Artificial General Intelligence, Noncomputability, and Dynamical Systems: A Critical Reexamination

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October 30, 2023

Abstract

Achieving genuine (human-level) artificial general intelligence (AGI) is one of the major goals of computer science, engineering, psychology, and mathematics. In this article, we critically reexamine the relation between natural intelligence and artificial intelligence at a fairly general theoretical level. After identifying four major structural themes in natural intelligence, we move to the issue of AGI implementation through physical computing machines. Motivated by Penrose's G¨ddelian argument refuting the thesis of AGI realizability via Turing machines, we formulate several theses on the noncomputable essence of AGI systems and suggest that infinitary noncomputability might constitute a viable path toward future AGI implementations, especially if coupled with nonlocality and a non-classical probabilistic structure such as the quantum case. A theoretical mathematical framework for non-Markovian stochastic dynamic systems is then presented and illustrated by describing multiagent AGI assemblages comprised of interconnected dynamic agents. We envision that such networked dynamical assemblages might be powered by noncomputable physics or arranged in an infinitary structure.

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***Index Terms*—Artificial General intelligence, natural intelligence, noncomputability, dynamical system theory.**

I. Introduction

Within the last ten years or so, a major shift in artificial intelligence (AI) research had taken place: bottom-up connectionism [1]–[3], most notably machine learning (ML) [4] and artificial neural networks (ANN) [5], have become the forerunners in the quest for building various core and multi-functional AI systems [6]. Initially at its inception moment in 1950 [7], it was believed that AI should be approached through a symbolic top-down strategy, i.e., a combination of cognitive psychology and some proper logical calculus were conjectured to be sufficient for constructing intelligent behaviour or generating suitable action using machines preprogrammed to perform well with varying environmental conditions [8]. But regardless to whether AI is seen as a strong/symbolic AI (top-down) or weak/connectionist AI (bottom-up), such terminological differences are now beginning to give away to a newer framework where ANNs and ML dominate low-level implementations of the system's core, while higher-level methods, e.g., pretraining, hyperparameter optimization, model selection, etc, are incorporated through some proper top-down-like strategy [9].[2] There is then a contemporary tendency to move toward increasingly data/environment-centered frameworks where ideas such as connectionism [5], emergent data-driven solutions, training, supervising, risk assessment, and reinforcement [4], [9], [10], all become the main key players, while trying to avoid paying too much attention to whether the core AI system is actually controlled in a strictly top-down manner or vice versa.

Motivated by this recent convergence of strong AI[3] and weak AI into a hybrid schema, namely learning-based-data-driven AI, an obvious question to pose at this stage is the following: How do traditional and current AI paradigms compare with *generic* intelligence? or to rephrase the same question in the old terminology, How does AI measure up against *natural intelligence*? The latter includes the subjects treated by cognitive psychology [11], philosophy of mind [12], logic [13], [14], mathematics [15], and linguistics [16], [17]. More specifically, we focus here on a recently proposed synthesis of all these domains known as *artificial general intelligence* (AGI) [18]–[21]. In the still relatively limited but growing recent literature on AGI, the latter is often defined as an AI system capable of "achieving human level intelligence."[4] Could AGI be achieved by mere intensification of one already known computing strategy or a combination of some of the currently used AI paradigms? It seems that AGI represents a singular form of AI research that may face some fundamental obstacles due to insurrmontable limitations inherent in the nature of mind, consciousness, and the laws of physics *as such*. There have been already some recent doubts about whether AGI is achievable using algorithms or standard computing machines [21]. Several fundamental investigations that started to appear from the late 1970s argued that Gödel's incompleteness theorems set some strict limits on the mechanization of thought, understanding, awareness, consciousnesses, though opinions vary widely regarding what it is exactly about natural intelligence that cannot be emulated via pure computational procedures [15], [22]–[24].

In our opinion, however, this subject was considered in a very comprehensive and definitive manner only by Roger Penrose, especially during the period 1989-1994, where he developed an ingenious Gödelian argument to show that no Turing machine or algorithm can in principle simulate the

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[2]In fact, unsupervised learning methods such as clustering and dimensionality reduction are hard to classify from the philosophical viewpoint: are they top-down or bottom-up? Clearly both.

[3]The historically earlier term *strong AI* is sometimes used in a manner similar to AGI. However, here strong AI means fully symbolic or top-down AI approach. The terms *strong AI* and *AGI* have different semantics in this article.

[4]In this article we prefer the term *natural intelligence* since its scope is broader and less restrictive. For example, the biological world already exhibits high degrees of intelligent behaviour that even the most advanced current AI chips cannot emulate. Nevertheless, most of the main features of natural intelligence to be reviewed in Sec. II are fully developed only in humans. So in a first approximation, it is still possible to use the terms *natural intelligence* and *human intelligence* interchangeably here.

self-awareness aspect of thought essential to the process of conscious understanding as experienced by human minds, especially in the field of mathematical productivity [25]–[29].

Our main objective in this article is to theoretically investigate potential limitations in nature that may set AGI apart from conventional AI. Our approach is inspired by and founded on some of Penrose's theories, especially his 1994 text [26]. However, in contrast to Penrose, we don't consider achieving self-consciousness or self-awareness the main objective of current and future researches into AGI. For us, AGI needs not attain self-consciousness for the simple reason that an AGI system is not necessarily an attempt to replace or copy the human mind.

This paper is structured as follows. In Sec. II, we outline some of the essential features of natural intelligence that an AGI system is projected to reproduce, at least partially. This is followed in Sec. III by our view on the specifics of that critical power a future AGI agent is expected to possess, namely the existence of a noncomputable level and the realizability of the intelligent function in terms of a nonlocal or memory dissipative dynamical process involving some sort of infinitary structure. Sec. IV then provides a high-level theoretical formalism for a possible realization of an AGI system comprised of an assemblage of interacting agents, each modeled as a non-Markovian dynamical system. Finally, we end up with conclusion.

