Channel Coding Method Based on Weighted Probability Model

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Abstract

Â X is a discrete memoryless binary source , Sequence X undergoes lossless transformation to satisfy the requirement â\euroœEach 1 is separated by one or more zeros“, ”Each 0 is separated by one or two 1s” or similar condition , These conditions are the decision conditions for error detection in channel transmission. A new channel coding method is proposed based on a weighted probability model for lossless coding , The encoding rate and encoding and decoding steps of different conversion methods are different , It is proved that the decoding error probability can reach 0 when the code length is long enough. In the simulation experiment of BPSK signal in AWGN channel, At 0.5 bit rate, the proposed method improves 1.1dB over Polar code when FER is 0.001, and 1.4dB over Polar code when FER is 0.0001.Â

Channel Coding Method Based on Weighted Probability Model

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Abstract: X is a discrete memoryless binary source , Sequence X undergoes lossless transformation to satisfy the requirement "Each 1 is separated by one or more zeros", "Each 0 is separated by one or two 1s" or similar condition , These conditions are the decision conditions for error detection in channel transmission. A new channel coding method is proposed based on a weighted probability model for lossless coding , The encoding rate and encoding and decoding steps of different conversion methods are different , It is proved that the decoding error probability can reach 0 when the code length is long enough. In the simulation experiment of BPSK signal in AWGN channel, At 0.5 bit rate, the proposed method improves 1.1dB over Polar code when FER is 0.001, and 1.4dB over Polar code when FER is 0.0001.

Keywords: Information entropy, Channel capacity, Data validation, Forward error correction

1 Introduction

Set the binary source sequence X with bit number n, So there are 2^n possibilities for sequence X,If the agreed sequence X must meet the requirement of "Each symbol 1 is separated by one or more zeros", "Each 0 is separated by one or two 1s "or similar conditions, So these conditions become the decision conditions for error detection during channel transmission . For any source sequence X the above conditions can be satisfied by transformation. Taking $X = (0,1,1,0,1,0,0,1,1,1,0,1,0,0,0,0\ldots)$ as an example , The lossless transformation and inverse transformation methods are as follows :

（a）Replace the symbol 1 in X with "10" from left to right to get the sequence$Y$,the

$$Y = (0,1,0,1,0,0,1,0,0,0,1,0,1,0,1,0,0,1,0,0,0,0,0\ldots)$$

Replacing the "10" in Y with 1 from left to right restores X.

（b）Replacing symbol 1 with "101"and symbol 0 with "01" in X from left to right yields sequence Y,the

$$Y = (0,1,1,0,1,1,0,1,0,1,1,0,1,0,1,0,1,1,0,1,1,0,1,1,0,1\ldots)$$

Replacing "101" in Y with 1 and "01" with 0 from left to right can restore X.

（c）Replacing symbol 1 with "010"and symbol 0 with "10" in X from left to right yields sequence Y,the

$$Y = (1,0,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,0,1,0,0,1,0\ldots)$$

Replacing "010" in Y with 1 and "10" with 0 from left to right can restore X.

Record the length of Y as l , The purpose of sequence conversion is to ensure that Y meets the conditions for data validation.（a）data verification conditions are :

$$\text{The number of adjacent 1 cannot exceed 1} \tag{1-1}$$

（b）the data verification condition is:

$$\text{No more than 1 adjacent 0 and no more than 2 adjacent 1} \tag{1-2}$$

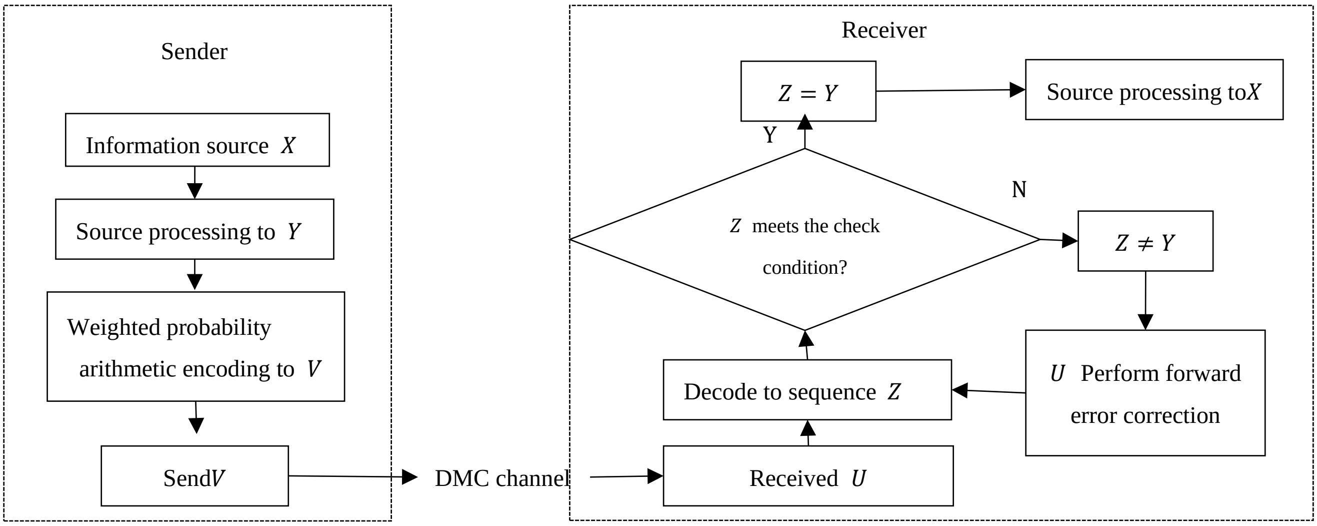
（c）the data verification condition is:

$$\text{No more than 2 adjacent 0 and no more than 1 adjacent 1} \tag{1-3}$$

(b)、(c) has one more verification condition than(a), Different transformation methods enable sequence Y to have more validation conditions, Call the transformation method source processing. Source processing results in a large amount of redundant information in Y, Due to the obvious linear contextual relationship between the symbols in Y, You can choose common entropy encoding (such as run-length encoding[1][2], dictionary encoding[3], Huffman encoding[4], or arithmetic encoding[5][6]) to compress Y, Code words with different bit lengths need to be assigned to "0", "10", "101", "01", or"010", "10", However, during decoding, the(1-1), (1-2), and (1-3) data verification conditions will be lost. So we constructed a new arithmetic coding method based on weighted probability[7][8], Defined as weighted probability arithmetic encoding. This method can still meet the data verification conditions when performing lossless decoding with Y, The analysis in Chapter 3 shows that the weighted probability arithmetic coding has a significantly better bitrate than standard arithmetic coding.

