A Closed-form Analytical Solution to Torque Free Precession: Euler-Poinsot Problem

Barkan Ugurlu [1]

[1]Affiliation not available

October 31, 2023

Abstract

This work aims to disclose a closed-form analytical solution to attitude motion for a rigid body subject to zero body-fixed torques, i.e., Euler-Poinsot problem. Revisiting Routh's study, the presence of multiple solutions are identified. To verify the proposed solution, numerical simulations and real-life experiment results are presented. A Code Ocean repository is also provided so that readers could test the algorithm individually.

A Closed-form Analytical Solution to Torque Free Precession: Euler-Poinsot Problem

Barkan Ugurlu[1]

***Abstract*— This paper aims to disclose a closed-form analytical solution to attitude motion for a rigid body subject to zero body-fixed torques, i.e., Euler-Poinsot problem. Revisiting Routh's study, the presence of multiple solutions are identified. To verify the proposed solution, numerical simulations and real-life experiment results are presented. A Code Ocean repository is also provided so that readers could test the algorithm individually.**

I. Introduction

The Euler-Poinsot problem tackles the description of attitude motion of a rigid body, subject to zero body-fixed torques. To this end, Poinsot proposed a visualization technique, called Poinsot construction, which can be used to geometrically describe the attitude of a rigid body [1], [2]. Routh challenged the Euler-Poinsot problem and provided analytical solutions to Euler's equation for the case of zero body-fixed torques [3]. Jupp also proposed a solution using a Lie-series perturbation procedure [4]. Recently, Routh's study was revisited and researchers provided further findings, e.g, analyses on its state transition matrix, motion constants, and ergodicity [5], Dzhanibekov effect [6], computer-aided visualization [7], and open-source code [8]. They also demonstrated the presence of multiple solutions concerning the problem.

In this study, Routh's solution was further elaborated so as to identify the multiple solution cases such that it could be used as a generalized *zero-input* solution. The accuracy of the solution was primarily tested via numerical simulations. It was also demonstrated that the proposed solution could represent uncontrolled rotational motion of an actual spacecraft. By solely using the initial angular velocity values, the solution accurately predicted the angular velocity trajectory data which was recorded while the actual spacecraft Aist was undergoing uncontrolled rotational motion [9].

The manuscript is organized as follows. In section II, the mathematical tools used in the manuscript is succinctly provided. The problem definition is given in section III. The closed-form analytical solution is explained in section IV. Simulation and experiment results are presented in section V and the manuscript is concluded in section VI.

II. Mathematical Preliminaries

Introduced by Carl Gustav Jakob Jacobi, Jacobi elliptic functions are a set of elliptic functions, which are used for the analytical description of a certain class of nonlinear system motion. While a unit circle is considered as a basis for trigonometric functions, a *normalized* ellipse is used as the basis for the definition of Jacobi elliptic functions. Following the descriptions provided in [10], the following equations defines an ellipse:

$$\left(\frac{x}{a}\right)^2 + (y)^2 = 1 \tag{1}$$

$$m = 1 - \frac{1}{a^2} \tag{2}$$

$$\tag{3}$$

Note that $a > 1$ and the shape of the ellipse can be controlled by the parameter m which obeys $0 < m < 1$. When $m = 0$, the ellipse take the form of a unit circle. The parameter u may be defined as in the following:

$$u = \int_0^{\gamma} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \tag{4}$$

Using these parameters, the elliptic functions are respectively described as $\mathrm{sn}(u,m) = \sin\gamma$, $\mathrm{cn}(u,m) = \cos\gamma$, and $\mathrm{dn}(u,m) = \sqrt{1 - m\sin^2\gamma}$. Furthermore, let (x,y) be a point on the ellipse; $\mathrm{sn}(u,m) = y$ and $\mathrm{cn}(u,m) = \frac{x}{a}$. Therefore, $\mathrm{sn}(u,m)^2 + \mathrm{cn}(u,m)^2 = 1$. In addition, $\mathrm{dn}(u,m) = \frac{r}{a}$; r is the radial distance of the ellipse.

