A Symmetric Uniform Formula and Sole Index Method for Sieving (Twin) Primes

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Abstract. All primes can be indexed by k, as primes must be in the form of 6k+1 or 6k-1. In this paper, we explore for the set of k such that either 6k+1 or 6k-1 is not a prime. Our method provides a uniform formula for k that can sieve primes and twin primes as well. The uniform presents symmetry in terms of k that works as single index in sieving. We also propose a new conjecture that is equivalent to Twin Prime Conjecture but possibly be easier to approach by merely exploring of sole index in term of k.
The uniform formula for prime sieving are as follows:
$k \in S_l \Rightarrow 6k-1 \notin \mathbb{P}$, where $S_l = [-I]_{6I+1} = [I]_{6I-1} \setminus \min([I]_{6I-1}), I \in \mathbb{N}$.
$k \in S_r \Rightarrow 6k+1 \notin \mathbb{P}$, where $S_r = [-I]_{6I-1} \cup ([I]_{6I+1} \setminus \min([I]_{6I+1})), I \in \mathbb{N}$.

Keywords: Twin Prime Conjecture; Computational Number Theory; Algorithm

1 Introduction

Twin Prime Conjecture and Prime testing [1–6] has been explored for a long time. As Twin Prime must be in the form $6k \pm 1$, we explore for what k such that either of $6k \pm 1$ is not a prime. We derive $\{k | 6k+1 \notin \mathbb{P} \vee 6k-1 \notin \mathbb{P}, k \in \mathbb{N}\}$ in an elementary method and the results can be used for approaching Twin Prime Conjecture. Major notations are listed as follows:

1. $\mathbb{P}$: the set of all prime integers.
2. $\mathbb{Z}$: integers.
3. $\mathbb{N}$: positive integers. In this paper, we only discuss $x \in \mathbb{N}$.
4. $x \in [i]_m = \{x | x \in \mathbb{N}, x \mod m = i, m \in \mathbb{N}, m \geq 2, 0 \leq i \leq m-1, i \in \mathbb{Z}\}$.
5. $[i, j]_m = [i]_m \cup [j]_m$.
6. $|S|$ returns the number of items in a set S.

7. $\min(S)$ returns the minimal value in a set S.
8. $gcd(m, n)$ returns the greatest common devisor of m and n.

Proposition 1. $x \in \mathbb{P}, x > 3 \Rightarrow x \in [1, 5]_6$.

Proof. $\forall x \in [0, 2, 4]_6$, $2|x$, thus $x \notin \mathbb{P}$. $\forall x \in [3]_6, x > 3$, $3|x$, thus $x \notin \mathbb{P}$. $\forall x \in \mathbb{N}$, $x \in [0, 2, 4]_6 \cup [3]_6 \cup [1, 5]_6$. Therefore, $\forall x \in P$, $x \in [1, 5]_6$. □

Definition 1. $\mathsf{TwinPrime}(x, y) = \{(x, y)|x, y \in \mathbb{P}, y = x + 2\}$.

Twin Prime Conjecture can be stated as $|\{(x, y)|\mathsf{TwinPrime}(x, y)\}| = +\infty$.

Proposition 2. $\mathsf{TwinPrime}(x, y) \Rightarrow x \in [5]_6 \wedge y \in [1]_6 \Rightarrow \exists k \in \mathbb{N}, x = 6k-1, y = 6k + 1$.

Proof. Due to Proposition 1, $\mathbb{P} \subset [1, 5]_6$.
If $x, y \in [1]_6, y > x$, then $y - x \geq 6$;
If $x, y \in [5]_6, y > x$, then $y - x \geq 6$;
If $x \in [1]_6, y \in [5]_6, y > x$, then $y - x \geq 4$.
Because $y - x = 2$, we have $x \in [5]_6, y \in [1]_6$.
Thus, $\exists k \in \mathbb{N}$, such that $x = 6k - 1, y = 6k + 1$. □

Proposition 3. $|\{x|x = 6k - 1, k \in \mathbb{N}, x \in \mathbb{P}\}| \neq +\infty$
$\Rightarrow |\{(x, y)|\mathsf{TwinPrime}(x, y)\}| \neq +\infty$.

