A Linear Distflow Model Considering Line Shunts

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Abstract

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***Index Terms*—Distflow equations, line shunts, linear power flow model, power flow analysis**

I. Introduction

POWER flow model is the most important component of many power system problems, including planning and optimization problems. The transmission lines of both transmission and distribution systems are originally modelled with the Π circuit line model, whose shunt elements are usually assumed to be zero in distribution systems, which can be accepted previously when the majority of distribution lines are overhead lines with nearly zero shunt admittance. However, according to a field test report from Shenzhen Power Supply Company, when distribution networks are gradually modernised, overhead lines are replaced by underground cables for urban appearance, whose shunt admittance is 5-10 times larger [1], [2], even larger for submarine cables [3], [4]. Therefore, the "charging effect" caused by the capacitive susceptance of line shunt admittance would introduce model errors to power flow models with zero line shunts, lift the voltage magnitude at ending nodes [5], cause voltage violations, and affect the stability and reliability in power distribution systems (PDS). Thus, the shunt elements in the Π circuit line model cannot be ignored for the precise calculation of voltage in modern PDS.

Alternating current power flow (ACPF) is a general PF model which has a full consideration of system parameters. However, due to its complexity and non-convexity [6], ACPF cannot be applied directly. The Distflow model, as a simplification of ACPF for radial distribution systems is proposed by [7], assuming zero line shunts. The first approximation method is the linearization of Distflow model. Linear Distflow model is widely used when line loss is small [7]. Another linearized AC power flow model is proposed in [8]. Study [9] provides an implicit linearization method using the flat voltage solution, which can obtain the linear Distflow model. An approximated linear PF model for three-phase unbalanced systems is proposed in [10]. Another approximation method is the relaxation of Distflow model. Second order cone programming (SOCP) of optimal power flow (OPF) problems is proposed and the exactness of convex relaxations are proved in [11]. However, as all the aforementioned models assume zero line shunts, model errors brought by "charging effects" of underground/submarine power cables would be large and not acceptable, and sufficient conditions for the exactness of convex relaxations might not be guaranteed [12].

To address this issue, three recent studies have proposed non-linear branch flow model (BFM) based on SOCP with non-zero shunt elements. Study [13] computes the local solution of OPF problem based on BFM with line shunts (BFMS). Another BFM including line charging is proposed by [12] as a replacement of the SOCP relaxation, whose exactness is proved to hold [14]. However, the optimal solution cannot be guaranteed by using the BFM in [13], and the relaxation error would become non-negligible if the squared current magnitude is not bounded at the lower bound, like the reverse power flow [15]. Besides, a PF analytical method with non-zero line shunts is proposed by [16], but it cannot be applied easily as the multi-dimensional holomorphic embedding method is required.

In this case, linear models are highly required for higher efficiency, strong duality and better accuracy in case of reverse power flow. A linear BFM with line shunts (LBFS) is proposed in [17] to replace BFMS considering the effect of reverse power flow, but LBFS shows relatively large errors in branch flow as it regards apparent power flow as current magnitude. Thus, we propose a modified linear Distflow model with line shunts (LinDistS) in this brief with a decent calculation accuracy to address model errors caused by the capacitive susceptance of line shunts. The main contributions are summarized as follows:

- LinDistS is proposed to address model errors brought by "charging effects" of line shunts. Its superiority lies in preserving the structure of the original LinDist model with a modified factor added in the voltage drop equation. The error of voltage component in the linearization process is analyzed, which shows that the error mainly depends on the squared magnitude of voltage difference between neighboring nodes. Simulation results exhibit LinDistS's enhanced calculation efficiency than non-linear models and accuracy than the original LinDist model and other linear models.
- LinDistS is further extended to three-phase unbalanced

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systems. The strength of LinDistS lies in maintaining the linearity, and hence the convex nature after the extension. Case studies with comparisons of sufficient counterparts highlight the significance of introducing line shunts into the LinDist model in three-phase systems.

The rest of this brief is organized as follows. Section II presents the background of Distflow model. Then, the model formulation of LinDistS and its extensions are given in Section III. Numerical results are provided in Section IV and conclusions are drawn in Section V.