The channel transmission model based on weighted probability arithmetic encoding and decoding is shown in Figure 1. The sender uses weighted probability arithmetic encoding for lossless compression of Y, When decoding at the receiving end, data verification and error correction verification are achieved through (1-1), (1-2), and (1-3) conditions. Convert the encoding result L_l of sequence Yinto a binary sequence of length m $V = (v_1, v_2, \ldots, v_m)$, V transmitted through DMC channel[9][10], $U = (u_1, u_2, \ldots, u_m)$ is the received binary sequence, When transmitting without error$U = V$.

Figure 1 Weighted Probability Arithmetic Decoding Channel Transmission Model



Decoded binary sequence at the receiving end in Figure 1 $Z = (z_1, z_2, \dots, z_l)$,When the sequence Z satisfies conditions such as (1-1), (1-2) and (1-3), there is a probability P_{err} that $Z \neq Y$,So P_{err} is the average decoding error probability [11][12] of weighted probability arithmetic decoding. When Z does not meet the conditions such as (1-1), (1-2), and (1-3), there must be $Z \neq Y$.

2 Weighted probability arithmetic encoding and decoding

Definition2.1 Set discrete memoryless source $Y = (y_1, y_2, \dots, y_l)$, $y_i \in A = \{0,1,\dots,k\}$, $i = 1,2,\dots,l$,Then the weighted probability mass function is $\varphi(y_i) = rp(y_i)$, $p(y_i)$ is the probability mass function,$0 \leq p(y_i) \leq 1$,r is the weight coefficient, and

$$F(y_i) = \sum_{y \leq y_i} p(y) \tag{2-1}$$

Definition2.2 The weighted cumulative distribution function is defined as

$$F(y_i, r) = \sum_{y \leq y_i} rp(y) = r \sum_{y \leq y_i} p(y) = rF(y_i) \tag{2-2}$$

Since Y is a binary sequence, So $A = \{0,1\}$,Then $y_i \in \{0,1\}$,When $y_i = 0$,$F(y_i - 1, r) = rF(y_i - 1) = rp(-1) = 0$;When $y_i = 1$,$F(y_i - 1, r) = rF(y_i - 1) = \varphi(0) = rp(0)$.Set $R_0 = 1, L_0 = 0$,then the ith（$i = 1,2,3,\dots$）symbol y_i The weighted probability arithmetic encoding formula for i is defined as:

$$\begin{gathered} R_i = R_{i-1}\varphi(y_i) \\ L_i = L_{i-1} + R_{i-1}F(y_i - 1, r) \\ H_i = L_i + R_i \end{gathered} \tag{2-3}$$

Among them R_i、L_i and H_i is a real arithmetic variable, After y_l finish coding,L_l is the encoding result, which is equivalent to V in Figure 1. According to (2-3) weighted probability arithmetic coding steps:

Algorithm(1):Weighted Probability Arithmetic Coding Based on Y
1:Initialization parameter:$R \leftarrow 1, L \leftarrow 0, i \leftarrow 1$
2: Gets the ith symbol y_i in the sequence Y
3:If $y_i = 0$,$R \leftarrow R\varphi(0)$
4: If $y_i = 1$,$R \leftarrow R\varphi(1)$ and $L \leftarrow L + R\varphi(0)$
5: $i \leftarrow i + 1$,If $i \leq l$, repeat 2 to 5
6: Converts the real number L to a binary sequence V of length m
7: Send V,r,l,$p(0)$ or $p(1)$（because $p(0) + p(1) = 1$, Known one can be decoded）

When r,l,$p(0)$ or $p(1)$ is known at the receiving end, according to (2-3) If $y_i = 0$,$F(0 - 1, r) = rF(-1)$,because$-1 \notin \{0,1\}$, so

$F(-1) = 0.L_i = L_{i-1}$ and $R_i = R_{i-1}\varphi(0)$.When $y_i = 1$,$L_i = L_{i-1} + R_{i-1}F(0,r) = L_{i-1} + R_{i-1}\varphi(0)$ and $R_i = R_{i-1}\varphi(1)$.Because L_i is monotonically constant, So $y_i = 1$ if $L_l \geq L_{i-1} + R_{i-1}\varphi(0)$,Otherwise $y_i = 0$.So when decoding y_i, order

$$T_i = L_{i-1} + R_{i-1}\varphi(0) \tag{2-4}$$

T_i is the decision threshold for decoding y_i.Let the receiving binary sequence U,When lossless decoding,U is equivalent to the real number L_l, then the decoding steps are:

Algorithm(2)：Weighted Probability Arithmetic Decoding Based on U
1: Initialization parameter:$R \leftarrow 1$,$L \leftarrow 0$, $i \leftarrow 1$,$T \leftarrow 0$
2: $T \leftarrow L + R\varphi(0)$
3: When $U \geq T$,$z_i \leftarrow 1$,$R \leftarrow R\varphi(1)$ and $L \leftarrow L + R\varphi(0)$
4: When $U < T$,$z_i \leftarrow 0$,$R \leftarrow R\varphi(0)$
5: $i \leftarrow i + 1$,if $i \leq l$ repeat 2 to 5
6: Get sequence $Z = (z_1, z_2, \ldots, z_l)$

Lossless coding and decoding is possible when $U = V$,X is obtained by Y inverse source processing, as shown in experiment (1) in Chapter 5.From definition 2.1, it is not difficult to conclude that there are three basic cases of weight coefficient r: $0 < r < 1$;$r = 1$;$r > 1$.Take source processing $r > 1$ scheme (a) as an example, at this time Y is satisfied (1-1), and the probability of symbol 0 and symbol 1 in the sequence $Y$are respectively $p(0) = p$,$p(1) = 1 - p$. The process of encoding $y_{i+1} = 0$,$y_{i+2} = 1$ and $y_{i+3} = 0$ according to (2-3) is shown in Figures 2 and 3.