Derivatives of the elliptic functions can be expressed as below:

$$\frac{d}{du}\mathrm{sn} = \mathrm{cn}\ \mathrm{dn} \tag{5}$$

$$\frac{d}{du}\mathrm{cn} = -\mathrm{sn}\ \mathrm{dn} \tag{6}$$

$$\frac{d}{du}\mathrm{dn} = -m\ \mathrm{sn}\ \mathrm{cn} \tag{7}$$

To simplify the notation, the argument (u,m) is dropped. For the solution of the Euler-Poinsot motion, one needs the squares of the eqs. (5)-(7):

$$\left(\frac{d}{du}\mathrm{sn}\right)^2 = m\ \mathrm{sn}^4 - (1+m)\,\mathrm{sn}^2 + 1 \tag{8}$$

$$\left(\frac{d}{du}\mathrm{cn}\right)^2 = -m\ \mathrm{cn}^4 + (2m-1)\,\mathrm{cn}^2 + (1-m) \tag{9}$$

$$\left(\frac{d}{du}\mathrm{dn}\right)^2 = -\mathrm{dn}^4 + (2-m)\,\mathrm{dn}^2 + m - 1 \tag{10}$$

[1]The author is with the Biomechatronics Laboratory, Department of Mechanical Engineering, Ozyegin University, 34794 Istanbul, Türkiye. barkanu@ieee.org

III. Problem Statement: Euler-Poinsot Motion

For a given rigid body, Euler's equation for principal axes describes the attitude motion [3],

$$M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \quad (11)$$
$$M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \quad (12)$$
$$M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \quad (13)$$

where $\mathbf{M} = [M_x, M_y, M_z]^\intercal$ denote the body-fixed torques, I_{xx}, I_{yy}, I_{zz} are moment of inertia values for principal axes, $\omega = [\omega_x, \omega_y, \omega_z]^\intercal$ stand for the angular velocity. In this case, the products of inertia are deemed to be negligible. When the body-fixed torques are zero, the body is subject to Euler-Poinsot motion. In this case, the eqs. (11)-(13) take the following form:

$$I_{xx}\dot{\omega}_x = (I_{yy} - I_{zz})\omega_y\omega_z \quad (14)$$
$$I_{yy}\dot{\omega}_y = (I_{zz} - I_{xx})\omega_z\omega_x \quad (15)$$
$$I_{zz}\dot{\omega}_z = (I_{xx} - I_{yy})\omega_x\omega_y \quad (16)$$

A. Conserved Quantities

To express the conservation of energy, the expression $(14)\omega_x + (15)\omega_y + (16)\omega_z$ is computed:

$$I_{xx}\dot{\omega}_x\omega_x + I_{yy}\dot{\omega}_y\omega_y + I_{zz}\dot{\omega}_z\omega_z = 0 \quad (17)$$

If (17) is integrated with respect to time, the following is yielded:

$$\begin{aligned} 2E_k &= I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 && (18) \\ &= I_{xx}\omega_{x0}^2 + I_{yy}\omega_{y0}^2 + I_{zz}\omega_{z0}^2 && (19) \end{aligned}$$

In (19), E_k is the rotational kinetic energy, and $(\omega_{x0}, \omega_{y0}, \omega_{z0})$ stand for initial angular velocity values. Hence, given the initial angular velocity values, one can compute the conserved value of $2E_k$. To express the conservation of angular momentum, one can compute $(14)I_{xx}\omega_x + (15)I_{yy}\omega_y + (16)I_{zz}\omega_z$:

$$I_{xx}^2\dot{\omega}_x\omega_x + I_{yy}^2\dot{\omega}_y\omega_y + I_{zz}^2\dot{\omega}_z\omega_z = 0 \quad (20)$$

If (20) is integrated with respect to time, the square of the magnitude of angular momentum vector, ${H_b}^2$, can be obtained:

$$\begin{aligned} H_b^2 &= I_{xx}^2\omega_x^2 + I_{yy}^2\omega_y^2 + I_{zz}^2\omega_z^2 && (21) \\ &= I_{xx}^2\omega_{x0}^2 + I_{yy}^2\omega_{y0}^2 + I_{zz}^2\omega_{z0}^2 && (22) \end{aligned}$$

Similarly, given the initial angular velocity values, one can compute the conserved value of H_b^2 using (22). Without the loss of generality, one can consider $I_{xx} > I_{yy} > I_{zz}$, y-axis being the intermediate axis. Thus, ω_y is primarily isolated.