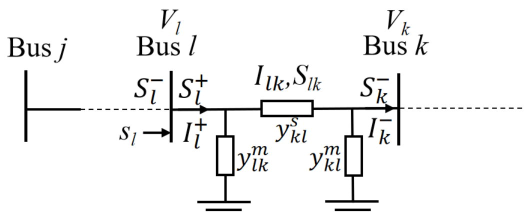

Fig. 1. Part of a radial distribution network with Π circuit model

II. Background

$Notations.$ $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex numbers and real numbers, respectively. For any $n \in \mathbb{C}$, n^* means its complex conjugate. Let L denotes the set of all branches and N denotes the set of buses of a single-phase power network.

A widely used power flow model, **Distflow Model**, is proposed in [7] with the assumption of zero shunt elements, namely $y^m = 0$. Distflow equations are used for the calculation of single-phase radial distribution networks (shown in Fig. 1) and have the following formulation:

$$S_l^- + s_l = \sum S_l^+ = \sum_{k:l\to k} (S_k^- + z_{lk}\ell_{lk}) \tag{1}$$

$$v_l - v_k = 2\mathrm{Re}(z_{lk}^* S_k^-) - |z_{lk}|^2 \ell_{lk} \tag{2}$$

$$v_l \ell_{lk} = |S_l^+|^2 \tag{3}$$

where $s_l := p_l + \mathbf{i}q_l$ is the complex power injections at bus l, in which $p_l, q_l \in \mathbb{R}$ denote its active and reactive part, $S_l^-/S_l^+ \in \mathbb{C}$ is the receiving/sending end power flow at bus l, and $v_l = |V_l|^2 \in \mathbb{R}$ means the squared voltage magnitude of bus l, $l \in N$. $\ell_{lk} = |I_{lk}|^2 \in \mathbb{R}$ is the squared current magnitude, $z_{lk} = 1/y_{lk}^s := r_{lk} + \mathbf{i}x_{lk}$ denotes the series impedance (resistance and reactance), and $y_{lk}^m := Gs_{lk} + \mathbf{i}Bs_{lk}$ denotes the shunt admittance (conductance and susceptance) of line $l \to k \in L$.

Linear Distflow equations (LinDist) [7] are modified from equations (Eqs.) (1)-(3) according to following assumptions:

- The losses on both series impedance and shunt admittance are neglected for each line.
- The shunt admittance of each line is assumed to be zero.

$$S_l^- + s_l = \sum_{k:l\to k} S_k^- \tag{4}$$

$$v_l - v_k = 2\mathrm{Re}(z_{lk}^* S_k^-) \tag{5}$$

A **linear power flow manifold model with line shunts** (LPFS) is also proposed according to the implicit linearization around the flat voltage profile described in [9]. The linear approximation matrix in [9] is modified to consider nonzero shunt elements:

$$A_{\mathrm{sh}} = \begin{bmatrix} \mathrm{Re}(\mathrm{diag}\overline{y}^m) + \mathrm{Re}(Y) & -\mathrm{Im}(\mathrm{diag}\overline{y}^m) - \mathrm{Im}(Y) \\ \mathrm{Im}(\mathrm{diag}\overline{y}^m) - \mathrm{Im}(Y) & \mathrm{Re}(\mathrm{diag}\overline{y}^m) - \mathrm{Re}(Y) \end{bmatrix} \tag{6}$$

The implicit linear manifold is described by: $A_{\mathrm{sh}} * (x - x^*) = 0_{2n}$ based on the flat voltage solution: $x^* = (1_n, 0_n, \mathrm{Re}(y^m), \mathrm{Im}(y^m))$. $x = (|V|, \theta, p, q)$ is the solution of power flow manifold, and θ denotes the phase angle of nodal voltage. The implicit linear model after the transformed coordinate is shown as:

$$A_{\mathrm{sh}} \begin{bmatrix} |V|^2/2 \\ \theta \end{bmatrix} = \begin{bmatrix} p - \mathrm{Re}(y^m) \\ q - \mathrm{Im}(y^m) \end{bmatrix} \tag{7}$$

As a linear PF model, LPFS would be compared with the proposed model in this brief.

III. Model Formulation

A. Single-phase Equivalent Model

According to Eqs. (1)-(3), we consider the Distflow equations with line shunts in complex form. As for the voltage drop equation (2), it is difficult to form a linear mathematical structure of Π circuit model with nonzero shunt elements. In this case, we start the derivation in a two-bus system (Fig. 1.).