FIG. 2 Schematic diagram of the coding process of the weighted model that can be lossless restored

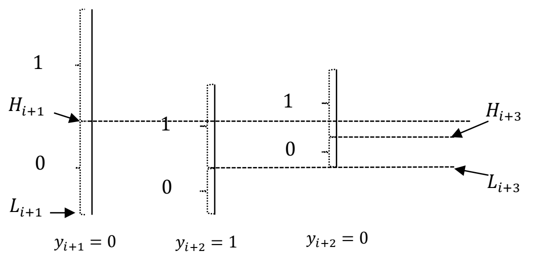
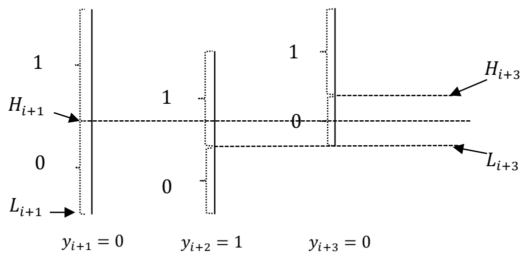


FIG. 3 Schematic diagram of the coding process of the weighted model that may not be restored



In Figure 3, due to L_i is monotonous and does not decrease, y_{i+1} is judged as 1 by (2-4) when $L_l \geq H_{i+1}$ which is a decoding error. Under the premise of lossless encoding and decoding r, there is a maximum value.According to Figure 2,$L_{i+3} = L_i + R_i r^2 p^2$，$R_{i+3} = R_i r^3 p^2 (1-p)$,$H_{i+3} = L_i + R_i r^2 p^2 + R_i r^3 p^2 (1-p)$.Because $y_{i+1} = 0$,$F(-1) = 0$,So $L_{i+1} = L_i$,$R_{i+1} = R_i rp$,$H_{i+1} = L_i + R_i rp$.Let $H_{i+3} \leq H_{i+1}$ get:

$$\varphi(0)\varphi(1) + \varphi(0) = r^2 p(1-p) + rp \leq 1 \tag{2-5}$$

Let the equation $ar^2 + br + c = 0$,where $a = p(1-p)$,$b = p$,$c = -1$,and $r > 0$.The positive real roots satisfying the equation are, $\frac{\sqrt{p^2+4(1-p)p}-p}{2(1-p)p}$,simplify:

$$r \leq \frac{\sqrt{4/p-3}-1}{2-2p} \tag{2-6}$$

Theorem2.3 Source sequence Y satisfies (1-1),p is the probability of symbol 0 in Y,When $0 < r \leq r_{max} < \frac{1}{p}$,$r_{max} = \frac{\sqrt{4/p-3}-1}{2-2p}$, weighted probability arithmetic encoding can lossless decode sequence Y through V.

proof: If L_i（$i \geq 1, i \in Z$）can be decoded lossless decoded $y_1, y_2, \ldots, y_i$,because the sequence Y satisfies(1-1),two cases are proved by induction method.

（1）$y_i = 1$,there is no $y_{i+1} = 1$,When $y_{i+1} = 0$ according to (2–3) encoding $L_{i+1} = L_i$，$R_{i+1} = R_i rp$ can be obtained. because $L_{i+1} = L_i$,So L_{i+1} can lossless decode $y_1, y_2, \ldots, y_i$.Also due to $R_i rp > 0$，$L_{i+1} < L_i + R_i rp$, so the decoding is obtained $y_{i+1} = 0$.

（2）$y_i = 0$, according to (2-3) coding can be obtained $L_i = L_{i-1}$,$R_i = R_{i-1} rp$,L_{i-1}Lossless decoding $y_1, y_2, \ldots, y_i$.When $y_{i+1} = 0$

according to (2−3) encoding $L_{i+1} = L_i$,$R_{i+1} = R_i rp$,because $L_{i+1} = L_i$,So L_{i+1} can lossless decode $y_1, y_2, \ldots, y_i$. Also due to $L_{i+1} < L_i + R_i rp$, so the decoding is obtained $y_{i+1} = 0$.

When $y_{i+1} = 1$ according to (2−3) encoding $L_{i+1} = L_i + R_i r^2 p^2$, $R_{i+1} = R_{i-1} r^2 p(1-p)$ .Due to $rp < 1$,So $L_{i+1} < L_i + R_i rp$,Decoded $y_i = 0$ .Due to $y_i = 0$, and $L_{i+1} + R_{i+1} rp = L_i + R_i r^2 p^2$,So $L_{i+1} = L_{i+1} + R_{i+1} rp$,Decoded $y_{i+1} = 1$ .When $y_i = 0$,$y_{i+1} = 1$,If it can be shown that y_{i-1}can be decoded correctly, then L_{i+1}can decoded $y_1, y_2, \ldots, y_i, y_{i+1}$ lossless by induction.

When $y_{i-1} = 0$,$y_i = 0$,$y_{i+1} = 1$, encoding yields $L_i = L_{i-1} = L_{i-2}$, $L_{i+1} = L_{i-2} + R_{i-2} r^3 p^3$ .Due to $rp < 1$,So $L_{i+1} < L_{i-2} + R_{i-2} rp$,Decode $y_{i-1} = 0$.When $y_{i-1} = 1$,$y_i = 0$,$y_{i+1} = 1$ encoding yields $L_{i-1} = L_{i-2} + R_{i-2} rp$,$L_i = L_{i-1}$,$L_{i+1} = L_i + R_i rp = L_{i-2} + R_{i-2} rp + R_{i-1} r^3 p^2 (1-p)$.Obviously, $L_{i+1} > L_{i-2} + R_{i-2} rp$, decoding $y_{i-1} = 1$.That is,y_{i-1} can be decoded correctly.

