}. \tag{A.2}$$

Next, let the first column of $\mathbf{W}_1 - \mu\mathbf{I}$ be subtracted from its second and third columns, thereby yielding $\mathbf{W}_2 - \mu\mathbf{I}$. It follows that

$$\mathbf{W}_2 = \begin{pmatrix} 1 & 0 & 0 \\ \eta_3 & \eta_1 - \eta_3 & \eta_2 - \eta_3 \\ \eta_2 & \eta_3 - \eta_2 & \eta_1 - \eta_2 \end{pmatrix}. \tag{A.3}$$

The determinant of $\mathbf{P'} - \mu\mathbf{I}$ remains unchanged if a scalar multiple of any matrix column (row) is added to any other column (row) [23]. This implies that

$$\begin{aligned} \det(\mathbf{P'} - \mu\mathbf{I}) &= \det(\mathbf{W}_2 - \mu\mathbf{I}) \\ &= (1-\mu)((\eta_1 - \eta_3 - \mu)(\eta_1 - \eta_2 - \mu) + (\eta_2 - \eta_3)^2). \end{aligned} \tag{A.4}$$

Substitution of (17) to (19) into the latter equation yields

$$\begin{aligned} \det(\mathbf{P'} - \mu\mathbf{I}) &= (1-\mu)(\mu^2 - 2r\mu\cos\theta + r^2) \\ &= -(\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3), \end{aligned} \tag{A.5}$$

where $\lambda_1 = 1$ and $\lambda_2 = \lambda_3^* = re^{i\theta}$.

Appendix A.2. Markov chain with $N \times N$ transition matrix, where $N > 1$

Let $\mathbf{P} = [p_{i,j}]_{i,j=1}^{N}$ denote the transition matrix of the original Markov chain prior to disaggregation. Without loss of generality[6], let state s_N be disaggregated using the novel technique with parameters r and θ satisfying (20) to (22). We prove that the transition matrix $\mathbf{P'}$ obtained after disaggregation inherits the eigenspectrum $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $\mathbf{P}$, and gains the additional eigenvalues $\lambda_{N+1} = re^{i\theta}$ and $\lambda_{N+2} = re^{-i\theta}$.

Equations 14 to 16 imply that $\mathbf{P'} = \mathcal{U}(\mathbf{P}, r, \theta, k)$, where

$$\mathbf{P'} = \left(\begin{array}{ccc|ccc} p_{1,1} & \cdots & p_{1,N-1} & p_{1,N} & p_{1,N} & p_{1,N} \\ p_{2,1} & \cdots & p_{2,N-1} & p_{2,N} & p_{2,N} & p_{2,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ p_{N-1,1} & \cdots & p_{N-1,N-1} & p_{N-1,N} & p_{N-1,N} & p_{N-1,N} \\ \hline p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_1 & \eta_2 & \eta_3 \\ p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_3 & \eta_1 & \eta_2 \\ p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_2 & \eta_3 & \eta_1 \end{array}\right) \tag{A.6}$$

and $\eta_q \triangleq \eta_q(p_{N,N}, r, \theta)$ is defined for $q \in \{1, 2, 3\}$ by (17) to (19). The eigenvalues of $\mathbf{P'}$ are the roots of its characteristic polynomial $\det(\mathbf{P'} - \mu\mathbf{I})$. Let rows $N+1$ and $N+2$ of $\mathbf{P'} - \mu\mathbf{I}$ both be added to row N of the same matrix, thereby yielding $\mathbf{W_1} - \mu\mathbf{I}$. Equations (17) to (19) imply that $\eta_1 + \eta_2 + \eta_3 = p_{N,N}$; hence,

$$\mathbf{W}_1 = \left(\begin{array}{ccc|ccc} p_{1,1} & \cdots & p_{1,N-1} & p_{1,N} & p_{1,N} & p_{1,N} \\ p_{2,1} & \cdots & p_{2,N-1} & p_{2,N} & p_{2,N} & p_{2,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ p_{N-1,1} & \cdots & p_{N-1,N-1} & p_{N-1,N} & p_{N-1,N} & p_{N-1,N} \\ \hline p_{N,1} & \cdots & p_{N,N-1} & p_{N,N} & p_{N,N} - \mu & p_{N,N} - \mu \\ p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_3 & \eta_1 & \eta_2 \\ p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_2 & \eta_3 & \eta_1 \end{array}\right). \tag{A.7}$$