II. Fundamental Structures of Natural Intelligence

A. The Abstract and the Concrete

In a memorable classic, Claude Levy-Strauss famously identified the working principles of the "primitive" mind as "the science of the *concrete*" [30]. Regardless to the social and cultural controversy surrounding this anthropological study itself, we note that an implicit criterion was inserted into that peculiar position, commonly upheld by Levy-Strauss and most of his generation, according to which one considers the defining trait of "non-primitve" minds to be nothing other than the latter's mastery of the category of the *abstract.* We have then a fundamental bifurcation of thinking into the abstract and the concrete, somehow mirroring the Spinozist ontological parallelism of extension and thought as modes of substance [31]. Natural minds can function in dual modes, moving from concrete settings to abstract ones, and vice versa. We grasp objects through a complex cognitive perceptual process that involves building higher-order representations of sense data [12], themselves arranged and structured in a hierarchical fashion, after which we move to an actualization or realization of top-down categorical directives and concepts by stuffing and filling up the "empty" abstract schema with various grades of bottom-up sensual and mental content [32]. Conversely, one may start with a purely concrete ("filled up container") then abstract away the content in order to arrive at the higher-order form [33]. Therefore, the dynamic duality between the abstract and the concrete is not far from the traditional dialectical structure comprised of form and content that so dominated Greek and western philosophy [34]–[36]. A genuine AGI capacity must be able to at least reproduce this quite uncanny non-algorithmic dynamic splitting of the process of objective cognition into either abstract or concrete modalities, plus the possibility of some quantum superposition comprising the two. The difficulty stems from the non-reductive nature of this division: Neither it is possible to add or combine the abstract and the concrete into each other, nor to effectively differentiate or separate them using a computational automatic criterion. The abstract and the concrete each inhabit its own distinct ontology, while the relation between the two is too subtle to be fully captured by existing sophisticated apparatuses such as the Russell's ontology of types [37] or the hierarchy of metalanguages [38].

B. Improvisation vs Algorithms

From another very fundamental perspective, algorithmic machinic action can never be able to muster the often wild but spontaneous ability of advanced organisms to respond to changes in their environments by producing totally new and unexpected behaviour. Indeed, ML algorithms are inherently unable to come up with fully novel patterns or behavioural activities that had not been already written into their internal states *a priori* (preferably through a data-driven learning process.[5] This implies that natural intelligence is ontologically irreducible to all known forms of AI, especially the ML and ANN paradigms where in the latter algorithmic determinism rules supreme. Even the incorporation of stochastic fluctuations or randomness in the design of AI algorithms does not considerably change this conclusion since a probabilistic machine continues to exhibit most of the essential ontological and sub-ontological features of a Turing-like computing machinery. On the other hand, a creative AGI agent, whose frame of reference should be natural intelligence, not AI, should be equipped with an intrinsic competence allowing it to produce novelty, experiment with new approaches, and engage with odd behavioural possibilities. No known AI core system has ever successfully demonstrated the ability to perform creative improvisatory actions. In fact, detailed documented investigations into this scenario (human-level AGI) are still consigned to art and literature, e.g., science fiction and futurological research. There is then the natural question about whether AI should or need consider improvisation as a basic long-term objective for reaching AGI through the pathway of computational intelligence, the only path open to AI practitioners so far. We take up this question in the next section.

C. Nonlocality vs Locality

An AI "algorithm" may be pictured as "improvising" if a "novelty cost function" can be assigned to the computing

[5]For example, consider the formalization of this intuitive insight given in terms of no-free-lunch theorems in machine intelligence [4]. For binary ML under generic but reasonable conditions, it can be proved that there exists no universally good learning machine. In other words, every ML-based AI system has to be specific (very good at solving specific problems.) For that reason, it is hard to see how genuine AGI agents can be constructed using existing methods. Sec. III provides further discussion of this point but see also [21] for another perspective.

machinery, in addition to the other risk, loss, cost, etc, performance measures to be mutually integrated into the core system operation [4]. However, "novelty" should never be evaluated or assessed immanently, i.e., through means intrinsic to the AI agent itself and its surrounding environing field. Instead, accounting for creativity, whether in cognitive psychology, artistic production, or in a potential AI setting, will fundamentally involve *nonlocal* processes, whereby potentials and possibilities, objectives and teleologies, have been forecast and foreseen in advance; for otherwise it is ontologically impossible to determine whether what is being produced is a creative action rather than random or pretrained actions. Securing the existence of nonlocal predictive processes is inherently beyond the means of classical computing machines, which are often based on classical (Kolmogorov) probability and classical physics [39], [40]. Nonlocality in intelligent agent networks requires the establishment of regimes of long-spatiotemporal correlations allowing distinct sub-components of the system to exchange information among each other [41]–[44] and, more importantly, to evaluate and assess each other [45], [46], leading then to at least the possibility of creating a novel behaviour that surpasses the capacities of each sub-component when taken on its own [47]. Thus, while nonlocality and novelty are ontologically distinct concepts, they are closely related with each other [48].

Of course one may also argue that neuroscience is still operating with classical probability and classical physics. Indeed, this field continues to actively promote the idea that cognition and intelligence will be eventually accounted for using the computational paradigm of neural circuits trained through past experiences [49]–[51]. More interestingly, such classical neuroscience already admits the fundamental importance played by nonlocality in understanding and explicating brain and cognitive structures [52]. Nevertheless, it aspires to do so while using mainly classical probability and classical field theories, possibly with the inclusion of stochastic elements [51], [53]. But it is questionable whether non-classical structures can be kept outside the cognitive neuro-scientific framework, at least not for a long time. Several nonlocal approaches, including quantum formalisms, have been already advanced to explain the action of the brain and its relation to memory, cognition, consciousness, e.g., see [28], [52], [54]–[58]. While these scattered investigations (and several others not cited here) have not been fully successful in unequivocally changing the direction of brain science research *per se*, one may say that at least when viewed collectively they have opened a traceable pathway leading toward understanding or even constructing natural intelligence systems, hence building possible AGI in the future.