Extending theorem 2.3,Let $t + 2(t = 1,2,3, \ldots)$ symbols from the $i + 1$ position in the sequence Y be 0,1,…,1,0,where the number of adjacent symbols 1 is t.According to (2-3) there is:

$$H_{i+t+2} = L_i + R_i \varphi(0)^2 + R_i \varphi(0)^2 \varphi(1) + R_i \varphi(0)^2 \varphi(1)^2 + R_i \varphi(0)^2 \varphi(1)^3 + \cdots + R_i \varphi(0)^2 \varphi(1)^{t-1} + R_i \varphi(0)^3 \varphi(1)^t$$

$$H_{i+1} = L_i + R_i \varphi(0)$$

It is obtained by $H_{i+t+2} \leq H_{i+1}$:

$$\varphi(0) + \varphi(0)\varphi(1) + \varphi(0)\varphi(1)^2 + \cdots + \varphi(0)\varphi(1)^t \leq 1 \tag{2-7}$$

When $\varphi(1) = 1$:

$$\varphi(0) \leq \frac{1}{t+1} \tag{2-8}$$

The value of r_{max} when $t \geq 1$ can be obtained by solving the equation (2-7), as shown in Table 1, the larger t is, the closer $r_{max}$is to 1.

Table 1 Set the relationship between t and r_{max} when $p(0) = p(1)$ in sequence Y

t	r_{max}	t	r_{max}
1	1.236067	5	1.008276
2	1.087378	6	1.004034
3	1.037580	7	1.001988
4	1.017320	8	1.000986

Theorem 2.4 Let adjacent 1 in sequence Ynot exceed t（$t \geq 1$）,When $\varphi(1) = 1$ and $\varphi(0) \leq \frac{1}{t+1}$Weighted probability arithmetic encoding can losslessly decode sequence Y through V.

proof Let d be the number of consecutive symbols 1 in sequence Y,$0 \leq d \leq t$.When $d = 0$, according to (2-3) can be obtained $L_{i+t+2} = L_{i+t+1} = L_{i+t} = \cdots = L_i$,Because $L_{i+t+2} < L_i + R_i \frac{1}{(t+1)^{t+2}} < L_i + R_i \frac{1}{(t+1)^{t+1}} < \cdots < L_i + R_i \frac{1}{t+1}$, So $y_{i+1} = y_{i+2} = \cdots = y_{i+t+2} = 0$.When $1 \leq d \leq t$,$\varphi(1) = 1$,$\varphi(0) \leq \frac{1}{t+1}$. according to (2-3) can be obtained $L_{i+d+2} = L_i + R_i \frac{d}{(t+1)^2} \leq L_i + R_i \frac{t}{(t+1)^2}$. When decoding, because $L_i + R_i \frac{t}{(t+1)^2} < L_i + R_i \frac{1}{t+1}$, L_{i+d+2} can accurately decode $y_{i+1} = 0$ .Due to $y_{i+1} = 0$, so $L_{i+1} + R_{i+1}\varphi(0) = L_i + R_i \varphi(0)^2 = L_i + R_i \frac{1}{(t+1)^2}$,Due to $\frac{d}{(t+1)^2} \geq \frac{1}{(t+1)^2}$, so $L_{i+d+2} \geq L_i + R_i \frac{1}{(t+1)^2}$,$L_{i+d+2}$can accurately decode $y_{i+2} = 1$ .When $d \geq 2$,$L_i + R_i \frac{d}{(t+1)^2} \geq L_i + R_i \frac{d}{(t+1)^2}$,$L_{i+d+2}$ can accurately decode $y_{i+d+1} = 1$.Also due to $L_{i+d+2} = L_{i+d+1}$ and $L_{i+d+2} < L_i + R_i \frac{d+1}{(t+1)^2}$,$L_{i+d+2}$ can accurately decode $y_{i+d+2} = 0$.When $d = t + 1$,$L_i + R_i \frac{t+1}{(t+1)^2} = L_i + R_i \frac{1}{t+1} > L_i + R_i \frac{t}{(t+1)^2}$,so $y_{i+1} = 1$, Decoding error; Due to $d = t + 1 + c$（$c \geq 1$）,$L_i + R_i \frac{t+1+c}{(t+1)^2} = L_i + R_i \frac{1}{t+1} + R_i \frac{c}{(t+1)^2} > L_i + R_i \frac{1}{t+1} > L_i + R_i \frac{t}{(t+1)^2}$, so $y_{i+1} = 1$, Decoding error. When $0 \leq d \leq t$ is obtained, V can decode the sequence Y lossless. When $\varphi(1) = 1$ and $\varphi(0) < \frac{1}{t+1}$, it can be obtained from the above proof step that when $0 \leq d \leq t$,V can decode sequence Y lossless.

2.1 Coding Enhancement

When the sending and receiving end know the length n of the sequence X, according to 2.4,Source processing method (a) adopts $\varphi(1) = 1$,$\varphi(0) \leq \frac{1}{2}$ lossless encoding and decoding, source processing method (b) using $\varphi(1) = 1$,$\varphi(0) \leq \frac{1}{3}$ lossless encoding. Take source processing method (b) as an example, when $\varphi(1)=1,\varphi(0)=\frac{1}{3}$ encoding steps are as follows:

Algorithm(3): arithmetic coding lift based on Y weighted probability

1: Initialization parameter: $R \leftarrow 1$, $L \leftarrow 0$, $i \leftarrow 1$

2: Gets the ith symbol y_i in the sequence Y

3: If $y_i = 0, R \leftarrow \frac{R}{3}$

4: If $y_i = 1, L \leftarrow L + \frac{R}{3}$

5: $i \leftarrow i+1$, if $i \le l$, repeat 2 to 5

6: Converts the real number L to a binary sequence V of length m.

7: Send V

Modifying the above steps (3) and (4) according to Theorem 2.4 can be adapted to all kinds of source processing methods, Since the length n of sequence X is known at the receiver when decoding, and $\varphi(1)=1,\varphi(0)=\frac{1}{3}$ are known, only V needs to be sent.

