

Proper Definitions and Demonstration of Dirichlet Conditions

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Fourier theory is the backbone of the area of Signal Processing (SP) and Communication Engineering. However, Fourier series (FS) or Fourier transform (FT) do not exist for some signals that fail to fulfill a predefined set of Dirichlet conditions (DCs). We note a subtle gap in the explanation of these conditions as available in the popular signal processing literature. They lack a certain degree of explanation essential for the proper understanding of the same. For example, the original second Dirichlet condition is the requirement of bounded variations over one time period for the convergence of Fourier Series, where there can be at most *infinite* but countable number of maxima and minima, and at most *infinite* but countable number of discontinuities of finite magnitude. However, a large body of the literature replaces this statement with the requirements of *finite* number of maxima and minima over one time period, and *finite* number of discontinuities. The latter incorrectly disqualifies some signals from having valid FS representation. Similar problem holds in the description of DCs for the Fourier transform. Likewise, while it is easy to relate the first DC with the finite value of FS or FT coefficients, it is hard to relate the second and third DCs as specified in the signal processing literature with the Fourier representation as to how the failure to satisfy these conditions disqualifies those signals from having valid FS or FT representation.

These concepts are taught worldwide in the standard course of “Signals and Systems” to the second year undergraduate students of Electrical/Electronics/Communication/Computer Engineering. Lack of clarity in the explanation of DCs in the signal processing textbooks makes it hard for the young engineering students to understand these concepts. In fact, many instructors roll out

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this topic as a “reading assignment” for the students because it may be equally uncomfortable for signal processing instructors to provide suitable explanations for the second and third DCs. In this lecture note, we attempt to provide the required clarifications and provide a lucid but precise description and explanation on DCs. We support the text with the help of suitable examples. The entire signal processing community may benefit from this work. In particular, the presented work can help in teaching these concepts with better clarity to the young signal processing audience.

Relevance

Fourier theory provides a powerful tool for the analysis of signals. It has been widely used in the areas of signal processing and communication engineering. Fourier representation, such as FS and FT, provides an alternate representation of a time-domain signal in terms of its frequency content that is helpful in the analysis and processing of signals in umpteen number of applications and devices. While FS is evaluated for periodic signals, FT is generally used to analyze non-periodic (or aperiodic) signals, although it can be computed for the periodic signals by roping in impulse functions that are generalized functions.

It is not necessary that every signal will have a valid Fourier representation. In general, it is stated that a signal would have a valid FS or FT representation provided it satisfies a set of conditions given by Dirichlet. These are stated to be sufficient, but not necessary conditions. The Fourier theory and the associated DCs are essential tools that are utilized by the scientific community working in the area of signal processing and related applications. Almost every textbook on “Signals and Systems” as well as many widely-used manuscripts and books [1–10] specify these conditions in the context of Fourier representation and also provide examples, some of which are incorrectly perceived as not fulfilling these conditions.

Preliminaries

In this section, we present some mathematical preliminaries that are important to understand the concepts with clarity.

Definition 1: A signal $\tilde{x}(t)$ is periodic if, $\tilde{x}(t + T_0) = \tilde{x}(t)$, $\forall t$ where T_0 is the fundamental period in seconds and $\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$ is the fundamental frequency in radians/second.

Definition 2: Let $x : [a, b] \rightarrow \mathbb{C}$ be a function and any finite partition of $[a, b]$ is set $P = \{a = t_0 < t_1 < \dots < t_n = b\}$. The variation of x over $[a, b]$ is defined as

$$V(x, [a, b]) = \sup \{S_P : P \text{ is a partition of } [a, b]\}, \quad (1)$$

$$\text{where } S_P = \sum_{i=1}^n |x(t_i) - x(t_{i-1})|. \quad (2)$$

The function x has ‘bounded variation’ (BV) on $[a, b]$ denoted as $x \in BV([a, b])$, if $V(x, [a, b])$ is finite, i.e., $V(x, [a, b]) < \infty$.

To obtain the variation in (1) the supremum is taken over all possible partitions on $[a, b]$. Some important results related to a function of bounded variation are as follows:

1. If a function x is increasing on $[a, b]$, then $x \in BV([a, b])$ and $V(x, [a, b]) = x(b) - x(a)$.
2. If a function x is decreasing on $[a, b]$, then $x \in BV([a, b])$ and $V(x, [a, b]) = x(a) - x(b)$.
3. The function x has at most countably infinite discontinuities on $[a, b]$.
4. If x is differentiable and its derivative x' is continuous on $[a, b]$, then

$$V(x, [a, b]) = \int_a^b |x'(t)| dt. \quad (3)$$

Next, we illustrate this concept of bounded variation (BV) with a few examples.