Proof. Let $Set_1 = \{x|x = 6k - 1, k \in \mathbb{N}, x \in \mathbb{P}\}$. If $|Set_1| \neq +\infty$, $\exists x_{max} = \max(Set_1)$. When $y > x_{max} + 2, y \in \mathbb{P}, y = 6k + 1, k \in \mathbb{N}$, y's twin prime does not exist. ($\nexists x = 6k - 1, x \in \mathbb{P}$ because $y - 2 = 6k + 1 - 2 = 6k - 1 > x_{max} = \max(Set_1)$.) □

Proposition 4. $|\{y|y = 6k + 1, k \in \mathbb{N}, y \in \mathbb{P}\}| \neq +\infty$
$\Rightarrow |\{(x, y)|\mathsf{TwinPrime}(x, y)\}| \neq +\infty$.

Proof. The proof is similar to Proposition 3. □

Proposition 5. $|\{(x, y)|\mathsf{TwinPrime}(x, y)\}| = +\infty$
$\Rightarrow |\{x|x = 6k - 1, k \in \mathbb{N}, x \in \mathbb{P}\}| = +\infty \wedge |\{y|y = 6k + 1, k \in \mathbb{N}, y \in \mathbb{P}\}| = +\infty$.

Proof. It is due to Proposition 3 and Proposition 4. □

Proposition 6. $|\{x|x = 6k - 1, k \in \mathbb{N}, x \in \mathbb{P}\}| = +\infty$.

Proof. Straightforward. Suppose primes with forms $6k - 1$ is not infinite. List them as $p_1 < p_2 < ... < p_n$. Let $X = 6p_1p_2...p_n - 1$. X is with a form $6k - 1$, thus $X \notin \mathbb{P}$. $p_i \nmid X$. Thus, only primes with forms $6k + 1$ can be divisors of X. However, the multiplication of those primes must be with form $6k + 1$ instead of $6k - 1$. Thus, $X \in \mathbb{P}$ where contradiction occurs. □

Proposition 7. $|\{y|y = 6k + 1, k \in \mathbb{N}, y \in \mathbb{P}\}| = +\infty$.

Proof. Straightforward. Given arithmetic progression $ax+b$ where $(a,b) = 1$, $ax+b$ is prime infinitely often, due to Dirichlet's theorem. $\square$

Proposition 8. $x \in \{x|x = 6k-1, k \in \mathbb{N}, k = 6A^2, A \in \mathbb{N}\} \Rightarrow x \notin \mathbb{P}$.

Proof. $x = 6k-1 = 6*6A^2-1 = 36A^2-1 = (6A+1)(6A-1) \notin P$. $\square$

Proposition 9. $y \in \{y|y = 6k+1, k \in \mathbb{N}, k = 5B-1, 7C+1, B, C \in \mathbb{N}\} \Rightarrow y \notin \mathbb{P}$.

Proof. $y = 6k+1 = 6*(5B-1)+1 = 30B-5 = 5*(6B-1) \notin \mathbb{P}$. $y = 6k+1 = 6*(7C+1)+1 = 42C+7 = 7*(6C+1) \notin P$. $\square$

2 Analysis of $\{k|6k+1 \notin \mathbb{P}, k \in \mathbb{N}\}$

Suppose $\exists m, t \in \mathbb{N}, 6k+1 = m*t, k \in \mathbb{N}$. As $6k+1 \in [1]_2$, $m \in [1]_2, m \geq 3$, and $t \in [1]_2, t \geq 3$. $6k+1 = m*t \geq 3*3 = 9$, thus, $k \geq 2$.

$1 = t*m-6*k = (t-k)*m+(m-6)*k$ will be explored for its solutions.