2.2 Error detection and decoding

The error detection condition of source processing method (b) is (1-2), Decoding is the process of synchronously completing the inverse source processing, that is, the output of "101" is 1 and the output of "01" is 0, so the receiving end based on U linear error detection decoding steps are as follows:

Algorithm(4): Weighted Probability Arithmetic Error Detection and Decoding Based on U

1: Initialization parameter: $R \leftarrow 1, L \leftarrow 0, i \leftarrow 1, T \leftarrow 0, s \leftarrow 0, c \leftarrow 0$

2: $T \leftarrow L + \frac{R}{3}$

3: When $U < T$ and $s = 0, s \leftarrow 1, R \leftarrow \frac{R}{3}, c \leftarrow c+1$

4: When $U \ge T$ and $s = 0, s \leftarrow 2$, $L \leftarrow L + \frac{R}{3}$

5: When $U < T$ and $s = 1$, An "00" error occurred, decoding is aborted and is returned c

6: When $U \ge T$ and $s = 1, z_i \leftarrow 0, s \leftarrow 0, L \leftarrow L + \frac{R}{3}, i \leftarrow i+1$

7: When $U < T$ and $s = 2, s \leftarrow 3, R \leftarrow \frac{R}{3}, c \leftarrow c+1$

8: When $U \ge T$ and $s = 2$, An "11" error occurred, decoding is aborted and is returned c

9: When $U \ge T$ and $s = 3$, $z_i \leftarrow 1$, $s \leftarrow 0$, $L \leftarrow L + \frac{R}{3}$, $i \leftarrow i+1$

10: When $U < H$ and $s = 3$, An "100" error occurred, decoding is aborted and is returned c

11: if $i \le n$ repeat 2 to 11

12: Output binary sequence $Z = (z_1, z_2, \ldots, z_n)$

The above error detection and decoding process is also the process of checking and decoding in forward error correction, The linear decoding process can stop the decoding in time, In Chapter 3, the theoretical position of the error bit in U calculated by c is given as $\left\lceil -c \log_2 \frac{1}{3} \right\rceil$. Combined with the theoretical error position, different forward error correction methods or re-transmission methods are given according to the DMC channel model.

2.3 Forward error correction decoding based on bit flip

Set the sequence $U = \{u_1, u_2, \ldots, u_m\}$ and there is τ bit error, when $\tau = 1$, the bit flip[13][14] forward error correction decoding steps are as follows:

Algorithm(5): Forward error correction decoding when $\tau = 1$

1: Set error correction range control parameters: $\mathscr{r}, \rho, \sigma, \vartheta$

2: After Algorithm(4) is aborted, c is used to calculate the theoretical error bit position as pos $pos \leftarrow \left\lceil -c \log_2 \frac{1}{3} \right\rceil$

3: Error correction range based on pos set in sequence U, $start \leftarrow pos - \mathscr{r}, end \leftarrow pos + \rho$, $i \in [start, end]$

4: $i \leftarrow end$

5: Flip the ith bit of U, which is $u_i \leftarrow \overline{u_i}$

6: U is re−decoded through Algorithm(4) for error detection

7: If $\left\lceil -c\log_2\frac{1}{3}\right\rceil \ge end+\sigma+\vartheta$, All errors in $\left[start,\left\lceil -c\log_2\frac{1}{3}\right\rceil\right]$ are corrected , end

8: If $\left\lceil -c\log_2\frac{1}{3}\right\rceil \ge end+\sigma$, The first error from left to right in $\left[start,\left\lceil -c\log_2\frac{1}{3}\right\rceil\right]$ is corrected, end

9: $u_i \leftarrow \overline{u_i}$,$i \leftarrow i-1$

10: If $i > start$,$u_i \leftarrow \overline{u_i}$, Repeat 5 to 9

10: Error correction with $\tau = 2$ is started

When $\tau = 2$, the forward error correction decoding steps are as follows:

Algorithm(6):When $\tau = 2$,the forward error correction decoding

1: Set error correction range control parameters:γ,ρ,σ,ϑ

2: After Algorithm(4) is aborted, c is used to calculate the theoretical error bit position as pos $pos \leftarrow \left\lceil -c\log_2\frac{1}{3}\right\rceil$

3: Error correction range based on pos set in sequence U,$start \leftarrow pos-\gamma$,$end \leftarrow pos+\rho$,$i \in [start, end]$

4: $i \leftarrow end$

5: $u_i \leftarrow \overline{u_i}$

6: $j \leftarrow i-1$

7: $u_j \leftarrow \overline{u_j}$

8: U is re−decoded through Algorithm(4) for error detection

9: If $\left\lceil -c\log_2\frac{1}{3}\right\rceil \ge end+\sigma+\vartheta$, All errors in $\left[start,\left\lceil -c\log_2\frac{1}{3}\right\rceil\right]$ are corrected , end

10: If $\left\lceil -c\log_2\frac{1}{3}\right\rceil \ge end+\sigma$, The first error from left to right in $\left[start,\left\lceil -c\log_2\frac{1}{3}\right\rceil\right]$ is corrected, end

11: $u_j \leftarrow \overline{u_j}$,$j \leftarrow j-1$

12: If $j > start-1$, Repeat 7 to 12

13: $u_i \leftarrow \overline{u_i}$,$i \leftarrow i-1$

13: If $i > start$, Repeat 5 to 14

14: Error correction with $\tau = 3$ is started

Where γ and ρ are used to limit the error correction range, σ and ϑ are used to limit the error correction quality. The values of γ, ρ and σ,ϑ refer to the decoding error probability set in Chapter 3. Step (5) is repeated $C^1_{end-start+1}$ times when $\tau = 1$. When $\tau \ge 2$, multiple bits need to be flipped by nested loops, and step (5) needs to be repeated $C^2_{end-start+1}$ times. Therefore, the number of repetitions of step (5) when performing τ bits of error correction is:

$$C^1_{end-start+1} + C^2_{end-start+1} + \cdots + C^{\tau}_{end-start+1} \tag{2-9}$$

Obviously, the larger the τ is, the greater the number of repeated error detection decoding times. The value of τ can be set according to different SNR, and the number of passes can also be limited to control the error correction operation time.