D. Classical vs Quantum Probability

We have seen then that injecting an element of improvisation into an AGI system requires introducing nonlocality, despite the fact that a nonlocal action is hard to account for using only classical physics. This is why quantum effects in neuroscience had been proposed [59]. Yet the most direct and straightforward path toward setting up nonlocal brain correlations would be through the hypothesis of macroscopic quantum coherence extending along macroscopic scales [26], [54], [60]. But it should be noted that strictly speaking nonlocality *can* exist as a classical phenomenon [61], e.g., in nanoscale problems [62], [63], via strong near-field coupling [64] or, even more fundamentally, as an expression of the existence of a superspace structure transcending classical spacetime though indexed by the latter [44]. Nevertheless, it is still true that the most prominent source of nonlocality in nature is quantum processes, whether at the microscopic structure of matter-field interactions [43], [65], long-distance correlation via entanglement [66], or quantum memory effects [54], [60]. For that reason, a very promising alternative to classical computational AI might be in moving directly toward using non-classical computational procedures borrowed from quantum physics in order to realize cognitive and natural intelligent functions such as memory and decision making [54], [57], [67]. In a more ontologically relaxed approach, a "quantum-like" information processing paradigm was proposed in which quantum theory is used as a tool to perform AI-like computations without committing to a view on whether the underlying physico-chemical structure of the brain is quantum or classical [68]. In some of these approaches, the very structure of classical probability, so fundamental in mainstream AI and ML methods, is to be questioned. For instance, we now know that the formula of total probability (FTP) does not hold in quantum superposition regimes [69]. Since classical probability models, essentially the Kolmogorovian formalism, are all based on FTP, this has led to the suggestion that the very popular measure-theoretic probability theory [70] due to Kolmogorov is not the only possible probability theory that may be deployed in applications to physics, social science, and computing [58].

The collapse of the FTP in computing paradigms based on the alternative worldview offered by quantum probability is particularly worrying for conventional AI. Moreover, this is also relevant to the potential of a successful realization of AGI agents in the future. Indeed, in a quantum framework, the following changes to the classical AI and ML paradigms are expected:

1) The Boolean structure of classical logic, already integrated into the σ-algebraic set-theoretic structure of Kolmogorovian probability [70], should be revised in order to account for non-classical logical operations such as those entailed by quantum superposition and entanglement [71]. This has strong implications for top-down AI in particular since the latter is based on first-order logic [72].
2) A multi-functional AGI system is expected to be exhaustive and complete in the sense that it can produce all required actions when facing a generic situation. However, the total probability law's expression in terms of the FTP is the only mathematical framework available to classical AI in order to formally exhaust all possible events in a given probability space. Therefore, a non-classical AGI, e.g., quantum AGI or an artificial brain, may need to be designed with a completely different set of rules and options to account for the strictly

non-classical interference effects due, for example, to quantum superposition and long-distance correlations induced by quantum entanglement.

3) Non-Kolmogorovian probability models [58] might be incorporated into AI and AGI in order to explore – or even possibly create artificial – scenarios of universes of actions where classical probability rules such as the FTP are not available. These need not necessarily be quantum probability theories. There might be paradigms not based on the set-theoretic universe of measure theory, e.g., category theory [73], frequentist theories [74], contexualist formalisms [58], and others.
4) Closely related to the FTP is the Bayesian rule, which is not expected to survive unchanged in non-classical AI such as quantum AI and AGI. However, context-based paradigms of probabilistic thinking can be developed to expand the reach of probability theory beyond the traditional domain, e.g., see [58] and references cited therein.
5) Embedding multiple probability spaces into a larger non-Kolmogorovian probability superspace. This "trick" might be needed in order to still use classical Kolmogorovian probability theory but only "locally," i.e., in co-existence with other possibly incompatible probability models, but while all are embeddable into a larger abstract superspace. One of the advantages of such non-Kolmogorovian probability superspace is its ability to deal with contextuality and possibly improvisation in AGI systems.

Because of such possible fundamental changes in the mathematical structure inside where AI research can be conducted, it is reasonable to expect that emulating natural intelligent and AGI agents may eventually involve introducing some non-classical alterations in the basic fabric of all researches into intelligence as such. For example, could the incorporation of quantum probability help solving the problem of the dynamic splitting or bifurcation into abstract and concrete so essential for natural intelligence referred to above? Regardless to the eventual answer to such open questions whose complete or even partially complete answers are still awaiting us in the future, we believe that projected domains such as AGI, artificial brains, and strong AI will eventually involve a major departure from the classical mathematical and logical models that have governed the progress of the field since the 1930s. In the next section, we provide a more formal outline of some of these new structural changes expected to play a role in such a departure from the common approach.

III. Noncomputability and Dynamical Systems in Artificial General Intelligence

A. *Infinitary and infinitary noncomputability*

In this article, a computable procedure is defined as any algorithm implementable on a standard or stochastic Turing machine [75]. This definition is popular and is thought to encompass both classical and quantum computing [39], [76]. However, it should be noted that it is not the only possible one. Indeed, a noncomputable procedure need not be a method that cannot be implemented on a Turing machine. Noncomputablity (NC) as such may involve non-Turing machine, for example machines that use an infinite set of rules or even a completely different computing structure. A recent comprehensive reexamination of this subject may be found in [77], where numerous additional references on computing and the relation with fundamental physics can be found.

The Church-Turing thesis is the common belief that all important computational problems are precisely those implementable on either a deterministic or stochastic Turing machine [76], [78]–[80].[6] Motivated by this perspective, we distinguish two types of noncomputability (NC), infinitary NC and finitary NC, defined as follows:

Definition 1. (Infinitary and finitary noncomputability) An *infinitary noncomputable* (INC) procedure is one that cannot be implemented on a Turing machine yet may be realized by a method utilizing infinitary, e.g., ordinal, formal structures. On the other hand, a *finitary noncomputable* (FNC) algorithm is one where there is a strictly finitary proof that a Turing machine cannot implement its basic computation.