B. Isolating the Variable ω_y

Using (15), $\dot{\omega}_y^2$ can be expressed as in the following:

$$\dot{\omega}_y^2 = \left(\frac{I_{zz} - I_{xx}}{I_{yy}}\right)^2 \omega_z^2\omega_x^2 \quad (23)$$

Furthermore, (18) and (21) can be collectively processed to obtain ω_x^2 and ω_z^2, in terms of ω_y and the conserved quantities H_b^2 and $2E_k$:

$$\omega_x^2 = \frac{H_b^2 - 2E_kI_{zz} - I_{yy}\omega_y^2(I_{yy} - I_{zz})}{I_{xx}(I_{xx} - I_{zz})} \quad (24)$$
$$\omega_z^2 = \frac{2E_kI_{xx} - H_b^2 - I_{yy}\omega_y^2(I_{xx} - I_{yy})}{I_{zz}(I_{xx} - I_{zz})} \quad (25)$$

Plugging (24) and (25) into (23), one could obtain the final differential equation that solely comprises ω_y:

$$\dot{\omega}_y^2 = a_y\omega_y^4 + b_y\omega_y^2 + c_y \quad (26)$$

where the coefficients a_y, b_y, and c_y take the following form:

$$\begin{aligned} b_y &= \frac{(I_{yy}-I_{zz})(H_b^2-2E_kI_{xx}) - (I_{xx}-I_{yy})(H_b^2-2E_kI_{zz})}{I_{xx}I_{yy}I_{zz}} \\ a_y &= \frac{(I_{xx} - I_{yy})(I_{yy} - I_{zz})}{I_{xx}I_{zz}} \\ c_y &= \frac{(2E_kI_{xx} - H_b^2)(H_b^2 - 2E_kI_{zz})}{I_{xx}I_{zz}I_{yy}^2} \end{aligned} \quad (27)$$

Considering the case $I_{xx} > I_{yy} > I_{zz}$, it is observed that a_y is always positive. Furthermore, $(H_b^2 - 2E_kI_{zz}) = I_{xx}W_{x0}^2(I_{xx} - I_{zz}) + I_{yy}W_{y0}^2(I_{yy} - I_{zz})$ and $(H_b^2 - 2E_kI_{xx}) = I_{yy}W_{y0}^2(I_{yy} - I_{xx}) + I_{zz}W_{z0}^2(I_{zz} - I_{xx})$; thus, b_y is always negative. Likewise, c_y is always positive. Therefore, regardless of the initial angular velocity values, the coefficients of the differential equation (26) meets the criteria of $a_y > 0$, $b_y < 0$, and $c_y > 0$, concerning the intermediate axis.

C. Isolating the Variable ω_z

Using (16), ω_z^2 can be express as in the following:

$$\dot{\omega}_z^2 = \left(\frac{I_{xx} - I_{yy}}{I_{zz}}\right)^2 \omega_x^2\omega_y^2 \quad (28)$$

Similar to the previous case, (18) and (21) can be collectively processed to obtain ω_x^2 and ω_y^2, in terms of ω_z and the conserved quantities H_b^2 and $2E_k$:

$$\omega_x^2 = \frac{H_b^2 - 2E_kI_{yy} + I_{zz}\omega_z^2(I_{yy} - I_{zz})}{I_{xx}(I_{xx} - I_{yy})} \quad (29)$$
$$\omega_y^2 = \frac{H_b^2 - 2E_kI_{xx} + I_{zz}\omega_z^2(I_{xx} - I_{zz})}{I_{yy}(I_{yy} - I_{xx})} \quad (30)$$

Combining (29) and (30) in (28), the following differential equation is obtained:

$$\dot{\omega}_z^2 = a_z\omega_z^4 + b_z\omega_z^2 + c_z \tag{31}$$

where the coefficients a_z, b_z, and c_z are computed as follows:

$$\begin{aligned} b_z &= \frac{H_b^2(2I_{zz} - I_{xx} - I_{yy}) - 2E_k(I_{zz}(I_{xx} + I_{yy}) - 2I_{xx}I_{yy})}{I_{xx}I_{yy}I_{zz}} \\ a_z &= \frac{(I_{zz} - I_{xx})(I_{yy} - I_{zz})}{I_{xx}I_{yy}} \\ c_z &= \frac{(2E_kI_{xx} - H_b^2)(H_b^2 - 2E_kI_{yy})}{I_{xx}I_{yy}I_{zz}^2} \end{aligned} \tag{32}$$

Recall that $I_{xx} > I_{yy} > I_{zz}$, and thus a_z is always negative. However, unlike the case for the intermediate axis, the signs of b_z and c_z depend on the initial angular velocity values.