Proposition 10. $\forall m, n \in \mathbb{N}, m > n, gcd(m,n) = 1 \Leftrightarrow \exists s, t \in \mathbb{Z}, s \neq 0, t \neq 0, m*s+n*t = 1,$ *and* $t \equiv n^{-1} \mod m$.

Proof. Straightforward due to extended Euclid algorithm. $\square$

(1) $m > 6, m \in [1]_2$.

Observe $1 = t*m-6*k = (t-k)*m+(m-6)*k$, where $t \in [1]_2, t \geq 3$, $k \in \mathbb{N}, m \in [1]_2, m > 6$.

If $gcd(m, m-6) = 1$, then $\exists t-k, k \in \mathbb{Z}$, $t-k \neq 0, k \neq 0$ such that $(t-k)*m+(m-6)*k = 1$. That is, if $gcd(m, m-6) = 1$, then $\exists t \in [1]_2, t \geq 3, \exists k \in \mathbb{Z}$, $t \neq k, k \neq 0$ such that $(t-k)*m+(m-6)*k = 1$.

Next, observe $gcd(m, m-6) = 1$. Recall that $m > 6, m \in [1]_2$.

Proposition 11. $\forall a, b \in \mathbb{N}, a > b,$ $gcd(a, a-b) = gcd(a, b)$.

Proof. Let $gcd(a, a-b) = c$, $c|a, c|(a-b)$. Thus, $c|a-(a-b) = b$.
As $c|a, c|b$, thus, $c|gcd(a,b)$. That is, $gcd(a, a-b)|gcd(a,b)$
Let $gcd(a,b) = d$, $d|a, d|b$. Thus, $d|a-b$.
As $d|a, d|a-b$, thus $d|gcd(a, a-b)$. That is, $gcd(a,b)|gcd(a, a-b)$.
Thus, $gcd(a, a-b) = gcd(a,b)$. $\square$

Thus, $gcd(m, m-6) = 1 \Rightarrow gcd(m, 6) = 1 \Rightarrow m = 6D+1, 6D+5, D \in \mathbb{N}$.

$k \equiv (m-6)^{-1} \mod m$, thus

$$\begin{cases} k \equiv (6D-5)^{-1} \mod 6D+1 & D \in \mathbb{N} \\ k \equiv (6D-1)^{-1} \mod 6D+5 & D \in \mathbb{N} \end{cases} \tag{1}$$

More specifically, $(6D-5)D = (6D+1)D-6D = (6D+1)D-6D-1+1 = (6D+1)(D-1)+1$. Thus,

$$(6D-5)^{-1} \equiv D \mod 6D+1.$$

$(6D-1)(5D+4) = 30D^2+19D-4 = (6D+5)5D-6D-4 = (6D+5)5D-6D-5+1 = (6D+5)(5D-1)+1$. Thus,

$$(6D-1)^{-1} \equiv 5D+4 \mod 6D+5.$$

That is, Eq. 2 can be written as Eq. 5

$$\begin{cases} k \equiv D \mod 6D+1 & D \in \mathbb{N} \\ k \equiv 5D+4 \mod 6D+5 & D \in \mathbb{N} \end{cases} \tag{2}$$