We assume $y_{lk}^m = y_{kl}^m$ within a single branch because the two-bus system does not have transformers. According to the two-bus system in Fig. 1., the current through series impedance, z_{lk}, can be expressed as:

$$I_{lk} = I_k^- + V_k (y_{lk}^m)^* \tag{8}$$

where I_{lk} denotes the current of the line $l \to k$ and I_k^- is the receiving-end current at bus k. Then, the power flow through the same impedance is:

$$S_{lk} = V_l I_{lk}^* \tag{9}$$

and the voltage at node l is the summation of voltage at another node and the voltage drop on series impedance:

$$V_l = V_k + z_{lk} I_{lk} \tag{10}$$

Take (10) and multiply each side by its complex conjugate:

$$|V_l|^2 = V_l V_k^* + V_l z_{lk}^* I_{lk}^* \tag{11}$$

$$= (V_k + z_{lk} I_{lk}) V_k^* + S_{lk} z_{lk}^* \tag{12}$$

$$= |V_k|^2 + z_{lk} I_{lk} (V_l^* - z_{lk}^* I_{lk}^*) + S_{lk} z_{lk}^* \tag{13}$$

$$= |V_k|^2 + 2\mathrm{Re}(z_{lk}^* S_{lk}) - |z_{lk}|^2 |I_k^- + V_k (y_{lk}^m)^*|^2 \tag{14}$$

$$\geq |V_k|^2 + 2\mathrm{Re}(z_{lk}^* S_{lk}) - |z_{lk}|^2 (|I_k^-| + |V_k (y_{lk}^m)^*|)^2 \tag{15}$$

$$= v_k + 2\mathrm{Re}(z_{lk}^* S_{lk}) - |z_{lk}|^2 |I_k^-|^2 - v_k |z_{lk} (y_{lk}^m)^*|^2 - 2|z_{lk}|^2 |I_k^-| |V_k (y_{lk}^m)^*| \tag{16}$$

$$\approx (1 - |z_{lk} (y_{lk}^m)^*|^2) v_k + 2\mathrm{Re}(z_{lk}^* S_{lk}) \tag{17}$$

$$= (1 - |z_{lk} (y_{lk}^m)^*|^2 + 2\mathrm{Re}(z_{lk} y_{lk}^m)) v_k + 2\mathrm{Re}(z_{lk}^* S_k^-) \tag{18}$$

The linear approximation of voltage drop equation can be accessed by adopting an approximation in (15) (taking the lower bound of the inequality equation) and ignoring two parts, $|z_{lk}|^2|I_k^-|^2$ and $2|z_{lk}|^2|I_k^-||V_k y_{lk}^m|$, in (16).

Remark 1: The approximation from (14) to (15) is achieved with the assumption that current phasors' angles are equal. Taking the 33-bus system described in Section IV as an example, this mild assumption is based on the findings that the average angle difference, θ, between I_k^- and I_{kl}^m of the whole system is around 19.36°, and the average error between $|I_k^- + I_{kl}^m|$ and $|I_k^-| + |I_{kl}^m|$ is 1.32%, , visualized in Fig. 2 ($Error = 0.4\%$ for IEEE 123-bus system).

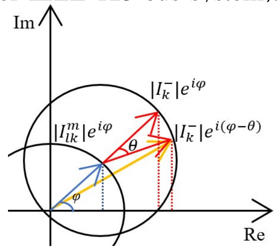

Fig. 2. Vector graph of branch currents

Remark 2: Both $|z_{lk}|^2|I_k^-|^2$ and $2|z_{lk}|^2|I_k^-||V_k y_{lk}^m|$ in (16) share the mathematical nature of the product of squared impedance magnitude and squared current magnitude. The value of squared impedance magnitude in p.u. can be neglected compared with the squared voltage magnitude in p.u, thus these two parts can be ignored in the linear approximation.