D. Problem Definition

By respectively isolating ω_y and ω_z, the solution of (14)-(16) is reduced to the solution of (26) and (31). Although being nonlinear, the eqs. (26) and (31) have no coupling terms; (26) only includes ω_y as the variable and (31) only includes ω_z as the variable. In the next subsection, it is demonstrated that they can be solved via Jacobi elliptic functions. Once (26) and (31) are solved, ω_x can be obtained rather easily.

IV. Analytical Solution

Observing the analogy between the eqs. (8)-(10), (26), and (31), it is natural to think that the analytical solutions may appear in the form of Jacobi elliptic functions.

A. Analytical Solution for ω_y

To obtain the analytical solution for ω_y, one must solve (26). As previously discussed, the coefficients of (26) obey $a_y > 0$, $b_y < 0$, and $c_y > 0$, regardless of the initial conditions. Considering the fact that $0 < m < 1$ for the elliptic functions, (26) can only be similar to (8). Therefore, the following function is proposed for the analytical solution of ω_y:

$$\omega_y = \mu_y \operatorname{sn}(u,m) \tag{33}$$

where u is described as $u = q_0 + qt$; t is the time variable [10], q and q_0 are constants. The parameter μ_y denotes the amplitude of ω_y. The first time differentiation of (33) and its square is expressed as below:

$$\dot{\omega}_y = \frac{d\omega_y}{dt} = \frac{d\omega_y}{du}\frac{du}{dt} = q\frac{d\omega_y}{du} \tag{34}$$

$$\dot{\omega}_y^2 = q^2\left(\frac{d\omega_y}{du}\right)^2 = q^2\mu_y^2\left(\frac{d}{du}\operatorname{sn}\right)^2 \tag{35}$$

Again, the function parameters (u,m) are dropped to simplify the notation. Plugging (8) into (35), the following is yielded:

$$\dot{\omega}_y^2 = m\mu_y^2q^2\operatorname{sn}^4 - (1+m)\mu_y^2q^2\operatorname{sn}^2 + \mu_y^2q^2 \tag{36}$$

Using the one-on-one correspondence between (26) and (36), the parameters a_y, b_y, and c_y can be computed as in the following:

$$a_y = m\mu_y^{-2}q^2 \tag{37}$$

$$b_y = q^4(1+m)^2 \tag{38}$$

$$c_y = \mu_y^2q^2 \tag{39}$$

Solving the eqs. (37)-(39), the parameter m appears to have two solutions:

$$m = \frac{1}{2a_yc_y}\left(\mp b_y\Delta_y - 2a_yc_y + b_y^2\right) \tag{40}$$

In (40), $\Delta_y = \sqrt{b_y^2 - 4a_yc_y}$. Despite the multiple mathematical solutions, only the one with the plus sign obeys the condition $0 < m < 1$, thus the other solution is discarded. Having obtained the parameter m, the rest of the parameters are computed subsequently:

$$q = \kappa_1\sqrt{\frac{-b_y}{m+1}} \tag{41}$$

$$\mu_y = \sqrt{\frac{c_y}{q^2}} \tag{42}$$

Despite the presence of multiple solutions, it is determined that $\kappa_1 = -\operatorname{sign}(W_{x0})\operatorname{sign}(W_{z0})$ yields a physically consistent solution. Furthermore, q_0 can be computed by using the initial value ω_{y0}:

$$q_0 = \operatorname{arcsn}(\frac{\omega_{y0}}{\mu_y}, m) \tag{43}$$

B. Analytical Solution for ω_z and ω_x

Unlike in the case of intermediate y-axis, the coefficients of (31) depend on the initial conditions. Yet, it is possible to compute a_z, b_z, and c_z, given the initial conditions. For a given set of initial conditions, if $c_z < 0$ the solution of ω_z is in the form dn; otherwise, it is cn. The ellipse parameters m, q, q_0 should remain the same [1], [11]; hence, only the computation of μ_z is required for two distinct cases.