(1.1) Let $k = D, m = 6D+1$.
Check
$1 = (t-k)*m + (m-6)*k = (t-D)(6D+1) + (6D-5)D$
$= 6D(t-1) + t = (6D+1)(t-1)+1$. Thus, $t = 1 \ngeq 3$.
(1.2) Therefore, let $k = (6D+1)*W_1 + D, W_1 \in \mathbb{N}$.
Check
$1 = (t-k)*m + (m-6)*k$
$= (t - 6DW_1 - W_1 - D)(6D+1) + (6D-5)(6DW_1 + W_1 + D)$
$= 6Dt - (6DW_1 + W_1 + D) + t - 6D(6DW_1 + W_1 + D) + 6D(6DW_1 + W_1 + D) - 5(6DW_1 + W_1 + D)$
$= (6D+1)t - 6(6DW_1 + W_1 + D)$
$= (6D+1)t - 6(6D+1)W_1 - 6D$
$= (6D+1)(t - 6W_1) - 6D - 1 + 1$
$= (6D+1)(t - 6W_1 - 1) + 1$. Thus, $t = 6W_1 + 1 \geq 3, t \in [1]_2$.
(1.3) Similarly, let $k = (6D+5)*W_2 + 5D + 4, W_2 \in \mathbb{Z}$, $m = 6D+5$.
Check
$1 = (t-k)*m + (m-6)*k$
$= (t-k)(6D+5) + (6D-1)k$
$= t(6D+5) - k(6D+5) + (6D+5)k - 6k$
$= t(6D+5) - 6((6D+5)W_2 + 5D + 4)$
$= (6D+5)(t - 6W_2) - 6(5D+4)$
$= (6D+5)(t - 6W_2 - 5) + 30D + 25 - 30D - 24$
$= (6D+5)(t - 6W_2 - 5) + 1$. Thus, $t = 6W_2 + 5 \geq 3, t \in [1]_2$.
Therefore, $6k+1 \notin \mathbb{P}$, if

$$k = \begin{cases} (6D+1)*W_1 + D & W_1 \in \mathbb{N}, D \in \mathbb{N} \\ (6D+5)*W_2 + 5D + 4 & W_2 \in \mathbb{Z}, D \in \mathbb{N} \end{cases} \tag{3}$$

(2) $3 \leq m \leq 6$. As $m \in [1]_2$, we have $m = 5, 3$.
If $m = 3$, then $6k+1 = 3t$, which is impossible.
If $m = 5$, then $6k+1 = 5t$. $k+1 = 5(t-k)$. Thus, $k \in [4]_5$.
Let $k = 5W_3 + 4, W_3 \in \mathbb{Z}$. Check $k+1 = 5W_3 + 4 + 1 = 5(W_3+1)$. $t - k = W_3 + 1$. $t = W_3 + 1 + k = W_3 + 1 + 5W_3 + 4 = 6W_3 + 5 \geq 3, t \in [1]_2$. Or, $6k+1 = 6(5W_3+4)+1 = 30W_3 + 25 = 5(6W_3+5) \notin \mathbb{P}$.

Combining (1) and (2) and in summary, $6k+1 \notin \mathbb{P}$, if

$$k = \begin{cases} (6D+1)W_1 + D & W_1 \in \mathbb{N}, D \in \mathbb{N} \\ (6D+5)W_2 + 5D + 4 & W_2 \in \mathbb{Z}, D \in \mathbb{N} \\ 5W_3 + 4 & W_3 \in \mathbb{Z} \end{cases} \tag{4}$$

That is,

$$k = \begin{cases} (6D+1)W_1 + D & W_1 \in \mathbb{N}, D \in \mathbb{N} \\ (6D+5)W_2 + 5D + 4 & W_2 \in \mathbb{Z}, D \in \mathbb{Z} \end{cases} \tag{5}$$

Or,

$$k \in \begin{cases} [D]_{6D+1} \backslash \{D\} & D \in \mathbb{N} \\ [5D+4]_{6D+5} & D \in \mathbb{Z} \end{cases} \tag{6}$$

Or,

$$k \in \begin{cases} [D]_{6D+1} \backslash \{D\} & D \in \mathbb{N} \\ [5D+4]_{6D+5} \cup [4]_5 & D \in \mathbb{N} \end{cases} \tag{7}$$

Or,

$$k \in \begin{cases} [D]_{6D+1} \backslash \{D\} \cup [5D+4]_{6D+5} & D \in \mathbb{N} \\ [4]_5 & \end{cases} \tag{8}$$

3 Analysis of $\{k|6k-1 \notin \mathbb{P}, k \in \mathbb{N}\}$

Next, we explore for what $k \in \mathbb{N}$, $6k-1 \notin \mathbb{P}$. That is, explore $\exists t \in \mathbb{N}$, such that $6*k-1 = t*m$. As $6*k-1 \in [1]_2$, $t \in [1]_2, m \in [1]_2$. $t \geq 3, m \geq 3$, thus $6*k-1 \geq 3*3 = 9$, $k \geq 2$.