Example-1:

Consider a signal

$$x_1(t) = \begin{cases} \sin(2\pi/t), & \text{if } t \in (0, 1] \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The derivative of $x_1(t)$ is given by $x_1'(t) = -2\pi \cos(2\pi/t)/t^2$. It is easy to see that $\int_0^1 |x_1'(t)| dt = \infty$. This implies that $V(x_1, [0, 1]) = \infty$. Hence, signal $x_1(t)$ does not have BV in the time interval $[0, 1]$, i.e., $x_1 \notin BV([0, 1])$.

Example-2:

Consider a signal

$$x_2(t) = \begin{cases} t \sin(2\pi/t), & \text{if } t \in (0, 1] \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The derivative of $x_2(t)$ is given by $x_2'(t) = -2\pi \cos(2\pi/t)/t + \sin(2\pi/t)/t$. It is easy to see that $\int_0^1 |x_2'(t)| dt = \infty$. This implies that $V(x_2, [0, 1]) = \infty$. Hence, signal $x_2(t)$ does not have BV in the time interval $[0, 1]$, i.e., $x_2 \notin BV([0, 1])$.

Example-3:

Consider a signal

$$x_3(t) = \begin{cases} t^2 \sin(2\pi/t), & \text{if } t \in (0, 1] \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

The signal $x_3(t)$ has BV in $[0, 1]$ because

$$\begin{aligned}
V(x_3, (0, 1]) &= \int_0^1 |x'_3(t)| dt \\
&= \int_0^1 |2t \sin(2\pi/t) - 2\pi \cos(2\pi/t)| dt, \\
&\leq \int_0^1 |2t \sin(2\pi/t)| + |2\pi \cos(1/t)| dt, \\
&\leq \int_0^1 (2t + 2\pi) dt = (1 + 2\pi).
\end{aligned} \tag{7}$$

In fact, one can easily show that the signal $x_r(t) = t^r \sin(2\pi/t)$ for $t \in (0, 1]$ and zero otherwise, has BV in $[0, 1]$ for $r \geq 2$.

Fourier Series and Dirichlet Conditions

The FS representation of a periodic signal $\tilde{x}(t)$ is defined as

$$\text{Analysis equation:} \quad X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) \exp(-jk\omega_0 t) dt, \tag{8}$$

$$\text{Synthesis equation:} \quad \tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k \exp(jk\omega_0 t), \tag{9}$$

According to the Dirichlet-Jordan Theorem [15], the above representation for a signal $\tilde{x}(t)$ is available provided it satisfies the following two DCs:

(A1) The signal $\tilde{x}(t)$ is absolutely integrable over one time period, i.e.,

$$\int_{-T_0/2}^{T_0/2} |\tilde{x}(t)| dt < \infty. \tag{10}$$

(A2) It has BV over one time period. Signals with BV have (i) at most countably infinite number of maxima and minima, and (ii) at most countably infinite number of finite discontinuities. This is to note that vice-versa need not be true because signals with BV form a subset of signals satisfying (i) and/or (ii).

However, the signal processing and other literature [1–10] states the following three DCs over one time period:

(B1) The signal $\tilde{x}(t)$ is absolutely integrable, i.e.,

$$\int_{-T_0/2}^{T_0/2} |\tilde{x}(t)| dt < \infty. \tag{11}$$

(B2) It has a finite number of maxima and minima.

(B3) It has a finite number of discontinuities of finite magnitude.

Furthermore, the first DC is stated as the weak Dirichlet condition, while the second and third DCs are stated as strong Dirichlet conditions [4].

The first DC is stated identically in the literature, i.e., (A1) and (B1) are identical, and ensures the convergence of the integral leading to the computation of FS coefficients (8). In other words, it ensures the existence of the FS coefficients or the analysis equation.

Signals that satisfy second DC as stated originally (A2) with bounded variations over one time period, also satisfy the conditions: (a) at most countably *infinite* maxima and minima, and/or (b) at most countably *infinite* number of finite discontinuities. In fact, (A2) is required to ensure the recovery of the original signal using the Fourier coefficients (except at the points of discontinuities) via the synthesis equation (9). In other words, it ensures the convergence of FS (9).