With regard to the power balance equation (1), power injections on shunts at both sides should be considered. The linear approximation of power balance equation is achieved by neglecting the losses on series admittance, $z_{lk}\ell_{lk}$. Then the formulation of **LinDist with line shunts** (LinDistS) is shown as follows according to the Π circuit model in Fig. 1.

$$S_l^- + s_l = \sum_{k:l\to k} (S_k^- + (v_k + v_l)(y_{lk}^m)^*) \tag{19}$$

$$v_l - \gamma_{lk}^* * v_k = 2\mathrm{Re}(z_{lk}^* S_k^-) \tag{20}$$

$$\gamma_{lk}^* = (1 - |z_{lk}|^2|y_{lk}^m|^2 + 2\mathrm{Re}(z_{lk} y_{lk}^m)) \tag{21}$$

Note that LinDistS shares the similar mathematical structure of LinDist with a fixed factor, γ_{lk}^*, to modify the node-branch matrix in the voltage drop equation.

B. Error Analysis for Voltage Component in Linearization

The quantitative analysis for the linearization error of voltage magnitude in the voltage drop equation, eqs. (11)-(18), is provided. As for the branch flow balance equation (19), the error of ignoring power loss on series impedance is widely discussed by previous studies [18].

The nonlinear (real) and linear squared voltage magnitude at bus l are presented as follows (bus l and bus k are the two ends of a branch):

$$|V_l^R|^2 = |V_k|^2 + 2\mathrm{Re}(z_{lk}^* S_{lk}) - |z_{lk}|^2|I_k^- + V_k(y_{lk}^m)^*|^2 \tag{22}$$

$$|V_l^L|^2 = (1 - |z_{lk}(y_{lk}^m)^*|^2)|V_k|^2 + 2\mathrm{Re}(z_{lk}^* S_{lk}) \tag{23}$$

Therefore, the linearization error e_l^v can be expressed as:

$$\begin{aligned} e_l^v &= [|V_l^L|^2 - |V_l^R|^2] && (24)\\ &= [-|z_{lk}(y_{lk}^m)^*|^2|V_k|^2 + |z_{lk}|^2|I_k^- + V_k(y_{lk}^m)^*|^2] && (25)\\ &= [-|z_{lk}(y_{lk}^m)^*|^2|V_k|^2 + |z_{lk}|^2|I_{lk}^*|^2] && (26)\\ &= [-|z_{lk}(y_{lk}^m)^*|^2|V_k|^2 + |V_l - V_k|^2] && (27)\end{aligned}$$

where [...] represents the absolute sign. Notice that the product of squared magnitude of series impedance and shunt admittance in p.u. value is small. For example, the largest value is 2.02×10^{-5} for 33-bus system, and $|z_{lk}(y_{lk}^m)^*|^2|V_k|^2 = 2.23\times10^{-5}$ can be ignored even if V_k reaches the upper bound, 1.05 p.u. Thus, the linearization error can be approximated as:

$$e_l^v \approx [|V_l - V_k|^2] \tag{28}$$

which is a quadratic term representing the influence of V on the losses. As shown in case studies, because the squared magnitude of voltage difference in distribution networks is normally small due to the local reactive balance and loss reduction [18], the linearization error can be acceptable.

C. Extension to Three-phase Unbalanced Case

For three-phase systems, the Krichhoff's voltage law, Eq. (10), is rewritten as:

$$\tilde{V}_l = \tilde{V}_k + z_{lk}^{abc}\tilde{I}_{lk} \tag{29}$$

where $\tilde{V}_l := [V_l^a, V_l^b, V_l^c]^T \in \mathbb{C}^{3\times1}$, $\tilde{I}_{lk} := [I_{lk}^a, I_{lk}^b, I_{lk}^c]^T \in \mathbb{C}^{3\times1}$ and $z_{lk}^{abc} \in \mathbb{C}^{3\times3}$ is the total line series impedance matrix [19]. The branch current through the line impedance is:

$$\tilde{I}_{lk}^* = \tilde{S}_{lk} \oslash \tilde{V}_l \tag{30}$$

where $\odot$ and $\oslash$ are the element wise product and division, respectively.