Next, let column N of $\mathbf{W}_1 - \mu\mathbf{I}$ be subtracted from columns $N+1$ and $N+2$ of the same matrix, thereby yielding $\mathbf{W}_2 - \mu\mathbf{I}$. It follows that

$$\mathbf{W}_2 = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{M} & \mathbf{N} \end{pmatrix}, \tag{A.8}$$

$$\mathbf{M} = \begin{pmatrix} p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_3 \\ p_{N,1}/3 & \cdots & p_{N,N-1}/3 & \eta_2 \end{pmatrix} \tag{A.9}$$

and

$$\mathbf{N} = \begin{pmatrix} \eta_1 - \eta_3 & \eta_2 - \eta_3 \\ \eta_3 - \eta_2 & \eta_1 - \eta_2 \end{pmatrix}. \tag{A.10}$$

[6] We motivate this claim by observing that the numbering of the Markov chain's states are arbitrary, and that the columns above and below matrix $\mathbf{U}$ (as well as the rows to the left and right of matrix $\mathbf{U}$) in (15) are modified in an identical fashion during disaggregation.

The determinant of $\mathbf{P}' - \mu\mathbf{I}$ remains unchanged if a scalar multiple of any matrix column (row) is added to any other column (row). Furthermore, since the determinant of the block triangular matrix $\mathbf{W}_2 - \mu\mathbf{I}$ is the product of the determinants of its diagonal blocks [23], it follows that

$$\det(\mathbf{P}' - \mu\mathbf{I}) = \det(\mathbf{W}_2 - \mu\mathbf{I}) = \det(\mathbf{P} - \mu\mathbf{I})\det(\mathbf{N} - \mu\mathbf{I}). \tag{A.11}$$

This expression implies that $\mathbf{P}'$ inherits the eigenspectrum $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $\mathbf{P}$ and gains additional eigenvalues equal to the roots of $\det(\mathbf{N} - \mu\mathbf{I})$. Substitution of (17) to (19) into (A.10) yields

$$\begin{aligned} \det(\mathbf{N} - \mu\mathbf{I}) &= \mu^2 - 2r\mu\cos\theta + r^2 \\ &= (\mu - \lambda_{N+1})(\mu - \lambda_{N+2}), \end{aligned} \tag{A.12}$$

where $\lambda_{N+1} = \lambda_{N+2}^* = re^{i\theta}$.

Appendix B. Proof of Theorem 3.4

We first prove the theorem for novel disaggregation of the elementary single-state Markov chain with transition matrix $\mathbf{P} = [1]$. Thereafter, we complete the proof by addressing novel disaggregation of the original Markov chain possessing an $N \times N$ transition matrix, where $N > 1$.

Appendix B.1. Elementary single-state Markov chain

The single-state Markov chain possesses the stationary probability vector $\boldsymbol{\pi} = [1]$. We prove that novel disaggregation of this Markov chain, with parameters r and θ satisfying (20) to (22), produces a Markov chain with transition matrix $\mathbf{P}'$ and corresponding stationary probability vector $\boldsymbol{\pi}' = [1/3, 1/3, 1/3]^T$.

Suppose that disaggregation is carried out as stated above. Equations 14 and (15) imply that $\mathbf{P}' = U(1, r, \theta)$, where $\mathbf{P}'$ is given by (A.1) and $\eta_q \triangleq \eta_q(1, r, \theta)$ is defined for $q \in \{1, 2, 3\}$ by (17) to (19). By observing that $\eta_1 + \eta_2 + \eta_3 = 1$, it is directly verified that $\mathbf{P}'\boldsymbol{\pi}' = \boldsymbol{\pi}'$. Hence, $\boldsymbol{\pi}'$ is a stationary probability vector of the Markov chain obtained after disaggregation.

Appendix B.2. Markov chain with $N \times N$ transition matrix, where $N > 1$

Let $\mathbf{P} = [p_{i,j}]_{i,j=1}^N$ denote the transition matrix of the original Markov chain prior to disaggregation. This matrix may be expressed as

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,N} \\ \hline \mathbf{w}_{N,1}^T & p_{N,N} \end{array}\right). \tag{B.1}$$

Furthermore, let $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_N]^T$ denote any stationary probability vector of the original Markov chain, such that $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$. Without loss of generality[7],

[7] We motivate this claim using the same argument presented in Appendix A.2.

let state s_N be disaggregated using the novel technique with parameters r and θ satisfying (20) to (22), and let $\mathbf{P}'$ denote the transition matrix of the Markov chain obtained after disaggregation. Equation (15) implies that

$$\mathbf{P}' = \left(\begin{array}{c|ccc} \mathbf{P}_{1,1} & \mathbf{v}_{1,N} & \mathbf{v}_{1,N} & \mathbf{v}_{1,N} \\ \hline \mathbf{w}_{N,1}^T/3 & & & \\ \mathbf{w}_{N,1}^T/3 & & \mathbf{U} & \\ \mathbf{w}_{N,1}^T/3 & & & \end{array}\right), \tag{B.2}$$

where $\mathbf{U} \triangleq U(p_{N,N}, r, \theta)$ is given by (16).