1) Case 1: $c_z < 0$: In this case, the proposed solution is in the form of dn; see (10):

$$\omega_z = \mu_z \operatorname{dn}(u,m) \tag{44}$$

where μ_z denotes the amplitude of ω_z. The first time differentiation of (44) and its square are expressed as below:

$$\dot{\omega}_z = \frac{d\omega_z}{dt} = \frac{d\omega_z}{du}\frac{du}{dt} = q\frac{d\omega_z}{du} \tag{45}$$

$$\dot{\omega}_z^2 = q^2\left(\frac{d\omega_z}{du}\right)^2 = q^2\mu_z^2\left(\frac{d}{du}\operatorname{dn}\right)^2 \tag{46}$$

Combining (10) and (46), the following is obtained:

$$\dot{\omega}_z^2 = -\mu_z^2 q^2 \,\mathrm{dn}^4 + \mu_z^2 q^2 (2-m)\,\mathrm{dn}^2 + \mu_z^2 q^2 (m-1) \quad (47)$$

Using the one-on-one correspondence between (31) and (47), the parameters a_z, b_z, and c_z can be calculated. As the parameter μ_z is solely needed, one can use one of the parameters to compute it. Choosing a_z, the parameter μ_z is yielded as below:

$$\mu_z = -\operatorname{sign}(\omega_{x0}) q \sqrt{-\frac{1}{a_z}} \quad (48)$$

Having obtained ω_y and ω_z, it is rather straightforward to compute ω_x by using the eq. (14) in the following form:

$$\omega_x = \frac{I_{yy} - I_{zz}}{I_{xx}} \mu_y \mu_z \int \mathrm{sn}\,\mathrm{dn}\, dt \quad (49)$$

Recall that $dt = \frac{1}{q} du$, the integration in (49) can be taken with the help of (6) to obtain ω_x:

$$\omega_x = \frac{\mu_y \mu_z}{q} \frac{(I_{zz} - I_{yy})}{I_{xx}} \mathrm{cn}(u,m) \quad (50)$$

2) Case 2: $c_z > 0$: In this case, the proposed solution is in the form of cn; see (9):

$$\omega_z = \mu_z \,\mathrm{cn}(u,m) \quad (51)$$

The first time differentiation of (51) and its square are computed as follows:

$$\dot{\omega}_z = \frac{d\omega_z}{dt} = \frac{d\omega_z}{du}\frac{du}{dt} = q\frac{d\omega_z}{du} \quad (52)$$

$$\dot{\omega}_z^2 = q^2 \left(\frac{d\omega_z}{du}\right)^2 = q^2 \mu_z^2 \left(\frac{d}{du}\mathrm{dn}\right)^2 \quad (53)$$

Plugging (10) into (53), the following is obtained:

$$\dot{\omega}_z^2 = -\mu_z^2 q^2 m\,\mathrm{cn}^4 + \mu_z^2 q^2 (2m-1)\,\mathrm{cn}^2 + \mu_z^2 q^2 (1-m) \quad (54)$$

Similar to the previous case, the parameters a_z, b_z, and c_z can be calculated by using the one-on-one correspondence between (31) and (54). Choosing a_z, the parameter μ_z is obtained as in the following:

$$\mu_z = -\operatorname{sign}(\omega_{x0}) q \sqrt{-\frac{m}{a_z}} \quad (55)$$

Following a similar procedure, ω_x can be expressd via (14):

$$\omega_x = \frac{I_{yy} - I_{zz}}{I_{xx}} \mu_y \mu_z \int \mathrm{sn}\,\mathrm{cn}\, dt \quad (56)$$

Using (7), the closed-form solution of (56) is acquired:

$$\omega_x = \frac{1}{m}\frac{\mu_y \mu_z}{q} \frac{(I_{zz} - I_{yy})}{I_{xx}} \mathrm{dn}(u,m) \quad (57)$$

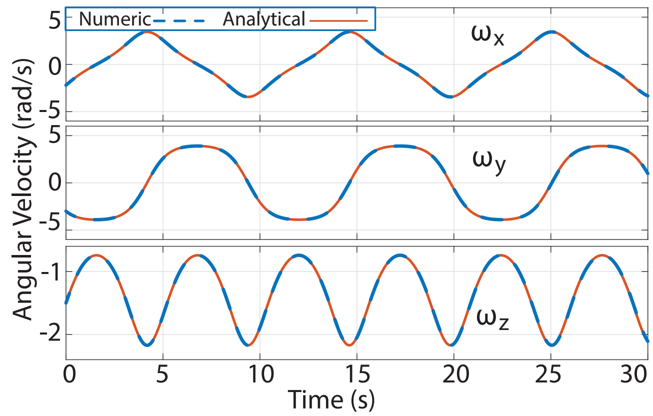

Fig. 1: Numerical simulation 1: $\omega_{x0} = 2.2$ rad/s, $\omega_{y0} = -3.0$ rad/s, $\omega_{z0} = -1.5$ rad/s.