The signal processing textbooks lack a clear explanation in multiple ways as listed below:

1. Superficially, the two sets of conditions (A2) vis-à-vis (B2)-(B3) may look similar, but they are not identical. Signals that satisfy (B2)-(B3) have bounded variations. But there may be signals that have bounded variations with at most countable *infinite* maxima and minima, and countable *infinite* discontinuities. Thus, the restriction to *finite* maxima and minima, and *finite* number of discontinuities incorrectly eliminates some signals that will have convergent FS.
2. While the first condition implies that signals belong to the L_1 space, the second condition (A2) is needed for only these signals belonging to the L_1 space. Interestingly, according to the Hunt theorem [13,14], DCs are not required for signals that belong to L_p with $p > 1$. This is the reason that these conditions are stated to be sufficient but not necessary conditions. Further, the space L_1 , i.e., $p = 1$ is excluded from the Hunt theorem because, in 1925, Kolmogorov provided an example of a periodic signal $\tilde{x} \in L_1(0, 2\pi)$ that satisfies (A1), but its FS diverges everywhere [13,14].
3. It may not be appropriate to say that the first condition (B1) is weak and the second and third DCs (B2)-(B3) are strong conditions. Both the original DCs (A1) and (A2) are required for the signals belonging to L_1 and they have their own significance in terms of the analysis and synthesis of signals using the Fourier basis.

Further, it is to be noted that one can do analysis using the Fourier tool for signals that satisfy (A1), but fail (A2). Researchers may find use cases of such signals in applications, where only analysis using Fourier tools is helpful and synthesis back to the original signal is not required. For example, two popular continuous wavelets, Morlet and Mexican-Hat are not invertible, but they find immense applications in the analysis of geophysical signals [16–18].

From the above discussion, it is evident that the concept of convergence of FS is required to be added with better clarity to the textbooks for academicians, students, and researchers. In the next section, we will show examples from popular signal processing literature that satisfy (B1) but violate (B2) or (B3) and hence, are stated to fail FS convergence. However, these signals satisfy (A2) and hence, their FS would converge. These examples can easily highlight the distinction between (A2) and (B2)-(B3).

Convergence of the Fourier series

In this section, the results related to the convergence of FS are presented. Generally, three types of convergence, namely, point-wise, uniform, and norm convergence are important and widely discussed in the literature. These definitions of convergence are connected with how the partial sum [15]

$$S_N(t) = \sum_{k=-N}^N X_k \exp(jk\omega_0 t) \quad (12)$$

converges to $\tilde{x}(t)$ as $N \rightarrow \infty$, i.e., whether $\lim_{N \rightarrow \infty} S_N(t) = \tilde{x}(t)$ is valid in some sense of convergence or not. The pointwise, uniform, and L_p -norm convergence are defined as

$$\begin{aligned} S_N &\rightarrow \tilde{x} \text{ pointwise, if } \lim_{N \rightarrow \infty} S_N(t) = \tilde{x}(t), \quad \forall t \in [0, T_0], \\ S_N &\rightarrow \tilde{x} \text{ uniformly, if } \lim_{N \rightarrow \infty} \left\{ \sup_{t \in [0, T_0]} |\tilde{x}(t) - S_N(t)| \right\} = 0, \\ \text{and } S_N &\rightarrow \tilde{x} \text{ in } L_p \text{ norm, if } \lim_{N \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} |\tilde{x}(t) - S_N(t)|^p dt = 0, \end{aligned}$$

where $p \in \mathbb{R}$, $p \geq 1$ and L_p norm is defined as

$$\|\tilde{x}\|_p = \left[\frac{1}{T_0} \int_0^{T_0} |\tilde{x}(t)|^p dt \right]^{1/p}. \quad (13)$$

Theorem 1 (Dirichlet–Jordan): If \tilde{x} has period T_0 and has bounded variation on $[0, T_0]$, then the Fourier series for \tilde{x} converges for all values of t as

$$\lim_{N \rightarrow \infty} S_N(t) = \frac{1}{2} [\tilde{x}(t+) + \tilde{x}(t-)], \quad (14)$$

where this convergence is uniform on every closed interval on which \tilde{x} is continuous.

Theorem 2 (Parseval's equality): if $\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |\tilde{x}(t)|^2 dt$ is finite, then

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |\tilde{x}(t)|^2 dt. \quad (15)$$

The convergence of FS for an L_2 -function along with Riesz and Fischer theorem [15] provides necessary and sufficient conditions for L_2 -convergence.

Theorem 3: If \tilde{x} is an L_2 -function on $[0, T_0]$, then its Fourier series converges to \tilde{x} in L_2 -norm. Conversely, If $\sum |X_k|^2$ converges, then there exists an L_2 -function \tilde{x} which has Fourier coefficients $\{X_k\}$.

The next theorem was proved by Carleson in 1966 [11] for L_2 -functions, and was extended by Hunt in 1968 [12] for L_p functions for all $p \in (1, \infty)$.