(1) $2 \leq m < 6, m \in [1]_2$.

Observe $1 = 6*k - t*m = (6-m)*k + (k-t)*m$, where $t \in [1]_2, t \geq 3, k \in \mathbb{N}, m \in [1]_2, 2 \leq m < 6$. Thus, $m = 3, 5$.

If $gcd(m, 6-m) = 1$, then $\exists k-t, k \in \mathbb{Z}, k-t \neq 0, k \neq 0$ such that $(6-m)*k + (k-t)*m = 6*k - t*m = 1$.

Proposition 12. *$\forall a, b \in \mathbb{N}$, $gcd(a, a+b) = gcd(a, b)$.*

Proof. Let $gcd(a, a+b) = c$, $c|a, c|a+b$. Thus, $c|a+b-a = b$.
As $c|a, c|b$, thus, $c|gcd(a,b)$. That is, $gcd(a, a+b)|gcd(a,b)$
Let $gcd(a,b) = d$, $d|a, d|b$. Thus, $d|a+b$.
As $d|a, d|a+b$, thus $d|gcd(a, a+b)$. That is, $gcd(a,b)|gcd(a, a+b)$.
Thus, $gcd(a, a+b) = gcd(a,b)$. □

$gcd(6-m, m) = 1 \Rightarrow gcd(m, 6) = 1 \Rightarrow m = 5$. $6*k-1 = 5t \in [0]_5$, thus $k \in [1]_5$.

Let $k = 5X_1+1, X_1 \in \mathbb{N}$. $6k-1 = 6(5X_1+1)-1 = 30X_1+5 = 5(6X_1+1) \notin \mathbb{P}$.

(2) $m > 6, m \in [1]_2$.

$1 = 6*k - t*m = (6+m)*k + (-t-k)*m, t, k \in \mathbb{N}, m \in [1]_2, m > 6$.

If $gcd(6+m,m)=1$, then $\exists t,k\in\mathbb{N}$, such that $(6+m)*k+(-t-k)*m=6*k-t*m=1$.

$gcd(6+m,m)=1$
$\Rightarrow gcd(m,6)=1$
$\Rightarrow m\in[1,5]_6$
$\Rightarrow m=6I+1,6I+5,I\in\mathbb{N}$.

$k\equiv(6+m)^{-1} \mod m$
$\Rightarrow k\equiv 6^{-1} \mod m$
$\Rightarrow k\equiv 6^{-1} \mod 6I+1,\ \ k\equiv 6^{-1} \mod 6I+5$.

$6(5I+1)=(6I+1)5+1$. Thus, $6^{-1}\equiv 5I+1 \mod 6I+1$.

Let $k=(6I+1)X_2+5I+1, X_2\in\mathbb{Z}$.

$6k-1=6((6I+1)X_2+5I+1)-1$
$=6(6I+1)X_2+30I+6-1$
$=6(6I+1)X_2+5(6I+1)$
$=(6I+1)(6X_2+5)\notin\mathbb{P}$.

$6*(I+1)=(6I+5)+1$. Thus, $6^{-1}\equiv I+1 \mod 6I+5$.

Let $k=(6I+5)X_3+I+1, X_3\in\mathbb{N}$.

$6k-1=6((6I+5)X_3+I+1)-1$
$=6((6I+5)X_3+6I+6-1$
$=(6I+5)(6X_3+1)\notin\mathbb{P}$.