Take (29) and multiply each side by its complex conjugate:

$$\begin{aligned} \tilde{V}_l \odot \tilde{V}_l^* &= \tilde{V}_l \odot \tilde{V}_k^* + (z_{lk}^{abc})^* \tilde{I}_{lk}^* \odot \tilde{V}_l && (31)\\ \tilde{v}_l &= (\tilde{V}_k + z_{lk}^{abc}\tilde{I}_{lk}) \odot \tilde{V}_k^* + \tilde{z}_{lk}^*\tilde{S}_{lk} && (32)\\ &= \tilde{V}_k \odot \tilde{V}_k^* + \tilde{z}_{lk}^*\tilde{S}_{lk} + \tilde{z}_{lk}\tilde{S}_{lk}^* - [|z_{lk}^{abc}\tilde{I}_{lk}|^{\otimes 2}] && (33)\\ &= \tilde{v}_k + \tilde{z}_{lk}^*\tilde{S}_{lk} + \tilde{z}_{lk}\tilde{S}_{lk}^* - [|z_{lk}^{abc}(\tilde{I}_k^- + (\tilde{y}_{lk}^m)^* \odot \\ &\quad \tilde{V}_k)|^{\otimes 2}] && (34)\\ &\geq \tilde{v}_k + \tilde{z}_{lk}^*\tilde{S}_{lk} + \tilde{z}_{lk}\tilde{S}_{lk}^* - [(|z_{lk}^{abc}\tilde{I}_k^-| + |z_{lk}^{abc}(\tilde{y}_{lk}^m)^* \\ &\quad \odot \tilde{V}_k|)^{\otimes 2}] && (35)\\ &\approx (\mathbb{1} - |\tilde{z}_{lk}^*(\tilde{y}_{lk}^m)^*|^{\otimes 2}) \odot \tilde{v}_k + \tilde{z}_{lk}^*\tilde{S}_{lk} + \tilde{z}_{lk}\tilde{S}_{lk}^* && (36)\\ &= (\mathbb{1} - |\tilde{z}_{lk}^*(\tilde{y}_{lk}^m)^*|^{\otimes 2} + \tilde{z}_{lk}^*(\tilde{y}_{lk}^m)^* + \tilde{z}_{lk}\tilde{y}_{lk}^m) \odot \tilde{v}_k \\ &\quad + \tilde{z}_{lk}^*\tilde{S}_k^- + \tilde{z}_{lk}\tilde{S}_k^{-*} && (37)\end{aligned}$$

where $\tilde{v}_l := [|V_l^a|^2, |V_l^b|^2, |V_l^c|^2]^T \in \mathbb{C}^{3\times1}$, $\tilde{y}_{lk}^m \in \mathbb{C}^{3\times1}$ means the phase shunt admittance, obtained from diagonal terms of the admittance matrix, and $[|\ldots|^{\otimes 2}]$ denotes the squared element-wise magnitude, like $\tilde{v}_l$. $\tilde{z}_{lk}^* := \alpha \odot z_{lk}^{abc} \in \mathbb{C}^{3\times3}$. According to the nearly balanced voltages for three-phase buses [10], α is defined as:

$$\alpha = \begin{bmatrix} 1 & e^{-j2\pi/3} & e^{j2\pi/3} \\ e^{j2\pi/3} & 1 & e^{-j2\pi/3} \\ e^{-j2\pi/3} & e^{j2\pi/3} & 1 \end{bmatrix} \tag{38}$$

The approximation methods adopted in Eqs. (31)-(37) are similar to the single-phase version, Eqs. (11)-(18), including

Remark 1 & 2. Together with the power balance equation, the **LinDistS** for three-phase systems is shown as:

$$\tilde{S}_l^- + \tilde{s}_l = \sum_{k:l\to k} (\tilde{S}_k^- + (\tilde{y}_{lk}^m)^* \odot (\tilde{v}_k + \tilde{v}_l)) \tag{39}$$

$$\tilde{v}_l - \tilde{\gamma}_{lk}^* \odot \tilde{v}_k = \tilde{z}_{lk}^* \tilde{S}_k^- + \tilde{z}_{lk} \tilde{S}_k^{-*} \tag{40}$$

$$\tilde{\gamma}_{lk}^* = \mathbb{1} - |\tilde{z}_{lk}^* (\tilde{y}_{lk}^m)^*|^{\otimes 2} + \tilde{z}_{lk}^* (\tilde{y}_{lk}^m)^* + \tilde{z}_{lk} \tilde{y}_{lk}^m \tag{41}$$

Then, the three-phase wye-connected power load can be described as:

$$\tilde{p}_l^{ZIP} = \tilde{p}_l (a_p \tilde{v}_l + b_p \tilde{v}_l/2 + b_p/2 + c_p) \tag{42}$$

$$\tilde{q}_l^{ZIP} = \tilde{q}_l (a_q \tilde{v}_l + b_q \tilde{v}_l/2 + b_q/2 + c_q) \tag{43}$$