Theorem 4 (Carleson [11]): For each $\tilde{x} \in L_2([0, T_0])$, the Fourier series $S_N(t)$ converges to \tilde{x} almost everywhere.

Theorem 5 (Hunt [12]): If \tilde{x} is an L_p -function for $p > 1$ on $[0, T_0]$, then its Fourier series converges to it at almost all points.

The phrases about convergence at ‘almost all points’ or ‘almost everywhere’ imply that the FS converges everywhere except at a countable set of points. Since the Lebesgue measure is zero on the set of countable points, the signal $\tilde{x}(t)$ is said to be equal to its series representation (Synthesis equation (9)) almost everywhere (excluding those countable points at which the divergence occurs). Thus, the convergence at ‘almost all points’ or ‘almost everywhere’ implies pointwise convergence at all but a set of countable points where divergence occurs. This convergence is a weaker version of pointwise convergence. It should also be noted that the condition of BV is not specified in Theorems 2–5. In the following subsections, we present some examples that are widely used in the signal processing literature to illustrate signals that fail (B2) or (B3) and are incorrectly stated as not having convergent FS representation [1–4].

Case 1: A function that satisfies only the first DC (A1/B1)

Example-4

Consider a signal $x(t) = \sin\left(\frac{2\pi}{t}\right)$ for $0 < t \leq 1$ and zero otherwise, as shown in Fig. 1 (top left) and use it to construct a periodic signal with fundamental period $(0, 1]$ as

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t - m). \quad (16)$$

Clearly, $x(t) \notin BV([0, 1])$ because $V(x, [0, 1]) = \infty$. However, $x(t) \in L_p(\mathbb{R})$ for $p \in [1, \infty]$. It has finite energy because $\int_0^1 [\sin^2\left(\frac{2\pi}{t}\right) + \cos^2\left(\frac{2\pi}{t}\right)] dt = 1$. Thus, it lies in $L_2(\mathbb{R})$. Therefore, by Theorems 2–5, its FS converges to the original signal at almost all points, although this signal is incorrectly stated as not having convergent FS representation. We compute the Fourier coefficients as

$$X_k = \int_0^1 \sin\left(\frac{2\pi}{t}\right) e^{-jk\omega_0 t} dt, \quad (17)$$

where $\omega_0 = 2\pi$ rad/s. By substituting $t = e^{-v}$, we obtain

$$X_k = \int_0^\infty e^{-v} \sin(2\pi e^v) e^{-jk\omega_0 e^{-v}} dv, \quad (18)$$

which is integration of an exponentially decaying function. One may observe from (17) and (18) that its Fourier coefficients are upper bounded to one, i.e., $|X_k| \leq 1$. Although its closed form expression is difficult to obtain, we used the *numerical integration* to compute the Fourier coefficients as shown in Fig. 1 (bottom left). For the purpose of illustration, Fig. 1 (right) depicts the corresponding plots obtained for the signal $x(t) = \cos\left(\frac{2\pi}{t}\right)$, $0 < t \leq 1$.

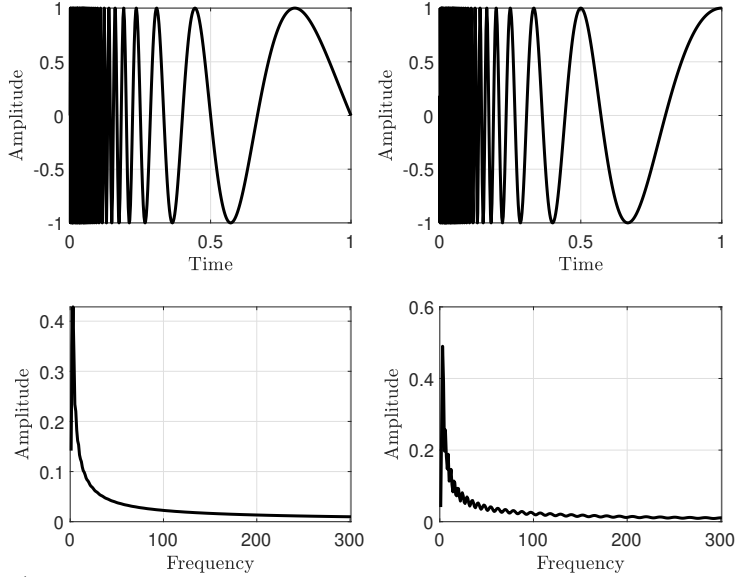


Figure 1: Plot of one period of signals over the time interval $0 < t \leq 1$ and corresponding spectrum: (i) signal $\sin\left(\frac{2\pi}{t}\right)$ (top left) and its spectrum (bottom left), (ii) signal $\cos\left(\frac{2\pi}{t}\right)$ (top right) and its spectrum (bottom right).