We prove that

$$\boldsymbol{\pi}' \triangleq [\tilde{\boldsymbol{\pi}}^T, \pi_N/3, \pi_N/3, \pi_N/3]^T \tag{B.3}$$

is a stationary probability vector of the Markov chain obtained after disaggregation, where $\tilde{\boldsymbol{\pi}} \triangleq [\pi_1, \pi_2, \ldots, \pi_{N-1}]^T$. Specifically, we prove that $\mathbf{P}'\boldsymbol{\pi}' = \boldsymbol{\pi}'$ by showing that

$$[\mathbf{P}_{1,1}, \mathbf{v}_{1,N}, \mathbf{v}_{1,N}, \mathbf{v}_{1,N}]\boldsymbol{\pi}' = \tilde{\boldsymbol{\pi}} \tag{B.4}$$

and

$$[\mathbf{A}, \mathbf{U}]\boldsymbol{\pi}' = \frac{1}{3}[\pi_N, \pi_N, \pi_N]^T, \tag{B.5}$$

where $\mathbf{A} \triangleq [\mathbf{w}_{N,1}, \mathbf{w}_{N,1}, \mathbf{w}_{N,1}]^T/3$.

Since $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$, (B.1) implies that $[\mathbf{P}_{1,1}, \mathbf{v}_{1,N}]\boldsymbol{\pi} = \mathbf{P}_{1,1}\tilde{\boldsymbol{\pi}} + \pi_N \mathbf{v}_{1,N} = \tilde{\boldsymbol{\pi}}$. It follows that

$$\begin{aligned} [\mathbf{P}_{1,1}, \mathbf{v}_{1,N}, \mathbf{v}_{1,N}, \mathbf{v}_{1,N}]\boldsymbol{\pi}' &= \mathbf{P}_{1,1}\tilde{\boldsymbol{\pi}} + \mathbf{v}_{1,N}(\pi_N/3 + \pi_N/3 + \pi_N/3) \\ &= \tilde{\boldsymbol{\pi}}, \end{aligned} \tag{B.6}$$

thereby yielding (B.4).

Next, we find that $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$ and (B.1) imply

$$\mathbf{w}_{N,1}^T\tilde{\boldsymbol{\pi}} + p_{N,N}\pi_N = \pi_N, \tag{B.7}$$

or $\mathbf{w}_{N,1}^T\tilde{\boldsymbol{\pi}} = \pi_N(1 - p_{N,N})$. Furthermore, (17) to (19) imply that the rows of $\mathbf{U}$ sum to $p_{N,N}$. It follows that

$$\begin{aligned} [\mathbf{A}, \mathbf{U}]\boldsymbol{\pi}' &= \mathbf{A}\tilde{\boldsymbol{\pi}} + \frac{1}{3}\mathbf{U}[\pi_N, \pi_N, \pi_N]^T \\ &= \frac{1 - p_{N,N}}{3}[\pi_N, \pi_N, \pi_N]^T + \frac{p_{N,N}}{3}[\pi_N, \pi_N, \pi_N]^T \\ &= \frac{1}{3}[\pi_N, \pi_N, \pi_N]^T, \end{aligned} \tag{B.8}$$

thereby yielding (B.5).

Conflict of interest statement

There is no conflict of interest.

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References

[1] M. T. Chu, G. H. Golub, Inverse eigenvalue problems: Theory, algorithms and applications, 1st Edition, Oxford University Press, 2005.

[2] M. T. Chu, G. H. Golub, Structured inverse eigenvalue problems, Acta Numerica 11 (2002) 1–71. doi:10.1017/S0962492902000016.

[3] A. M. McDonald, M. A. van Wyk, G. Chen, The inverse Frobenius-Perron problem: A survey of solutions to the original problem formulation, AIMS Mathematics 6 (10) (2022) 11200–11232. doi:10.3934/math.2021650.