A linear approximation of constant current loads, $b_p\sqrt{v_l}$, (the same for reactive part) is implemented in the ZIP load model by defining $v = 1 + \Delta v$ and neglecting high order terms of Taylor series around zero: $b_p\sqrt{v_l} = b_p\sqrt{1+\Delta v} \approx b_p(1 + \frac{\Delta v}{2}) = b_p(\frac{v_l+1}{2})$. According to [19], the approximated wye-connected load of delta-connected load can be obtained:

$$\begin{bmatrix} s_l^a \\ s_l^b \\ s_l^c \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} e^{-\frac{j\pi}{6}} & 0 & -e^{-\frac{j5\pi}{6}} \\ -e^{-\frac{j5\pi}{6}} & e^{-\frac{j\pi}{6}} & 0 \\ 0 & -e^{-\frac{j5\pi}{6}} & e^{-\frac{j\pi}{6}} \end{bmatrix} \begin{bmatrix} s_l^{ab} \\ s_l^{bc} \\ s_l^{ca} \end{bmatrix} \tag{44}$$

Equations (39)-(43) form the linear Distflow model with line shunts for unbalanced three-phase systems.

IV. Numerical Results

We tested the proposed models in four radial distribution systems, whose modifications are shown as follows:

- The 18-bus system, 33-bus system and 69-bus system: Test systems are modified with the consideration of nonzero shunt admittance: the shunt conductance is the same for the whole system, $Gs = 0.0005$ p.u; the shunt susceptance, B_s, is 1/4 of the series impedance magnitude in the same branch.
- IEEE 123-bus system (short as case123): The value of line shunt admittance is five times larger than that of the original system to replace the overhead lines with underground cables [20]. The original system is modified according to the system description in [10], which is used for three-phase model analysis. In the single-phase analysis, it is transferred into a single-phase system through the positive sequence equivalent and the application of Kron's reduction in case neutral is also present.

The test systems are expanded to 100 scenarios with randomly different load levels (including light, normal and heavy conditions) to verify the calculating performance in large scale PDS. The computation error is calculated as $\epsilon = |Test - Real|/Real$ ($Real$ is solved from ACPF).

Four linear models are introduced to test LinDistS's performance: (i) LBFS from [17]; (ii) LPFS from [9]; (iii) A linear branch flow model (LBFMS) is modified through ignoring squared branch current terms, ℓ_{jk}, in [14]; (iv) A linear load flow model (LLFS) is extended to consider line shunts through the modification of the admittance matrix in [21].

TABLE I
Error of Nodal Voltage Magnitude (%)

Test Models	Test Indices	Case18	Case33	Case69	Case123
LinDist	$\overline{V}\epsilon$	1.23	2.40	1.10	0.32
	$V\epsilon_{max}$	3.16	5.50	2.23	0.69
LinDistS	$\overline{V}\epsilon$	0.07	0.09	0.08	0.05
	$V\epsilon_{max}$	0.29	0.18	0.23	0.07
LPFS	$\overline{V}\epsilon$	0.11	0.19	0.14	0.09
	$V\epsilon_{max}$	1.63	2.38	1.92	0.21
LBFMS	$\overline{V}\epsilon$	0.37	0.58	0.42	0.29
	$V\epsilon_{max}$	1.17	2.39	1.50	0.43
LLFS	$\overline{V}\epsilon$	0.06	0.08	0.09	0.08
	$V\epsilon_{max}$	0.18	0.21	0.26	0.12
LBFS	$\overline{V}\epsilon$	0.76	1.31	0.71	0.24
	$V\epsilon_{max}$	1.79	3.02	2.16	1.05

TABLE II
Average Error of Branch Flow (%)