Example-5

We consider another interesting example of a periodic signal with fundamental period $(0, 1]$ as

$$\tilde{x}(t) = \sin(\tan(\pi t)), \quad (19)$$

as shown in bottom Fig. 2. Clearly, in one period, $\tilde{x}(t) \in L_p(\mathbb{R})$ for $p \geq 1$, as $\int_0^1 |\sin(\tan(\pi t))|^p dt \leq \int_0^1 1^p dt = 1$, $\int_0^1 [\sin^2(\tan(\pi t)) + \cos^2(\tan(\pi t))] dt = 1$, which proves $\tilde{x}(t) \in L_2(\mathbb{R})$, but $\tilde{x}(t) \notin BV([0, 1])$ as $\tilde{x}'(t) = \pi \cos(\tan(\pi t)) \sec^2(\pi t)$ and $V(x, [0, 1]) = \int_0^1 |\tilde{x}'(t)| dt = \infty$. Theorems 2–5 lead to convergence in this

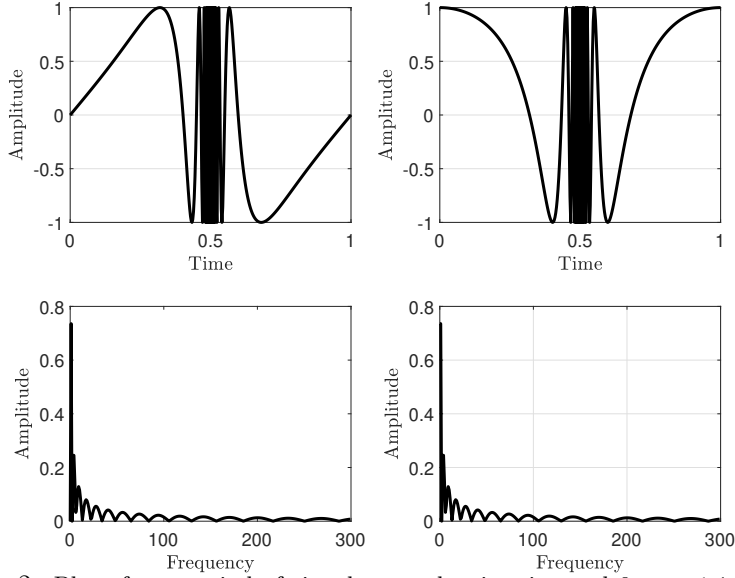


Figure 2: Plot of one period of signals over the time interval $0 < t \leq 1$ and corresponding spectrum: (i) signal $\sin(\tan(\pi t))$ (top left) and its spectrum (bottom left), (ii) signal $\cos(\tan(\pi t))$ (top right) and its spectrum (bottom right).

case. Its Fourier coefficients can be obtained as

$$X_k = \int_0^1 \sin(\tan(\pi t)) e^{-jk\omega_0 t} dt, \quad (20)$$

$$|X_k| = \left| \int_0^1 \sin(\tan(\pi t)) e^{-jk\omega_0 t} dt \right|,$$

$$|X_k| \leq \int_0^1 |\sin(\tan(\pi t)) e^{-jk\omega_0 t}| dt, \quad (21)$$

$$\leq \int_0^1 1 dt = 1, \quad (22)$$

thus they are bounded, i.e. $|X_k| \leq 1$, for all k . Although its closed form expression is difficult to obtain, we used the *numerical integration* to compute the Fourier coefficients as shown in Fig. 2 (bottom left). Similar analysis has been performed for the signal $\tilde{x}(t) = \cos(\tan(\pi t))$ and plots are shown in Fig. 2 (top right and bottom right).

Example-6

Here, we consider a signal $x(t) = \frac{1}{\sqrt{t}}$ for $0 < t \leq 1$, and zero otherwise. Now, we construct a periodic function

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t-m), \quad (23)$$

as shown in bottom Fig. 3. Clearly, $x(t) \in L_1(\mathbb{R})$ as $\int_0^1 \frac{1}{\sqrt{t}} dt = 2$, but $x(t) \notin L_2(\mathbb{R})$, and $x(t) \notin BV([0,1])$ as $V(x, [0,1]) = x(0) - x(1) = \infty$. However, $x(t) \in L_p(\mathbb{R})$ for $1 \leq p < 2$. From Theorems 1–4, it seems that the FS does not converge while Theorem 5 leads to the convergence in this case. Its Fourier coefficients can be obtained as

$$\begin{aligned} X_k &= \int_0^1 \frac{1}{\sqrt{t}} e^{-jk\omega_0 t} dt, \\ X_k &= \int_0^1 2e^{-jk\omega_0 z^2} dz, \end{aligned} \quad (24)$$

and they are bounded as $|X_k| \leq 2, \forall k$ and $X_0 = 2$. Although its closed form expression is difficult to obtain, we used the *numerical integration* to compute the Fourier coefficients as shown in Fig. 3 (bottom right).