Examples of FNC procedures are those famous open problems in mathematics where it has been effectively proved that no ordinary (hence finitary) Turing machine can be constructed such that it serves as an effective solution procedure [79], e.g., Hilbert's Tenth Problem [80], the plane tiling problem [26], and the word problem [81]. An FNC algorithm may or may not be also an INC one. However, INC methods seem to encompass a distinct set of problems since it is often easier to prove that a problem can be solved using infinitary method [38]. For example, the existence of nonmeasurable sets can be proved using the axiom of choice [82], which is an infinitary rule *par excellence* [70]. At the same time, producing a proof that a given computational problem is NC using only finitary methods is often very difficult and takes considerable time to achieve. Note that the significance of FNC stems from the Church-Turing thesis: since most important computable problems are thought to be those that are Turing-computable, then only FNC problems are significant as outstanding undecidable problems in computing. This may explain why in literature there has been more emphasis on FNC than INC, though in recent years this appears to have started to change, e.g., see the survey [77] and the literature on analog computing for example cited therein.

Curiously, Turing himself appears to have considered non-Turing machines as fundamental. Indeed, shortly after his famous 1937 paper on conventional Turing machines [79], he introduced ordinal and oracle machines [83]. The oracle machine paper [83] sharply contrasts with his more famous other articles such as the founding AI document [7] or the main connectionist paradigm text [3]. More recently, there is an interest in exploring Turing's nonstandard ideas on the role of noncomputability in both computer science and AI [77], [78]. In general, while this may not be really Turing's final theory, there is a growing evidence that true natural

[6] In this article, we often refer to both deterministic and stochastic Turing machines using the same name, simply as Turing machines when the distinction between the two is not conceptually important and based on the context.

intelligence may require introducing non-Turing-like types of machines (see [24]–[26] and more on this below). We are interested in defending the following thesis:

> **Thesis $\mathcal{A}$.** Genuine AGI systems, with capabilities approaching natural intelligence, may require the use of infinitary noncomputable (INC) procedures.

Thesis $\mathcal{A}$ is similar to several other theoretical frameworks proposed in literature proposing a departure from the Church-Turin thesis. However, such a departure is not as radical as it may appear at first sight. In fact, a recent analysis and survey [77] has shown that there is no wide agreement on whether the universe itself is Turing-computable, although, at least in our opinion, the latter view continues to be the mainstream's position: many researchers in fundamental physics, information theory, and AI firmly believe that NC is equivalent to not being Turing-computable. But admitting infinitary methods was already considered by Turing himself [83] as a game changer, though, curiously enough, he did not change the game in his later better known paper on the foundations of AI [7]. Indeed, in the latter work, the imitation game was presented in terms of standard computing machinery of the Turing type, finite state machines, von Neumann computers, and so on. Intelligence has been presumed to be outcome of performing computations. We propose to entitle this pervasive intellectual orientation attitude *the fundamental thesis of machine intelligence* (FTMI), defined as follows:

> **FTMI**. Intelligence is essentially generated by computations. *Computational* intelligence exhausts what constitutes intelligence *as such*.

By combining FTMI with the Church-Turing thesis, the following more familiar version of **FTMI** can be deduced:

> **FTMI***. Intelligence is essentially generated by Turing-computable methods.

It is particularly the **FTMI*** version of the fundamental thesis of machine intelligence what appears to be extremely popular in researches conducted within both physics and AI.

However, Roger Penrose had set out to produce a series of powerful arguments aiming at refuting **FTMI***, utilizing methods borrowed from Gödel's [84] and Turing's works [85], though arriving at final conclusions not necessarily held by them [25], [26]. But Penrose's position with regard to what we collected under Thesis $\mathcal{A}$ is less clear. In general, his emphasis had been laid mainly on an FNC approach based on Gödel's incompleteness theorems [86], framed in terms of Turing's approach, which can be completed with a formalization utilizing only finitary means [26]. The noncomputable oracle machine [83] was briefly considered in [26], but only to be dismissed as non-threatening to his main objective in that book (refuting **FTMI***). He later came back to the topic of oracle machines, producing a more detailed and positive theory, but this time with the aim to to shed more light on the structure of the human mind [29]. In any case, there is still no universal agreement on whether Penrose's argument had effectively demolished **FTMI***, and the debate still continues. From our own perspective in this article, we wish to point out the plausibility of Thesis $\mathcal{A}$ in light of Penrose's work and other related subsequent developments.

We begin by noting that Penrose's Gödelian argument aiming at a refutation of **FTMI*** does not logically entail Thesis $\mathcal{A}$ since the latter is focused not only on NC as such but more on INC, where in the latter case computations that might eventually generate intelligence are expected to ultimately rely on infinitary means. It is not clear at this stage whether or not one should consider Penrose's view as supporting either an FNC or INC version of the thesis that noncomputability is at the core of true intelligence.[7] In order to gain a better understanding of the problem, we need to consider how computation relates to physics, a large topic with complex history. Here, our approach will depend on the fundamental role played by dynamical system theory in computing and intelligence.

B. Dynamical Systems and Intelligence

In Sec. II, we listed some of those fundamental structural themes characteristic of natural intelligence that are not easily realizable using present-day AI technology, especially the data-driven ML paradigm. Some of these features, like the dynamic splitting into abstract and concrete ontological categories, may be taken as suggesting the need to establish a noncomputational procedure surpassing the limitations of a traditional Turing machine (Sec. III). The important question we now turn to is how to realize a given AGI core system, whether computable or not. The answer is critical for understanding some of the limitations of conventional AI and the demand for exploring alternative ideas for future AGI research.