Test Models	Test Indices	Case18	Case33	Case69	Case123
LinDist	$\overline{P}\epsilon$	3.01	1.67	2.77	1.46
	$\overline{Q}\epsilon$	36.17	69.63	44.53	39.14
	$\overline{I}\epsilon$	20.92	30.36	26.08	10.67
	$P\epsilon_{loss}$	24.80	33.93	25.37	17.23
LinDistS	$\overline{P}\epsilon$	3.01	1.67	2.77	1.46
	$\overline{Q}\epsilon$	13.79	18.89	17.10	9.43
	$\overline{I}\epsilon$	10.02	11.42	10.40	6.12
	$P\epsilon_{loss}$	10.33	10.45	10.74	7.37
LPFS	$\overline{P}\epsilon$	2.98	1.73	2.70	1.54
	$\overline{Q}\epsilon$	11.85	17.45	19.31	8.49
	$\overline{I}\epsilon$	9.29	10.82	9.89	6.14
	$P\epsilon_{loss}$	11.76	8.76	15.73	8.61
LBFMS	$\overline{P}\epsilon$	3.01	1.67	2.77	1.47
	$\overline{Q}\epsilon$	15.93	16.74	23.06	9.21
	$\overline{I}\epsilon$	10.72	9.36	9.79	6.06
	$P\epsilon_{loss}$	10.47	9.17	10.51	6.92
LLFS	$\overline{P}\epsilon$	3.11	1.72	2.73	1.45
	$\overline{Q}\epsilon$	13.37	19.18	21.65	12.18
	$\overline{I}\epsilon$	8.24	11.82	11.94	7.63
	$P\epsilon_{loss}$	9.86	12.77	13.46	10.32
LBFS	$\overline{I}\epsilon$	19.43	16.81	18.17	8.38
	$P\epsilon_{loss}$	21.25	18.13	22.96	14.59

TABLE III
Computing Time Comparison of All Scenarios(s)

Test Models	Case18	Case33	Case69	Case123
ACPF	9.347	15.982	21.683	40.579
BFMS	3.182	4.801	5.735	11.594
LBFS	1.232	1.437	3.601	5.799
LinDistS	1.310	1.633	3.694	6.003

TABLE IV
Simulation Results of Three-phase 123-bus System (%)

Test Models	$\overline{V}$	$\overline{P}\epsilon$	$\overline{Q}\epsilon$	$P\epsilon_{loss}$	Time(s)
FBS	$\sim$	$\sim$	$\sim$	$\sim$	69.13
LPF	0.28	1.38	27.89	15.87	10.649
LLFSmo	0.12	2.17	10.32	11.24	12.006
LinDistS	0.07	1.38	7.85	7.13	11.396

According to Table I, LinDistS shows a 80% improvement than LinDist in accuracy of nodal voltage magnitude, which is highly required in large PDS. LinDistS has a better performance compared with other linear models. Both LPFS and LBFMS modify the power flow with line parameters in the voltage drop equation, which might affect the accuracy of voltage magnitude. The 33-bus system has long feeders, thus large value of series impedance and shunt admittance magnitude, resulting in a higher error. As for reactive power, LinDistS, LPFS and LBFMS have similar errors, as shown in Table II, and are much smaller than that of LBFS. LinDistS still significantly improves the accuracy of branch current and hence power loss compared with LinDist, and errors around 10% can be accepted as it is not so important in PDS where line capacities are large enough, and line losses reduction is not the major concern. The error of power loss maybe smaller than that of current magnitude as the current error varies from positive and negative values. LLFS shows a slightly higher accuracy than LinDistS in voltage magnitude in case18 and case33, but lower accuracy in power loss because of its load flow nature. When the scale of test systems increases, LLFS's error of voltage magnitude is higher than that of LinDistS.

Moreover, the accuracy of LinDistS are not significantly affected by the type of network topology and load levels, while the error of voltage magnitude would be large (0.11% and 0.38% for 300% and 500% in case33, respectively) when the system is at the maximum loading point. The accuracy of voltage magnitude is similar for different line R/X ratios, while the exactitude of power loss decreases as the ratio increases, for example $\overline{V}_\epsilon, P\epsilon_{loss}$ increase from 0.12%, 9.38% (average $R/X = 0.7$) to 0.14%, 13.24% (average $R/X = 2.3$) in the 33-bus system. According to Table III, the overall computing time of LinDistS is nearly the same as that of LBFS (and other linear models) because of the shared linearity. Besides, the proposed model dramatically decreases the computing time compared to BFMS and ACPF.

In the three-phase analysis, ZIP load parameters are set as $a_{p,q} : b_{p,q} : c_{p,q} = 3 : 4 : 3$. A linear power flow (LPF) model from [10], [19] and LLFS's three-phase extension model modified from [22] are introduced to evaluate LinDistS's performance. The power injections from distributed resources are computed by the BFM-SDP in [10], and the real power flows and voltage magnitudes of the modified 123-bus system are solved through the forward backward sweep algorithm (FBS). Thus, LPF, LLFS and LinDistS are compared under the same benchmark. Relevant results are shown in Table IV.