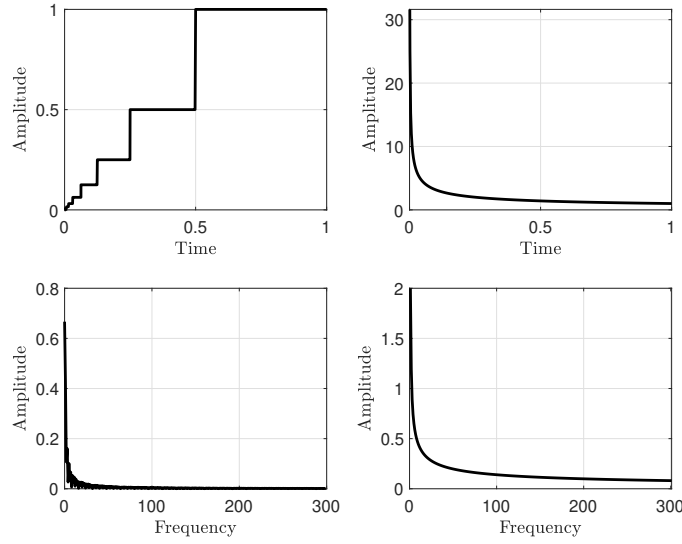


Figure 3: Plot of one period of signals and corresponding spectrum: (i) staircase signal (25) (top left) and its spectrum (bottom left), (ii) signal $\frac{1}{\sqrt{t}}$ (top right) and its spectrum (bottom right)

Case 2: A function that satisfies both the DCs (A1 and A2)

Example-7

Here, we consider a series $x_n(t) = \frac{1}{2^n}$ for $\frac{1}{2^{n+1}} \leq t < \frac{1}{2^n}$, $n = 0, 1, 2, \dots$, and zero otherwise. Now, we construct a periodic signal

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t-m), \text{ where } x(t) = \sum_{n=0}^{\infty} x_n(t), \quad (25)$$

as shown in top Fig. 3. This signal is also incorrectly stated as not having convergent FS representation since it does not satisfy DCs (B2 and B3). Clearly, $x(t) \in L_p(\mathbb{R})$ for $p \in [1, \infty]$, and $x(t) \in BV([0, 1])$ as $V(x, [0, 1]) = 1$. Using Theorems 1–5, its FS converges to it at almost all points.

We compute the Fourier coefficients as

$$\begin{aligned} X_k &= \sum_{n=0}^{\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \frac{1}{2^n} e^{-jk\omega_0 t} dt, \\ &= \frac{j}{k\omega_0} \sum_{n=0}^{\infty} \frac{1}{2^n} \left(e^{-\frac{jk\omega_0}{2^n}} - e^{-\frac{jk\omega_0}{2^{n+1}}} \right), \quad k \neq 0, \end{aligned} \quad (26)$$

$$\text{and } X_0 = \sum_{n=0}^{\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \frac{1}{2^n} dt = \frac{2}{3}, \quad (27)$$

where $\omega_0 = 2\pi$ rad/s. While its difficult to obtain a closed form expression, we observe from (26) that Fourier coefficients are upper bounded as

$$\begin{aligned} |X_k| &= \left| \frac{j}{k\omega_0} \sum_{n=0}^{\infty} \frac{1}{2^n} \left(e^{-\frac{jk\omega_0}{2^n}} - e^{-\frac{jk\omega_0}{2^{n+1}}} \right) \right|, \quad k \neq 0, \\ &\leq \frac{1}{|k|\omega_0} \sum_{n=0}^{\infty} \frac{1}{2^n} \left| \left(e^{-\frac{jk\omega_0}{2^n}} - e^{-\frac{jk\omega_0}{2^{n+1}}} \right) \right|, \quad k \neq 0, \\ &\leq \frac{1}{|k|\omega_0} \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\left| e^{-\frac{jk\omega_0}{2^n}} \right| + \left| e^{-\frac{jk\omega_0}{2^{n+1}}} \right| \right) = \frac{4}{|k|\omega_0}. \end{aligned}$$

Hence, $|X_k| \leq \frac{4}{|k|\omega_0} = \frac{2}{|k|\pi}$ for $k = \pm 1, \pm 2, \dots$, and $X_0 = \frac{2}{3}$.