We embrace below the now well-established trend of treating AI systems as essentially *dynamic agents* [72]. This view is fundamental for future realizations of AGI using networks or assemblages of multiple interacting agents [45], [46], [87]–[89].

Definition 2. (Intelligent dynamic agents) An intelligent dynamic agent (IDA) is a process in spacetime capable of performing *decisions* and outputting *actions* in real-time scenarios. The IDA is a continuous- or discrete-time dynamical process that can be embedded into higher-dimensional state spaces, possibly infinite-dimensional, but must involve local time as independent variables. The dynamics may be deterministic or stochastic, and in the latter it can be either Markovian or non-Markovian.

For both natural and artificial intelligence, the IDA is most commonly modeled as a stochastic (usually Markovian) dynamic system [51], [90], [91]. Note that non-Markovianity is closely related to nonlocality so non-Markovian systems are projected to play a prominent role in future dynamical realizations of AGI agents (cf. Sec. II-C). For a comprehensive and rigorous account of the dynamic approach to nature (biology, psychology) that also pays special attention to the role played by memory (non-Markovianity), see [92]. The IDA paradigm has become quite influential in AI teaching and research even while not always couched in the language of rigorous dynamic system theory [72]. IDAs include not

[7]However, the author's feeling is that Penrose's theory fits more with the finitary approach.

only single agents interacting with their environments using sensor arrays [93], [94], but also swarms or assemblages of *interacting* agents where intelligent behaviour can be seen to emerge out of the nexus of collective interactions [45], whether classical or quantum [91].

This dynamic view of intelligence was not well developed during the time of the founding fathers of computer science. Up to the early 1950s, figures like Gödel, Turing, Church, Post, and Chomsky worked with essentially a Hilbert-like formal system approach [95], where the principal content of the theory, whether applied to logic, foundations of mathematics, computing, AI, linguistics, cognition, is generated by executing a set of "static" or fixed rules whereby computations or derivations are logically performed in some ambient abstract mental space [16], [84], [96]–[98]. On the other hand, the rise of dynamic system theory in scientific studies of intelligence started to really take shape after the end of World War II. In general, the dynamical concept itself comes from both biology and mathematics. The evolutionary history of organisms inspires the idea that, similar to the situation with species in macroevolution [99], the intelligence of the *individual*, not the species – viewed as a creative reaction to nature – could be construed as a dynamic response to environmental pressure [100], [101]. According to this view, basic cognitive competences emerge from developmental equilibration processes responding to changes in the surrounding milieu but aiming at preserving formal invariants [102], [103] that provide the basis for the functional performance of the individual as an intelligent agent [104], [105]. In more recent years, influential theories of dissipative dynamical systems [47], [106], [107] also started to attract attention and are currently being actively deployed for conducting investigations into problems related to natural intelligence and cognition [108].

For the purpose of the specific discussion in this article, we would like to advance a second thesis, to be appended to $\mathcal{A}$, where we view the realization of advanced intelligent functions as essentially tied up to the evolutionary history of complex (most likely dissipative [42]) dynamical systems:

> **Thesis** $\mathcal{B}$. A genuine AGI system, with capabilities approaching natural intelligence, can be realized as a special complex dynamical system. Moreover, it is expected that such systems will be dissipative and non-Markovian.

Note that here we view $\mathcal{B}$ as independent of $\mathcal{A}$ although this cannot be proved without a complete theory combining physics, information, and computing, a topic outside the scope of this article.[8]

Much of the recent literature of neuroscience and theoretical brain studies appear to be already embracing the dynamical system doctrine of explaining intelligence, e.g., see [50], [51]. In AI research proper, there is already a trend to integrate the science of complexity with studies of cognition and intelligence [110], [111]. However, a majority of these studies does not emphasize or presuppose a noncomputable core in the dynamical system. In fact, the standard dynamical systems of both classical and quantum physical theories can be simulated using Turing-computable machines [26], [76], [77]. We then need to look more closely at how physics relates to computing and AGI.

C. The Role of Physics in Artificial General Intelligence

So far in this article, intelligence has been presented and analyzed from an abstract general perspective. Historically speaking, this is how the field stared. Indeed, since the early decades of AI research, it was believed that the core objective of achieving human-level intelligence could be attained by mere increase of computational power and the use of increasingly sophisticated algorithms. Strong AI for instance was treated as essentially a "programming problem" that requires only a substantial input from mathematical logic. We now know this is not the case. AI and AGI are unlikely to progress in the future without considering highly specialized physical and possibly biological systems that will either perform Turing computations or execute noncomputable actions that cannot be even modeled using a Turing or other equivalent machines [24], [26], [28], [77]. Let us see how our discussion so far may help shedding some light on this scenario. If we combine $\mathcal{A}$ with $\mathcal{B}$, the following corollary is obtained:

> **Principle of Dynamic Realization**. A genuine AGI system is an infinitary noncomputable dynamical system. In other words, we have
>
> $$\mathrm{INC} + \mathrm{IDA} \underset{\text{Analysis}}{\overset{\text{Realization}}{\rightleftarrows}} \mathrm{AGI} \tag{1}$$
>
> The movement from left to right represents the physical realization of a designed AGI agent via a physical process (dynamics in spacetime). Movement from right to left is the analysis of a given AGI function into two sublayers, a core INC algorithm and a model in terms of dynamic systems.

Additional remarks on the above principle will be given in the remaining parts of this paper. For now, it should be noted that this is not a theorem or an established law, but a formalization of a possible direction that a crossdisciplinary examination of AGI conducted within computer science, engineering, and physics, may take.

First, we clarify the issue of the possible existence of deterministic or stochastic noncomputable systems. An explicit NC system is exhibited in Appendix A, where the well known fact of the existence of several undecidable problems in mathematics [79]–[81] is exploited for the purpose of constructing a noncomputable (NC) discrete dynamical system. Since continuous-time systems can often (but not always) be effectively approximated by discrete dynamics, this serves as a demonstration of the essential distinction between the concepts of NC and deterministic systems: the existence of one does *not* exclude the other [25], [26]. Since physical processes, whether classical or quantum, are modeled by continuous-time dynamical systems, the conclusion is that even while these systems in their standard form are thought to be computable [39], [76], it is still possible *in principle* to discover or realize NC in future physical systems or theories [26], [77].