Note that the accuracy of LinDistS in voltage magnitude and branch flow is improved significantly from the original linear Distflow model (LPF) by considering line shunts in the three-phase unbalanced system, which obeys the conclusion from the single-phase analysis. Moreover, the calculation accuracy of LinDistS in three-phase analysis is enhanced from that in the single-phase analysis. For example, the calculation error of voltage magnitude is 0.0684%, which is lower than that in the equivalent 123-bus system with ZIP loads, 0.0902%. Besides, LinDistS shows a higher calculation accuracy in both voltage magnitude and power loss than LLFSmo. This highlights the accuracy of LinDistS in three-phase unbalance systems.

V. Conclusion

We showed that a small change in linear Distflow equations, by introducing a fixed factor in voltage drop equation, can address the model errors brought by "charging effects" of underground/submarine power cables with the consideration of nonzero line shunts. LinDistS is structure-preserving after the extension to there-phase unbalanced systems. Theoretical analysis and simulation results prove its strength in calculation accuracy and fast computing efficiency by comparing with sufficient counterparts.

Appendix A
Extension to Consider Weakly Meshed Topology and ZIP Load Model

The following equation of voltage drop phasor angle difference θ_{lk} is introduced to consider weakly meshed topology:

$$|V_l||V_k|sin\theta_{lk} = x_{lk}P_{lk} - r_{lk}Q_{lk} \tag{45}$$

According to [23], Eq. (45) can be linearized to:

$$\theta_{lk} = x_{lk}P_{lk} - r_{lk}Q_{lk} \tag{46}$$

Furthermore, the following ZIP load model (in active and reactive power form) is considered in the formulation of LinDistS:

$$p_l^{ZIP} = p_l * (a_p h^2|V_l|^2 + b_p h|V_l| + c_p) \tag{47}$$

$$q_l^{ZIP} = q_l * (a_q h^2|V_l|^2 + b_q h|V_l| + c_q) \tag{48}$$

where $h = 1/V_{norm} = 1$ for per unit representation [22]. By using $v_l = |V_l|^2$, the ZIP load equations can be reformed as (taking active power as an example):

$$p_l^{ZIP} = p_l * (a_p v_l + b_p\sqrt{v_l} + c_p) \tag{49}$$

Notice the ZIP model is linear in v_l except for the constant current loads ($p_l * b_p$). A linear approximation of $b_p\sqrt{v_l}$ (the same for reactive part) is implemented by defining $v = 1+\Delta v$ and neglecting high order terms of Taylor series around zero:

$$b_p\sqrt{v_l} = b_p\sqrt{1+\Delta v} \approx b_p(1 + \frac{\Delta v}{2}) = b_p(\frac{v_l+1}{2}) \tag{50}$$

The error for this approximation is calculated by defining a function, $100 \times ||(\sqrt{v_l} - (v_l+1)/2)||$. For example, the error for $v_l = 1.05^2$ is around 0.12% and decreases when v_l approaches 1. Thus, the formulation of LinDistS considering weakly-meshed topology and ZIP load model is given:

$$\sum_{l:j\to l}(P_l^-) + p_l^{ZIP} = \sum_{k:l\to k}(P_k^- + \frac{Bs_{lk}}{2}(v_l + v_k)) \tag{51}$$

$$\sum_{l:j\to l}(Q_l^-) + q_l^{ZIP} = \sum_{k:l\to k}(Q_k^- - \frac{Bs_{lk}}{2}(v_l + v_k)) \tag{52}$$

$$v_l - \gamma_{lk}^* * v_k = 2(r_{lk}P_k^- + x_{lk}Q_k^-) \tag{53}$$

$$\theta_{lk} = x_{lk}(P_k^- + v_k\frac{Gs_{lk}}{2}) - r_{lk}(Q_k^- - \frac{Bs_{lk}}{2}v_k) \tag{54}$$

where the value of $p_l^{ZIP} = p_l(a_p v_l + b_p v_l/2 + b_p/2 + c_p)$ (the same equation for q_l^{ZIP}) is negative for power load.

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