3 Mathematical Analysis of Weighted Probability Arithmetic Coding

3.1 Weighted probability arithmetic coding information entropy

The source sequence $Y = (y_1, y_2, \ldots, y_l)(y_i \in A = \{0,1\}, i = 1,2, \ldots, l)$,when $r = 1$,$\varphi(y_i) = p(y_i)$.Defined by Shannon's information entropy[15][16], the entropy of Y is:

$$H(X) = -p(0)\log_2 p(0) - p(1)\log_2 p(1) \tag{3-1}$$

When $r \ne 1$, the self-information[17] of random variable y_i with probability $\varphi(y_i)$ is defined as:

$$I(y_i) = -\log_2 \varphi(y_i) \tag{3-2}$$

Let the sequence Y have c_0 symbols 0 and c_1 symbols 1,and $c_0 + c_1 = l$. When r is known, the total information of source sequence Y is:

$$-c_0\log_2 \varphi(0) - c_1\log_2 \varphi(1)$$

So the average amount of information for each symbol is:

$$-\frac{c_0}{l}\log_2\varphi(0)-\frac{c_1}{l}\log_2\varphi(1)=-\sum_{y=0}^{1}p(y)\log_2\varphi(y)$$

Definition 3.1 Record the weighted probability arithmetic encoding information entropy as $H(Y,r)$:

$$\begin{aligned}H(Y,r)&=-\sum_{y=0}^{1}p(y)\log_2\varphi(y)\\&=-\log_2 r-\sum_{y=0}^{1}p(y)\log_2 p(y)\\&=-\log_2 r+H(Y)\end{aligned}\tag{3-3}$$

According to definition 3.1, after determining r, the length of V is $nH(Y,r)(bit)$ after weighted probability arithmetic coding. So the minimum limit for weighted probability arithmetic lossless coding is:

$$\begin{aligned}H(Y,r_{max})&=-\sum_{y=0}^{1}p(y)\log_2\varphi(y)\\&=-\log_2 r_{max}-\sum_{y=0}^{1}p(y)\log_2 p(y)\\&=-\log_2 r_{max}+H(Y)\end{aligned}\tag{3-4}$$

proof According to theorem 2.3, r_{max} is the maximum value of weighted probability arithmetic lossless encoding, and $r_{max}>1$.When $r>r_{max}$,V cannot restore the sequence Y,so $H(Y,r_{max})$ is the minimum limit of weighted probability arithmetic lossless coding.

Assuming that the symbols of the binary source sequence X with a length of n are uniformly distributed, and after processing with method (b), in sequence Y,$c_0=n$,use $\varphi(0)=\frac{1}{3}$,When $\varphi(1)=1$ encoding

$$H(Y,r)=-\log_2\frac{1}{3}-\log_2 1=-\log_2\frac{1}{3}\tag{3-5}$$

The total information is $-n\log_2\frac{1}{3}$.The self information of symbol 1 is 0, so symbol 0 in sequence Y determines the length of encoded V.Therefore, the error detection decoding in section 2.2 uses the number c of the statistical symbol 0,and the error bit position in U is found by $\left\lceil -c\log_2\frac{1}{3}\right\rceil$.

3.2 Coded bit rate

The average amount of information carried by each bit in the sequence Y is $H(Y,r)$（bit/ symbol）.The average amount of information carried by each bit in the source sequence X is $H(X)$（bit/ symbol）, which can determine the encoding rate of weighted probability arithmetic encoding:

$$R=\frac{H(X)}{H(Y,r)}\tag{3-6}$$

When $H(X)=1$, $\varphi(0)=\frac{1}{3}$, $\varphi(1)=1$ $R=-\log_2\frac{1}{3}$;When $H(X)=1$, $\varphi(0)=\frac{1}{2}$, $\varphi(1)=1$ $R=-\log_2\frac{1}{2}=1$. Different source processing methods can get different weighted probability arithmetic coding rate is different.

3.3 Average decoding error probability

Let Erepresent the set of binary sequences satisfying $(1-1)$, E has $f(l)$ sequences and $Q,Y\subset E$.When $l=1$,$E=(0,1)$,$f(l=1)=2$,The complementary event is $\bar{E}=\emptyset$.When$l=2$,$E=(00,01,10)$,$f(l=2)=3$,$\bar{E}=(11)$.When$l=3$,$E=(000,001,010,100,101)$,$f(l=3)=5$,$\bar{E}=(011,110,111)$. When $l\geq 3$:

$$f(l)=f(l-1)+f(l-2)\tag{3-7}$$

The probability of getting E is:

$$\mathrm{P}(E)=\frac{f(l)}{2^l} \tag{3-8}$$

$f(l)$ sequences Y in E are uniformly distributed, then:

$$\mathrm{P}(Q=Y)=\frac{1}{f(l)},\mathrm{P}(Q\neq Y)=\frac{f(l)-1}{f(l)} \tag{3-9}$$

So, the probability of $Q\in E$ and $Q\neq Y$ is:

$$\mathrm{P}(Q\neq Y|Q\in E)=\mathrm{P}(Q\in E)\mathrm{P}(Q\neq Y)=\frac{f(l)-1}{2^l} \tag{3-10}$$

$\mathrm{P}(Q\neq Y|Q\in E)$is the average decoding error probability, so $P_{err}=\mathrm{P}(Q\neq Y|Q\in E)$ is the probability of error detection and decoding in section 2.2.