[8]However, on this multidisciplinary general framework of a physics-based information theory of computing, see [23], [26], [39], [76]–[78], [109].

This issue is critical for our purposes here. Indeed, objections against propositions such as Theses $\mathcal{A}$ and $\mathcal{B}$ often claim that AGI, science, and engineering should only be concerned with computable systems since only the latter contain finite resources, while engineering systems utilizing infinite resources cannot be constructed in the laboratory. At first this appears to be certainly true, but it is advisable to be more careful when the expression "finite resources" is used. If I have access to a given number of ASICs, cables, wires, PCBs, *and* an artificial brain (regardless now to what this engineered brain is), then indeed I do have a "finite" list of resources. Nevertheless, the artificial brain "item" in this list may itself rely on a special internal *physical* infinitary process that can perform an NC operation. This is one of the key points of this article: new physics or novel discoveries in the natural world may bring into the AGI field unique dynamical systems capable of performing NC operations.

D. Comparison with Penrose's Position

While Penrose's main arguments were that only NC processes may explain or realize human consciousness, our goal is to highlight the importance of NC systems in providing some of the natural intelligence operations and themes discussed in Sec. II which are difficult to realize with classical AI methods but could become crucial for future AGI. As stated in Sec. III-C, The Principle of Dynamic Realization partially agrees with the main spirit of Penrose's arguments (Thesis $\mathcal{B}$), but also differs from his position in some important aspects:

1) In contrast to Penrose, we are not interested here in the problem of how to explain consciousness, where the latter is defined as self-awareness. Self-reflection, the fundamental ontological structure of consciousness [112], was not listed in the main themes of natural intelligence summarized in Sec. II.
2) Penrose does not focus on infinitary resources such as the axiom of choice, the well-ordering principle, transfinite induction, which are essential known infinitary principles used in higher mathematics [24], [38], [82].
3) Penrose's formulation highlights the importance of Objective Reduction (OR) [28], i.e., the process of reducing the quantum state to one of its eigenstates (wavefunction collapse, quantum measurement, etc.) On the other hand, proposals such as OR are outside our scope. Instead, we focus on the abstract and formal theoretical structure of the physics involved, here delimited in terms of an INC core integrated into the IDA as per the formula (1), Sec. III-C.

We further note that while Penrose's justification of Thesis $\mathcal{B}$ is very convincing, the status of his proposal that some sort of a quantum-gravitational noncomputable physics is involved in the OR process is still not fully clear or conclusive, but see [28], [55]. For these reasons, in Sec. IV we propose a theoretical model to explain how we envision INC components injected into an IDA framework for the purpose of building future AGI agents utilizing some future (still unknown) physics. The specific noncomputable OR process proposed by Penrose is not considered as essential for our presentation of AGI systems in this article, but if proved true, it can be incorporated into our general abstract theoretical framework.

IV. A Theoretical Formalism for Generic AGI System Assemblages

A formal model of an extremely general network or assemblage of interacting IDAs realizing an AGI framework is outlined next. Our objective here is to introduce the minimal theoretical structure needed to mathematically describe information processing and flow in generalized neural networks defined as assemblage of coupled or interacting subsystems dynamically exchanging information, energy, and matter with each other. For AGI assemblages, we are often mainly interested in information processing. Spiking neural networks [113] might be considered a current forerunner to the generalized AGI agent assemblage we envision here. We allow the AGI assemblage and each of its subsystems to be either classical or quantum. To work with both types of systems using a single mode, we utilize the stochastic dynamic methods of Prigogine [114] and Sudarshan [115], which provide a physical description of the same system using a master equation in statistical densities living in either function or operator spaces but be extended to superspaces going beyond Hilbert space such as rigged Hilbert spaces [107], [116], [117].

A. The Single Agent System Level Description

Let ρ_t be a either a quantum density operator [118] or distribution function [107] characterizing the state of a physical dynamical system defined on a state space $\mathcal{X}$. The system possesses an internal state $X_t \in \mathcal{X}$ and input $U_t \in \mathcal{U}$ at time t, where $\mathcal{U}$ is the space of input excitation fields (collection of physical inputs carrying information from the environment and possibly other interacting agents represented by other dynamical processes.) Note that the internal state X_t is different from the physical state captured by ρ_t; the latter is a statistical density defined on the state space $\mathcal{X} \ni X_t$. Note that while the evolution equations in terms of the density operator ρ_t (the quantum case) are linear [119], the underlying dynamics when expressed in terms of the internal states X_t can be highly nonlinear.

In order to take memory effects into account, hence generating nonlocal behaviour, we also introduce a *history operator* $\mathscr{H}$ such that $\mathscr{H}V_t$ becomes the past temporal history or time-slice of past instantiations of the quantity V_t. We introduce three distinct history operators: $\mathscr{H}_{\mathrm{s}}, \mathscr{H}_{\mathrm{i}}, \mathscr{H}_{\mathrm{p}}$ for updating past time-slices of the physical state, internal states, and the presynaptic input excitations, respectively. A generic memory operator is nonlocal-in-time. Through the evolution of various coupled degrees of freedom in complex systems, this nonlocality-in-time is transformed into nonlocality-in-spacetime, making the resulting dynamical system effectively nonlocal.