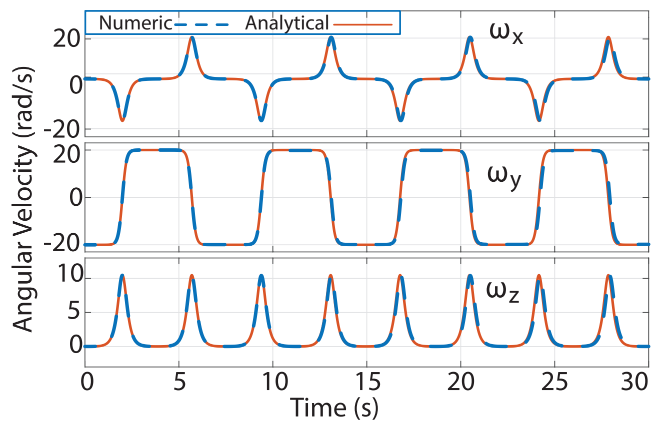

Fig. 2: Numerical simulation 2: $\omega_{x0} = 0.01$ rad/s, $\omega_{y0} = 20.0$ rad/s, $\omega_{z0} = 0.01$ rad/s.

V. Results

A. Numerical Simulations

In order to verify the proposed closed-form analytical solution, a series of numerical simulations were conducted. In these simulations, NASA's Cassini spacecraft was considered as the rigid body model. Its moment of inertiae about its principle axes are $I_{xx} = 8802$ kgm^2, $I_{yy} = 8155$ kgm^2, and $I_{zz} = 4715$ kgm^2, respectively [12]. The products of inertia are negligible. The results are displayed in Figs. 1 and 2, where dashed blue lines indicate numerical solutions and solid red lines indicate the proposed analytical solution. Numerical solutions are obtained via MATLAB ode45 solver in which the step size was 1 ms.

In the first numerical simulation, the initial angular velocities were set as $\omega_{x0} = 2.2$ rad/s, $\omega_{y0} = -3.0$ rad/s, $\omega_{z0} = -1.5$ rad/s. In the second numerical simulation, they were set as $\omega_{x0} = 0.01$ rad/s, $\omega_{y0} = 20.0$ rad/s, $\omega_{z0} = 0.01$ rad/s to simulate Dzhanibekov effect [6]. In both simulations, the difference between both solution is indistinguishable. For longer simulations, the numerical solution inevitably drifts while the proposed analytical solution exhibits no such behavior at all. In [8], a Code Ocean repository is provided

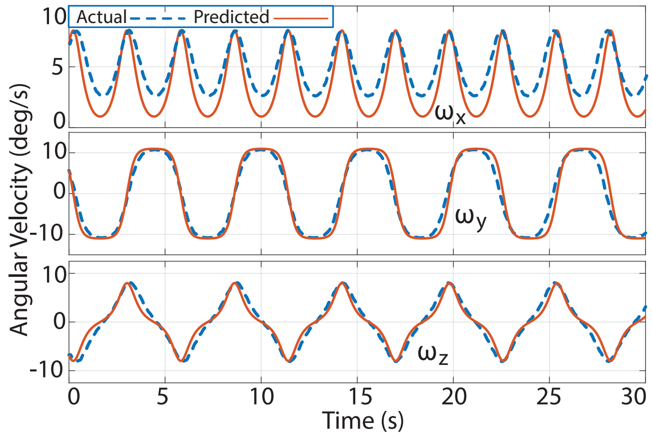

Fig. 3: Angular velocity trajectories of Aist spacecraft [9].

so that the reader could test the proposed analytical solution for different initial conditions and inertia values.