One may also argue that if FS of (16) and (25) do not converge, then they will create exceptions (counter-examples) to the well established results in Theorem 3, Theorem 4 and Theorem 5, and thus invalidate them.

For the sake of completeness, we provide some examples that neither satisfy (both) DCs, nor the theorem by Hunt. Therefore, their FS do not converge. Say, for example,

(i) $x_1(t) = \frac{1}{t}$, $t \in (0, 1]$

- (ii) $x_2(t) = \frac{1}{t} \sin\left(\frac{2\pi}{t}\right)$, $t \in (0, 1]$
 (iii) $x_3(t) = \sum_{n=0}^{\infty} x_n(t)$, where $x_n(t) = a^n$ for $\frac{1}{c^{n+1}} \leq t < \frac{1}{c^n}$, $c > 1$,
 $n = 0, 1, 2, \dots, \infty$, diverges for $|a| \geq c$ and converges for $0 \leq |a| < c$.

Dirichlet Conditions for Fourier Transform

Fourier transform (FT) is the limiting case of Fourier series as the time period T_0 of a periodic signal $\tilde{x}(t)$ approaches ∞ . In that case, the signal becomes aperiodic and the Fourier transform is the limiting case of the envelope of Fourier series. The Fourier transform (FT) and inverse FT are defined as [1–10]

$$\text{Analysis Equation: } X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt, \quad (28)$$

$$\text{Synthesis Equation: } x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df, \quad (29)$$

respectively. Similar to FS, the DCs for FT can be stated as below:

- (C1) The signal $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty. \quad (30)$$

- (C2) It has BV over $t \in (-\infty, \infty)$, i.e., the signal $x(t) \in BV(\mathbb{R})$. Such signals will have (i) at most countably infinite number of maxima and minima, and (ii) at most countably infinite number of finite discontinuities.

However, the DCs for FT are defined in the signal processing, communication and other literature as [1–10]:

- (D1) The signal $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty. \quad (31)$$

- (D2) It has a finite number of maxima and minima within any finite interval.

- (D3) It has a finite number of finite discontinuities within any finite interval.

These DCs are stated to be the sufficient, but not necessary conditions for the convergence of FT. Similar to the subtle gaps observed in the statements of DCs of FS, (A2) vis-à-vis (B2)-(B3), we observe a few gaps in (C2) vis-à-vis (D2)-(D3). Firstly, signals that satisfy (D2)-(D3) have bounded variations. But there may be signals that have bounded variations with at most countable *infinite* maxima and minima, and countable *infinite* discontinuities. Thus, the restriction to *finite* maxima and minima, and *finite* number of discontinuities incorrectly eliminates some signals that will have convergent FT. For example,

FT of the signals $x(t)$ defined in (4) and (25) exist because they satisfy (C1) and (C2). However, they are incorrectly stated as not having convergent FT because they do not satisfy (D2)-(D3). Secondly, there exist signals with valid FT representation, despite not satisfying DCs (C1) and/or (C2). For example, if $x(t) \in L_2(\mathbb{R})$, then its FT converges in L_2 -norm [1–4]. Next, we present some interesting examples in support of the above statements regarding DCs.

Example-8

We consider a signal $x(t) = \sin\left(\frac{2\pi}{t}\right)$ for $-\infty < t < \infty$, which does not satisfy DCs, i.e., $x(t) \notin L_1(\mathbb{R})$ and $x(t) \notin BV(\mathbb{R})$. This signal is incorrectly stated as not having convergent FT representation. Because $x(t) \in L_2(\mathbb{R})$, its FT converges in L_2 -norm. One may also note that $x(t) \in L_\infty(\mathbb{R})$. To show that $x(t) \in L_2(\mathbb{R})$, we compute its energy E_x as

$$E_x = \int_{-\infty}^{\infty} \sin^2\left(\frac{2\pi}{t}\right) dt = 2 \int_0^{\infty} \sin^2\left(\frac{2\pi}{t}\right) dt. \quad (32)$$

On substituting $2\pi/t = v$, we obtain

$$E_x = 4\pi \int_0^{\infty} \left(\frac{\sin(v)}{v}\right)^2 dv = 2\pi^2. \quad (33)$$

The Fourier transform of this signal is given by

$$X(f) = \int_{-\infty}^{\infty} \sin\left(\frac{2\pi}{t}\right) e^{-j2\pi ft} dt, \quad (34)$$

and numerical integration can be used to compute the spectrum.