The general dynamical law of the process may be stated in the form of first-order differential equation of the density operator/distribution ρ_t, i.e., the generalized master equation (motivated by the Markovian treatments in [114], [115], [118], [119]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \mathcal{L}\{\mathscr{H}_{\mathrm{s}}\rho_t, \mathscr{H}_{\mathrm{i}}X_t, \mathscr{H}_{\mathrm{p}}U_t, t\}, \tag{2}$$

where t is a local time variable and $\mathcal{L}$ is the dynamical evolution superoperator.[9] The solution of (2) provides the history of the evolution of the physical state (density ρ_t). Information processing is achieved by accessing either the internal states directly or using Hidden Markovian Network estimation [90]. However, note that due to the existence of three memory operators the agent governed by (2) exhibits a complex nonlocal behaviour and the equation is difficult to solve in this extremely general form (approximations require introducing additional restrictive assumptions which we would like to avoid in a very general treatment like ours.)

The system (2) applies only locally by representing one IDA. Other IDAs also exist, with possible mutual interactions. To simplify the presentation, we only state the rules for one agent. However, as in other population-based AI methods, assemblages of interacting agents can be set to evolve in time, with engineered or programmed interaction Hamiltonian such that their global behaviour may lead to a solution to a problem, hence exhibiting intelligent behaviour [45], [87]–[89], [91], [100]. Moreover, each subsystem described by a law like (2) can be non-Markovian due to the existence of nontrivial history operator $\mathcal{L}$ [118], [120]. When interpreted as a stochastic dynamic system, it also becomes dissipative or irreversible flow [107], [121].

B. The AGI Dynamical Assemblage

A generalized neural network can be expressed as a set of coupled processes where each sub-system is realized by a process of the form (2). Every process (generalized neuron or just neuron for simplicity) will have its own (locally accessible) state space $\mathcal{X}_m$, where m is discrete or continuous index of the mth neuronal process and $M \ni m$ is the index space. The reduced physical state of the mth generalized neuron is $\rho^m_{t_m}$, where t_m is the local time of the mth neuron. A global time operator τ with the form

$$(t, T) = \tau(t_m, X^m_{t_m}, m \in M) \tag{3}$$

such that τ acts on the local time array $[t_m]_{m\in M}$ of all IDAs' local time variables, eventually outputting a single global time t to describe the entire (global assemblage) time dynamics. In addition, side data stored in a proper mathematical object T are generated for use in temporal scheduling of information transfer and flow through the networked assemblage, e.g., via frameworks like the event-driven information flow paradigm, dataflow graphs, neurodynamics, spiking neural networks, neuromorphic computing [5], [89], [122], [123].

The global density operator (or distribution function) of the assemblage (the global state) will be denoted by $\mathfrak{R}_\tau$. The dynamical law may be written as

$$\mathfrak{R}_\tau = \Phi_\tau\left\{\rho^m_{t_m}, U^m_t, X^m_t, m \in M\right\}, \tag{4}$$

[9]Because in the quantum case ρ_t is already an operator, we prefer to use the term *superoperator* [107], which transforms operators to operators.

where Φ_τ is the collective (global) dynamical evolution operator of the entire assemblage. Note that due to the demand to account for nonlocality through non-Markovianity, the operator Φ need neither be a semigroup [119] nor analyzable to products of semigroups. The complete dynamics of a networked assemblage of dissipative non-Markovian networks is not as well understood as the Markovian case.

C. Incorporating Machine Intelligence into the Dynamic Multi-Agent Assemblage

Using the methods of machine learning [6] and reinforcement learning [10], various risk, reward, and policy functionals [4] can be constructed on the assemblage $\mathfrak{R}$ [9]. For example, let $\mathcal{E}$ be the environment, $\mathcal{A}$ a proper mathematical object containing the learning/reward/policy parameters, and C_t the value of the cost function at the global time instant t corresponding to a learning a task $\mathcal{T}$. Then we may write

$$C_t = \mathfrak{Y}_{\mathfrak{R}_\tau}\{\mathcal{E}, \mathcal{A}, \mathcal{T}\}, \tag{5}$$

where $\mathfrak{Y}_{\mathfrak{R}_\tau}$ is the global AGI operator of the assemblage $\mathfrak{R}$ evaluated with the help of the time operator τ. The operator $\mathfrak{Y}_{\mathfrak{R}_\tau}$ contains enough information about all the internal states, presynaptic excitation fields, temporal scheduling data T, local time variables, global time, and so on. All real-time data flow are contained in the environment object $\mathcal{E}$, which also has access to the local and global time parameters of the assemblage $\mathfrak{R}$. For an AGI system, the cost C and task $\mathcal{T}$ are expected to be rather complex multidimensional objects in order to incorporate all possible interaction scenarios with the environment $\mathcal{E}$.

The assemblage $\mathfrak{R}$ can be programmed in conjunction with the above described learning process through the adjustment of the details in which the various dynamical processes (agents) composing the networked assemblage $\mathfrak{R}$ (4) are coupled to each other. This can be achieved in myriad ways, but our goal here is only to provide the high-level view. For this model, it is enough to modify the three history operators $\mathscr{H}_{\mathrm{s}}, \mathscr{H}_{\mathrm{i}}, \mathscr{H}_{\mathrm{p}}$, which are now to be continued beyond the single agent/process picture (2) by connecting with every other agent (generalized neuron sub-system with form similar to (2)), eventually extending to the full assemblage (4). The modifications of these history operators under the learning algorithm cost (5) can be compared with how the synaptic connections are adjusted in present-day neural networks [5].

D. Noncomputability in the AGI Assemblage

There are various modes by which an INC level of structuration can be inserted into this generalized neuromorphic assemblage of dynamically interacting AGI agents:

1) The index space M could become infinite (a set with infinite cardinality) instead of being a finite grid, e.g., when M is countable, $\mathfrak{R}$ becomes an infinite lattice, i.e., effectively moving beyond ANNs and graph theory.
2) If the assemblage's base space M becomes continuous, then $\mathfrak{R}$ can be mathematically modeled as a *continuous random field* [124]. The construction and analysis of

continuous random fields is one of the most challenging problems in mathematics [125]. Furthermore, with non-Markovianity in the underlying "nodes" or agents of the form (2) in the continuous assemblage, the associated random field becomes nonlocal.