B. Real-life Experiment Results

In [9], the researchers have reported their findings regarding the uncontrolled rotational motion of an actual spacecraft Aist. Using the initial angular velocity values ($\omega_{x0} = 6.763$ deg/s, $\omega_{y0} = -5.672$ deg/s, $\omega_{z0} = 6.752$ deg/s), angular velocity trajectories were estimated using the proposed analytical solution. Aist's moment of inertia values were not reported; instead, its successor Aist-2's inertia values were slightly tweaked [13]: $I_{xx} = 275$ kgm^2, $I_{yy} = 235$ kgm^2, and $I_{zz} = 172$ kgm^2, respectively. The products of inertia are omitted. The result is presented in Fig. 3, where dashed blue and solid red lines respectively stand for the actual angular velocity and the proposed analytical solution's prediction. As may be observed, the actual angular velocity trajectories are well predicted with sufficient accuracy, although there is noticeable error for the x-axis angular velocity. This could be due to the fact that Aist's moment of inertia values were guessed by considering the inertia of its successors spacecraft. Despite this error, it is observed that the proposed solution could predict the attitude motion of spacecrafts which are subject to Euler-Poinsot motion.

VI. Conclusion

In this manuscript, a closed form analytical solution for torque free precession is proposed. In this motion type, a rigid body with no body-fixed torques may perform rotational motion and given the initial conditions, the angular velocity trajectories of the rigid body can be obtained using the proposed solution. The validity of the proposed solution was demonstrated via numerical solutions and real-life spacecraft data. In [8], a Code Ocean repository is provided where a MATLAB implementation concerning the proposed solution can be found. Interesting readers could play with the code; they can test the solution with different initial conditions and inertia values.

References

[1] L. Poinsot, *Theorie Nouvelle de la Rotation des Corps*. Bachelier, Paris, 1834.

[2] H. Goldstein, C. P. Poole, and J. L. Safko, "Classical mechanics, 3^{rd}*ed.*." Addison Wesley, 2000, pp. 74–110.

[3] E. J. Routh, "Motion of a body under the action of no forces," in *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies - Being part II of a treatise on the whole subject*. MacMillan and Co., Ltd, 1905, pp. 74–110.

[4] A. H. Jupp, "On the free rotation of a rigid body," *Celestial Mechanics*, vol. 9, no. 1, pp. 3–20, Mar. 1974.

[5] J. E. Hurtado and A. J. Sinclair, "State transition matrix, motion constants, and ergodicity of the Euler–Poinsot problem," *Nonlinear Dynamics*, vol. 85, no. 3, pp. 2049–2063, May 2016.

[6] C. Peterson and W. Schwalm, "Euler's rigid rotators, Jacobi elliptic functions, and the Dzhanibekov or tennis racket effect," *American Journal of Physics*, vol. 89, no. 4, pp. 349–357, Apr. 2021.

[7] C. Murakami, "Analytical solution of the Euler-Poinsot problem," *Journal of Geometry and Symmetry in Physics*, vol. 60, pp. 25–46, 2021.

[8] Barkan Ugurlu, "Analytical solution to torque free precession [source code]," 2023. [Online]. Available: https://codeocean.com/capsule/6311625/tree/v3

[9] V. I. Abrashkin, K. E. Voronov, A. V. Piyakov, Y. Y. Puzin, V. V. Sazonov, N. D. Semkin, A. S. Filippov, and S. Y. Chebukov, "Uncontrolled rotational motion of the aist small spacecraft prototype," *Cosmic Research*, vol. 55, no. 2, pp. 128–141, Mar. 2017.

[10] W. A. Schwalm, *Lectures on Selected Topics in Mathematical Physics: Elliptic Functions and Elliptic Integrals*. IOP Publishing, 2015. [Online]. Available: https://doi.org/10.1088/978-1-6817-4230-4

[11] L. Landau and E. Lifshitz, *Mechanics: Course of Theoretical Physics*. Elsevier, 1976.

[12] A. Y. Lee and J. A. Wertz, "In-flight estimation of the Cassini spacecraft's inertia tensor," *Journal of Spacecraft and Rockets*, vol. 39, no. 1, pp. 153–155, Jan. 2002.

[13] V. F. Petrishchev and M. G. Shipov, "Optimization of spatial turns of AIST-2 small spacecraft on the basis of the principle of minimum control," *VESTNIK of Samara University, Aerospace and Mechanical Engineering, (in Russian)*, vol. 14, no. 4, pp. 72–79, Jan. 2016.