[4] L. Ciampolini, S. Meignen, O. Menut, T. David, Direct solution of the inverse stochastic problem through elementary Markov state disaggregation, HAL Ouvertes. hal-01016804 (2014) 1–12.

[5] Z. Zhao, X.-Q. Jin, Z.-J. Bai, A geometric nonlinear conjugate gradient method for stochastic inverse eigenvalue problems, SIAM Journal on Numerical Analysis 54 (4) (2016) 2015–2035. doi:10.1137/140992576.

[6] A. McDonald, M. van Wyk, Solution of the inverse Frobenius-Perron problem for semi-Markov chaotic maps via recursive Markov state disaggregation, in: Proceedings of the 25th IEEE European Signal Processing Conference, Kos, Greece, 2017, pp. 1654–1658. doi:10.23919/EUSIPCO.2017.8081480.

[7] A. McDonald, A. van Wyk, Construction of semi-Markov ergodic maps with selectable spectral characteristics via the solution of the inverse eigenvalue problem, in: Proceedings of the 2017 Asia Pacific Signal and Information Processing Assoiation. Annual Summit and Conference, Kuala Lumpur, Malaysia, 2017, pp. 987–993. doi:10.1109/APSIPA.2017.8282167.

[8] A. M. McDonald, M. A. van Wyk, A novel approach to solving the generalized inverse Frobenius-Perron problem, in: Proceedings of the 2020 IEEE International Symposium on Circuits and Systems, Online, 2020, pp. 1–5. doi:10.1109/ISCAS45731.2020.9181115.

[9] H. Minc, Nonnegative Matrices, 1st Edition, John Wiley & Sons, 1988.

[10] M. T. Chu, K. R. Driessel, Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations, SIAM Journal on Mathematical Analysis 22 (5) (1991) 1372–1387. doi:10.1137/0522088.

[11] M. T. Chu, Q. Guo, A numerical method for the inverse stochastic spectrum problem, SIAM Journal on Matrix Analysis and Applications 19 (4) (1998) 1027–1039. doi:10.1137/S0895479896292418.

[12] Z. Zhao, Z. J. Bai, X. Q. Jin, A Riemannian inexact Newton-CG method for constructing a nonnegative matrix with prescribed realizable spectrum, Numerische Mathematik 140 (4) (2018) 827–855. doi:10.1007/s00211-018-0982-2.

[13] T. T. Yao, Z. J. Bai, Z. Zhao, A Riemannian variant of the Fletcher–Reeves conjugate gradient method for stochastic inverse eigenvalue problems with partial eigendata, Numerical Linear Algebra with Applications 26 (2) (2018) 1–19. doi:10.1002/nla.2221.

[14] G. Steidl, M. Winkler, A new constrained optimization model for solving the nonsymmetric stochastic inverse eigenvalue problem, Linear and Multilinear Algebra (2020) 1–30doi:10.1080/03081087.2020.1841077.

[15] M. M. Lin, An algorithm for constructing nonnegative matrices with prescribed real eigenvalues, Applied Mathematics and Computation 256 (2015) 582–590. doi:10.1016/j.amc.2015.01.033.

[16] C. R. Johnson, C. Marijuan, P. Paparella, M. Pisonero, Operator theory, operator algebras, and matrix theory, Springer, 2018, Ch. The NIEP, pp. 199–220. doi:10.1007/978-3-319-72449-2_10.

[17] P. D. Egleston, T. D. Lenker, S. K. Narayan, The nonnegative inverse eigenvalue problem, Linear algebra and its applications 379 (2004) 475–490. doi:10.1016/j.laa.2003.10.019.

[18] R. Orsi, Numerical methods for solving inverse eigenvalue problems for nonnegative matrices, SIAM Journal on Matrix Analysis and Applications 28 (1) (2006) 190–212. doi:10.1137/050634529.

[19] H. R. Suleımanova, Stochastic matrices with real characteristic numbers, Soviet Math. Dokl. 66 (1949) 343–345.

[20] H. Perfect, Methods of constructing certain stochastic matrices, Duke Mathematical Journal 20 (3) (1953) 395–404. doi:10.1215/S0012-7094-53-02040-7.

[21] P. Brémaud, Markov chains: Gibbs fields, Monte Carlo simulation, and queues, 1st Edition, Springer Science and Business Media, New York, 1999.

[22] W. T. Bayha, The generalized rotation matrix and the eigenvalue problem, American Journal of Physics 52 (4) (1984) 370–371. doi:10.1119/1.13673.

[23] G. Strang, Linear algebra and its applications, 4th Edition, Cengage Learning, 2006.