Theorem 3.2 $\lim_{l\to\infty}P_{err}=\lim_{l\to\infty}(Q\neq Y|Q\in E)=0$.

proof $l\to\infty$ then $\mathrm{P}(Y\neq Q)\to 1$,Get $\mathrm{P}(Y\neq Q|Y\in E)\to\mathrm{P}(E)$. According to the Fibonacci sequence [24],Let $F(0)=0,F(1)=1$,and when $l\geq 2,l\in N^*$ $F(l)=F(l-1)+F(l-2)$.So when $l\geq 1,l\in N^*,f(l)=F(l)+F(l+1)$. From the general term formula of the Fibonacci sequence:

$$f(l)=\frac{1}{\sqrt{5}}\left(\left[\left(\frac{1+\sqrt{5}}{2}\right)^l-\left(\frac{1-\sqrt{5}}{2}\right)^l\right]+\left[\left(\frac{1+\sqrt{5}}{2}\right)^{l+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{l+1}\right]\right)$$

You get:

$$\mathrm{P}(Q\in E)=\frac{1}{\sqrt{5}}\left(\left[\left(\frac{1+\sqrt{5}}{4}\right)^l-\left(\frac{1-\sqrt{5}}{4}\right)^l\right]+2\left[\left(\frac{1+\sqrt{5}}{4}\right)^{l+1}-\left(\frac{1-\sqrt{5}}{4}\right)^{l+1}\right]\right)$$

Because of $\frac{1+\sqrt{5}}{4}<1,\frac{1-\sqrt{5}}{4}<1$,so $l\to\infty$ $\mathrm{P}(E)\to 0$,then $\lim_{l\to\infty}\mathrm{P}(Q\neq Y|Q\in E)=0$.

Let Frepresent a set that satisfies (1−2) binary sequences, where F has $g(l)$sequences and $Q,Y\subset F$.When $l=1,F=(0,1),g(l=1)=2$,The complementary event is $\bar{F}=\emptyset$ .When $l=2$, $F=(01,10,11)$, $g(l=2)=3$, $\bar{F}=(00)$ .When $l=3$, $F=(010,101,011,110),f(l=3)=4,\bar{F}=(000,001,100,111)$.When $l\geq 4$:

$$g(l)=g(l-2)+g(l-3) \tag{3-11}$$

$\mathrm{P}(F)=\frac{g(l)}{2^l}$, get:

$$\mathrm{P}(Q\neq Y|Q\in F)=\mathrm{P}(F)\mathrm{P}(Q\neq Z)=\frac{g(l)-1}{2^l} \tag{3-12}$$

$f(l)$ and $g(l)$ are both monotonically increasing, and $g(l)\leq f(l)$,then $\frac{g(l)-1}{2^l}\leq\frac{f(l)-1}{2^l}$ .When $l\to\infty$, $\frac{f(l)-1}{2^l}\to 0$, $P_{err}=\mathrm{P}(Q\neq Y|Q\in F)$,You get $\lim_{l\to\infty}P_{err}=0$.According to (3-10) and (3-12), P_{err} can be calculated as shown in Table 2.

Table 2 calculates $\mathrm{P}(Q\neq Y|Q\in E)$ and $\mathrm{P}(Q\neq Y|Q\in F)$ based on l

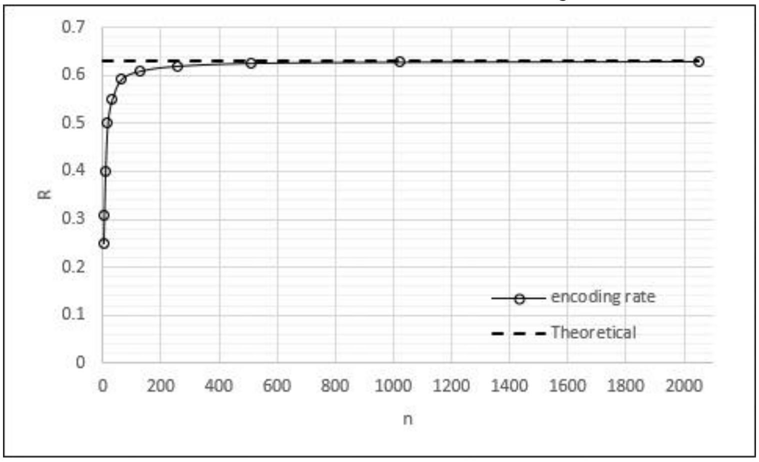
l（bit）	$\mathrm{P}(Q\neq Y\|Q\in E)$	$\mathrm{P}(Q\neq Y\|Q\in F)$
20	2.08778E-02	6.67572E-04
64	1.86132E-06	8.99225E-12
176	9.14278E-17	8.24769E-32
256	3.95995E-24	4.01612E-46

Table 2 shows that the larger l is, the lower the probability of error detection and decoding. Different source processing methods can get different results, and the weighted probability arithmetic coding average decoding error probability is different..Therefore, in the process of error correction in Section 2.3, while considering computational efficiency，the greater the value of r,ρ and σ, the smaller the P_{err} r, the lower the error probability of step (5) in section 2.3.According to the above reasoning and proof, When the source processing method is determined, the bit rate and transmission rate of weighted probability arithmetic coding are also determined, When l approaches infinity, the error probability after error correction and decoding in this paper is 0, indicating that the channel capacity can be reached.

4 Simulation experiment and result analysis

The simulation experiment of AWGN channel BPSK signal[18][19] ,experiment (1) consider error-free transmission, and it is concluded that the decoding can be lossless when $r \le r_{max}$ and the decoding error when $r > r_{max}$.Based on method (b) processing randomly generated binary sequences, using $\varphi(0) = \frac{1}{3}$,$\varphi(1) = 1$ can be lossless encoded and decoded, When the length n of the source sequence $X$is large enough, then $l$is large enough. The experimental results show that the actual encoding rate R approaches the theoretical value $-\log_2 \frac{1}{3}$, as shown in Figure 4. The experiment shows that Theorem 2.3, 2.4, and Definition 3.1 are correct.