3) We may also require that the state space $\mathcal{X}^m$ is infinite dimensional, effectively moving beyond computable systems in finite number of dimensions. If the infinite-dimensionl space can never be effectively approximated by a finite grid or graph, then an encounter with the "actual infinite" [24] can be realized by tapping into the physics of the infinite dimensional space [77].
4) We may incorporate nonlocality into the structure of the assemblage by requiring that all systems be non-Markovian. This provides intrinsic resources in the intelligent assemblage allowing it to harness long-distance information exchange and correlations.
5) Infinitary arrangements might be inserted into the structure of the intelligent neural assemblage $\mathfrak{R}$'s dynamical physical process. For instance, one may use infinitary principles such as the axiom of choice or the well-ordering principle, powered by a noncomputable physical process, in order to arrange an infinite number of IDAs in a "very large" superspace (the position space of a continuous assemblage for instance, see above), making the resulting global system fundamentally noncomputable by a Turing machine even if, locally speaking, each individual agent of the form (2) is Turing computable.

Finding a proper specific physical mechanism, such as Penrose's OR process, capable of realizing some or all of these INC-level structures and operations can be very challenging but so is achieving AGI supremacy. In recent years, several proposals for going beyond the finitary Turing paradigm were proposed, see [77]. Our goal in this section has been develop some of the initial structures of the problem using a rigorous theoretical framework.

V. Conclusion

The main objective of this work was suggesting that current AI paradigms, especially dominant bottom-up approaches such as neural networks and machine leaning, might face insuperable barriers blocking or slowing progress toward building genuine (human-level) AGI agents, possibly due to the existence of a set of fundamental inherent limitations in AI as such when it comes to demonstrating AGI capacities. We suggested injecting new structural dimensions such as nonlocality, non-classical probability, and noncomputability into the basic fabric of dynamics-based implementable AGI platforms. We hope that achieving genuine AGI performance such as improvisation and mastering the abstract/concrete switch might be attained by future generalized neural-like generations of intelligent machinery taking into account such new structural dimensions.

Abbreviation	Meaning
AGI	Artificial General Intelligence
ANN	Artificial Neural Network
NC	Noncomputability/Noncomputable
INC	Infinitary Noncomputability/Noncomputable
FNC	Finitary Noncomputability/Noncomputable
OR	Objective Reduction
ML	Machine Learning
IDA	Intelligent Dynamical Agent
FTP	Formula of Total Probability

TABLE I: List of the main abbreviations used in this paper.

Appendix A
An Explicit Construction of a Noncomputable Dynamical System

To simplify the discussion, let us ignore for a while the difference between INC and FNC introduced earlier and consider merely a generic NC system, i.e., a dynamical system that cannot be simulated by a Turing machine. It is not difficult to exhibit such a scenario. The idea is to exploit the already established fact that some problems are non-Turing solvable, for example the tiling problem [26] and Hilbert's Tenth Problem [80]. Let $a_i \in \mathbb{N}$ be a sequence of integers indexed by $i \in \mathbb{N}$. We will work with discrete dynamical systems, where the evolution of the process is described by specifying how the state $\mathbf{X}_n$ at time step $n \in \mathbb{N}$ evolves. Assume that a generic *undecidable* problem P takes as an input a finite number of integers and the purpose is to decide whether its solution can be implemented on a Turing machine. (From [79], we already know that the set of such problems is nonempty.) This can be represented by a binary function F defined as follows:

$$F(a_1, a_2, \dots, a_n) = \begin{cases} 1, & \text{if P is decidable} \\ 0, & \text{if P is undecidable} \end{cases} \tag{6}$$

Here decidable (undecidable) means a solution exists (does not exists) such that it can be effectively implemented on a Turing machine [79]. It is assumed above that a computational procedure is established to generate a sequence of n integers a_i for any given $n \in \mathbb{N}$; i.e., we assume the existence of a function $f : \mathbb{N} \to \mathbb{N}^\infty$ such that at each n, only the first n slots in the infinite-dimensional vector $f(n)$ are nonzero. (The function f can be either deterministic or random.) Now let the nth state of a discrete dynamical system be defined as

$$\mathbf{X}_n := \begin{pmatrix} a_1 & a_2 & \dots & a_n & 0 & \dots \end{pmatrix} \in \mathbb{N}^\infty. \tag{7}$$

The dynamical evolution is governed by the following update rule:

$$\mathbf{X}_{n+1} = \begin{cases} f(n+1), & \text{if } F(a_1, a_2, \dots, a_n) = 1, \\ f(n+2), & \text{if } F(a_1, a_2, \dots, a_n) = 0. \end{cases} \tag{8}$$

It is clear that (8) effectively sets up a well-defined discrete dynamical system evolving the state $\mathbf{X}_n$ in the state space $\mathbb{N}^\infty \ni \mathbf{X}_n$. This system is well-defined but NC, i.e., can neither be realized with a Turing machine when $f(n)$ is deterministic, nor using a stochastic Turing machine if $f(n)$ is probabilistic.[10]

Appendix B
Bibliographical Remarks

Gödel's incompleteness theorem [86] was mentioned several times but not explained. The interested reader may consult the excellent expositions given in [23]–[26]. On the infinitary in mathematics, the fundamental text is Cantor's [126] and Russell's [127], [128], where the latter also discuss applications to physics and geometry. The text [24] combines ideas about the infinitary mode of expression with computing and Gödel's theorems. The comprehensive and detailed review [77] is very helpful in terms of gathering numerous resources about noncomputability and infinity in the interaction between computer science and physics (though it does not discuss AGI as such). We recommend that the reader consult the work [77] while reading this article for additional references not mentioned here, e.g., oracle machines, analog computing, and noncomputable physics.

[10]Penrose [26] introduced a similar idea, but using the tiling problem as an undecidable problem to be exploited in constructing examples of noncomputable systems.

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