Example-9

Interestingly, in contrast to the signal $x(t) = \sin\left(\frac{2\pi}{t}\right)$ for $-\infty < t < \infty$ of Example-8, the signal $x(t) = \cos\left(\frac{2\pi}{t}\right)$ for $-\infty < t < \infty$ does not satisfy DCs. Also, its FT does not converge because $x(t)$ does not belong to $L_p(\mathbb{R})$, $p \in [1, \infty)$, but belongs to only $L_\infty(\mathbb{R})$. We compute its energy E_x as

$$E_x = \int_{-\infty}^{\infty} \cos^2\left(\frac{2\pi}{t}\right) dt = 2 \int_0^{\infty} \cos^2\left(\frac{2\pi}{t}\right) dt. \quad (35)$$

The above energy is not finite because

$$\int_{-\infty}^{\infty} \left[\sin^2\left(\frac{2\pi}{t}\right) + \cos^2\left(\frac{2\pi}{t}\right) \right] dt = \int_{-\infty}^{\infty} 1 dt = \infty, \quad (36)$$

$$\text{and thus, } \int_{-\infty}^{\infty} \cos^2\left(\frac{2\pi}{t}\right) dt = \infty. \quad (37)$$

One can also observe that

$$\int_a^b \cos^2\left(\frac{2\pi}{t}\right) dt + \int_a^b \sin^2\left(\frac{2\pi}{t}\right) dt = (b-a), \quad (38)$$

which is finite for any finite duration $a \leq t \leq b$. Intuition regarding the above computations of energies can be easily made from the graph of the signals $\sin\left(\frac{2\pi}{t}\right)$ and $\cos\left(\frac{2\pi}{t}\right)$ plotted on the time interval $1 \leq t \leq 100$ as shown in the Fig. 4 top and bottom, respectively.

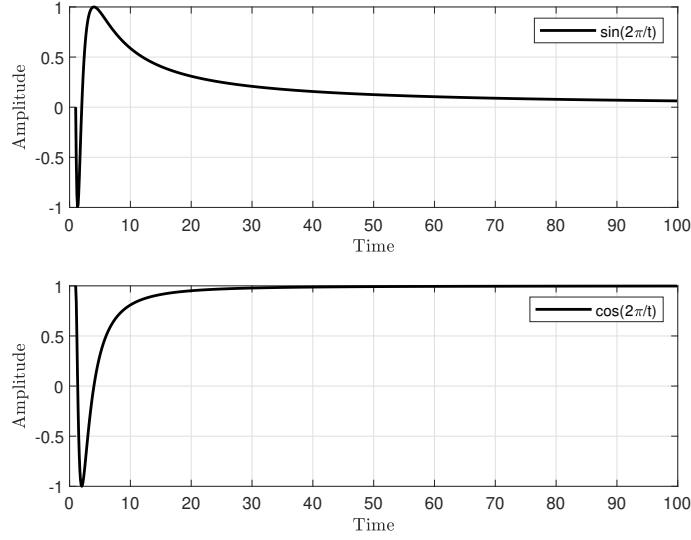


Figure 4: Plot of the signals $\sin\left(\frac{2\pi}{t}\right)$ and $\cos\left(\frac{2\pi}{t}\right)$ on the time interval $1 \leq t \leq 100$.

Example-10

Similar to Example-8, there is another interesting non-periodic signal that is frequently used, i.e., $x(t) = \frac{\sin(\pi t)}{\pi t}$ for $-\infty < t < \infty$, where $x(t) \notin L_1(\mathbb{R})$ and $x(t) \notin BV(\mathbb{R})$, i.e., it does not satisfy DCs. However, it is an $L_2(\mathbb{R})$ function. Its energy (E_x) and FT can be obtained as

$$E_x = \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 dt = 1, \quad (39)$$

$$X(f) = \begin{cases} 1, & \text{if } -1 \leq f \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

The two alternative sets of requirements, i.e., (i) fulfillment of DCs, (ii) finite energy, are sufficient to guarantee that a signal has a Fourier representation. The set of signals which are neither absolutely integrable nor square integrable over an infinite interval, can have Fourier representation if Dirac delta impulse function [19] is included in the Fourier representation [1–4].

Some Concluding Remarks

In this lecture note, we have developed a clear understanding of the Dirichlet conditions with detailed discussions and suitable examples. The clarifications provided here are not available across many popular books, articles, and lecture notes. Also, some of the examples considered in this note are inadvertently used in the existing literature to illustrate signals not having a valid Fourier representation. The Fourier analysis for these widely-used examples has been presented here to reveal some new findings. Therefore, as an outcome of this study, appropriate clarifications may be added to the existing literature to ensure proper dissemination of the intended information to students and researchers working across a plethora of applications utilizing Fourier theory.

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