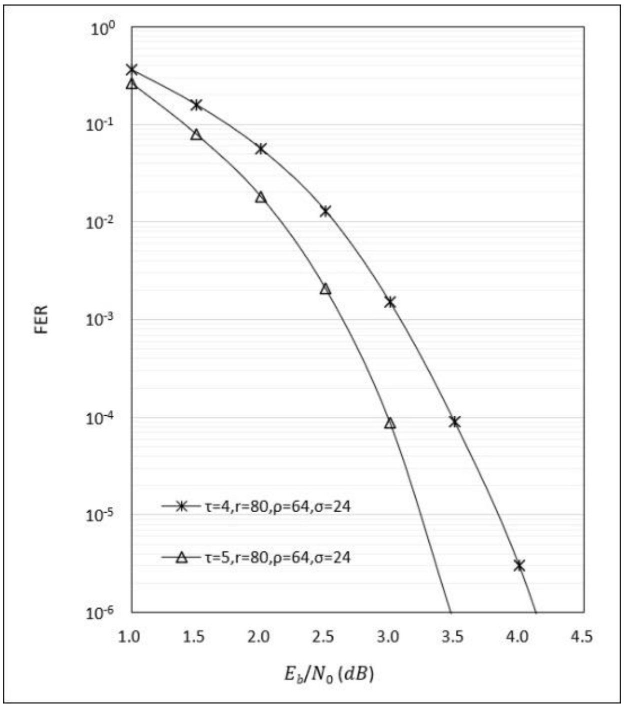
FIG. 4 Relation between R and n when $\varphi(0) = \frac{1}{3}$,$\varphi(1) = 1$



In Figure 4, the actual encoding rates corresponding to $n = 2,4,8,16,32,64,128,256,512,1024,2048$ are shown, indicating that the longer the code length, the closer it approaches the theoretical calculation limit.

Experiment (2) is incorrect transmission, set $\varphi(0) = \frac{1}{3}$,$\varphi(1) = 1$,, random data length of each frame is $n = 160$, encoded length is 320, denoted as $N = (320,160)$,According to Figure 4,According to Figure 4, at this time $R = 0.5$, the initial value of $E_b/N_0(dB)$ is 1.0, increasing by 0.5 each time and testing 10^6 frames, respectively $\tau = 5, r = 80, \rho = 64, \sigma = 24$ and set $\tau = 4, r = 80, \rho = 64, \sigma = 24$, The simulation experiment results are shown in Figure 5.

FIG. 5 FER corresponding to $\tau = 4$ and $\tau = 5$ when $N = (320,160)$

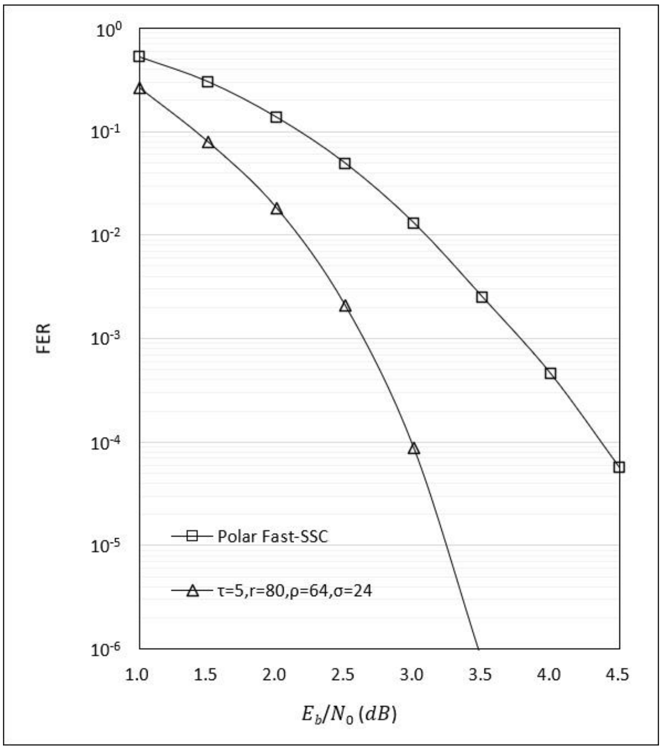


The increase of τ in Figure 5 can reduce the decoding error probability, and the operation time required for decoding in the experiment also increases. The frame error rate of the unmarked part in the fig

Ure is below 10^{-6}.When $\tau = 5, r = 80, \rho = 64, \sigma = 32, \vartheta = 24$, $N = (1360,800)$,At this point, the bit rate is $R = 0.6$,When $E_b/N_0(dB) \ge 3.0$, FER $\le 10^{-3}$ .When $\tau = 8, r = 96, \rho = 64, \sigma = 32, \vartheta = 24$, $N = (1360,8192)$,At this point, the bit rate is $R = 0.627836$,When $E_b/N_0(dB) \ge 4.0$,FER $\le 10^{-3}$.When $\tau = 5, r = 80, \rho = 64, \sigma = 32, \vartheta = 24$,$N = (96,24)$ At this point, the bit rate is $R = 0.25$,When $E_b/N_0(dB) \ge 1.0$,FER $\le 10^{-5}$.

Compared with Polar Fast-SSC[20][21], the experimental code rate of this method is 0.5,This article adopts $\varphi(0) = \frac{1}{3}, \varphi(1) = 1$ and $\tau = 5, \mathcal{r} = 80, \rho = 64, \sigma = 24, N = (320,160)$; The code length of Polar Fast-SSC is $N = (256,128)$, and the experimental results are shown in Figure 6.

FIG. 6 Comparison between the method in this paper and the polarization code



When $\mathrm{FER} = 10^{-3}$ in Figure 6, our algorithm improves by 1.1dB compared to polarization codes. When $\mathrm{FER} = 10^{-4}$, our algorithm improves by 1.4dB compared to polar codes. The simulation source code has been published on GitHub website, download address is: https://github.com/Jielin-Code/WJLErrRecoveryCode,In the program, The program has $\mathcal{r}, \rho, \sigma, \vartheta$ corresponding to the parameter START_LIMIT ,END_LIMIT and BLOCK_ERR_COMPARE_LIMIT,FIRST_ERR_COMPARE_LIMIT, Unit is byte,τ corresponds to the parameter ERRBITS_LIMIT, Unit is byte.

5 Concluding remarks

This article proposes a new approach to channel error detection and correction, and provides specific methods. Due to the numerous types of methods, this article provides two examples, corresponding to different code rates and decoding error probabilities. By proving that when the code length is long enough, the decoding error probability can reach 0. The method is simple and easy to implement in software and hardware. It can adapt to the interference of the channel and improve the error correction rate by increasing the values of $\tau, \mathcal{r}, \rho$ and σ.After controlling the size of $\tau, \mathcal{r}, \rho$ and σ, a channel coding method that integrates forward error correction and data verification retransmission can be constructed.

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