Together with the result in (1), $6k-1\notin\mathbb{P}$, if

$$k=\begin{cases} 5X_1+1 & X_1\in\mathbb{N}\\ (6I+1)X_2+5I+1 & X_2\in\mathbb{Z},I\in\mathbb{N}\\ (6I+5)X_3+I+1 & X_3\in\mathbb{N},I\in\mathbb{N}\end{cases} \tag{9}$$

That is,

$$k=\begin{cases} (6I+1)X_1+5I+1 & X_1\in\mathbb{Z},I\in\mathbb{N}\\ (6I+5)X_2+I+1 & X_2\in\mathbb{N},I\in\mathbb{N}\end{cases} \tag{10}$$

Or,

$$k\in\begin{cases} [5I+1]_{6I+1} & I\in\mathbb{N}\\ [I+1]_{6I+5}\backslash\{I+1\} & I\in\mathbb{N}\end{cases} \tag{11}$$

4 Analysis of $\{k|6k\pm 1\notin\mathbb{P},k\in\mathbb{N}\}$

(1) Due to Eq. 7, we have

$$k\in\begin{cases} [D]_{6D+1}\backslash\{D\} & D\in\mathbb{N}\\ [5D+4]_{6D+5}\cup[4]_5 & D\in\mathbb{N}\end{cases} \Rightarrow 6k+1\notin\mathbb{P} \tag{12}$$

Note that, recall Eq. 5, $(6D+5)W_2+5D+4=(6(D+1)-1)W_2+5(D+1)-1=(6E-1)*W_2+5E-1, W_2\in\mathbb{Z}, D\in\mathbb{Z}$, thus $E=D+1\in\mathbb{N}$.

Thus, Eq. 12 can be rewritten as follows:

$$k = \begin{cases} (6D+1)W_1 + D & W_1 \in \mathbb{N}, D \in \mathbb{N} \\ (6E-1)W_2 + 5E - 1 & W_2 \in \mathbb{Z}, E \in \mathbb{N} \end{cases} \Rightarrow 6k+1 \notin \mathbb{P} \tag{13}$$

Or,

$$k \in \begin{cases} [D]_{6D+1} \backslash \{D\} & D \in \mathbb{N} \\ [5E-1]_{6E-1} & E \in \mathbb{N} \end{cases} \Rightarrow 6k+1 \notin \mathbb{P} \tag{14}$$

Or,

$$k \in [D]_{6D+1} \backslash \{D\} \cup [5D-1]_{6D-1}, D \in \mathbb{N} \Rightarrow 6k+1 \notin \mathbb{P}.$$

(2) Due to Eq. 11, we have

$$k \in \begin{cases} [5I+1]_{6I+1} & I \in \mathbb{N} \\ [I+1]_{6I+5} \backslash \{I+1\} & I \in \mathbb{N} \end{cases} \Rightarrow 6k-1 \notin \mathbb{P} \tag{15}$$

Note that, recall Eq. 10, $(6I+5)X_2 + I + 1 = (6(I+1)-1) * X_2 + (I+1) = (6J-1)X_2 + J, X_2, I \in \mathbb{N}, J = I+1 \in \mathbb{N}, J \geq 2$.

Thus, Eq. 15 can be rewritten as follows:

$$k = \begin{cases} (6I+1)X_1 + 5I + 1 & X_1 \in \mathbb{Z}, I \in \mathbb{N} \\ (6J-1)X_2 + J & X_2 \in \mathbb{N}, J \in \mathbb{N}, J \geq 2 \end{cases} \Rightarrow 6k-1 \notin \mathbb{P} \tag{16}$$

Or,

$$k \in \begin{cases} [5I+1]_{6I+1} & I \in \mathbb{N} \\ [J]_{6J-1} \backslash \{J\} & J \in \mathbb{N}, J \geq 2 \end{cases} \Rightarrow 6k-1 \notin \mathbb{P} \tag{17}$$

Or,

$$k \in [5I+1]_{6I+1} \cup [I]_{6I-1} \backslash \{I\} \backslash [1]_5, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}.$$

Note that, $[1]_5 \subset [5I+1]_{6I+1} \cup \{1\}, I \in \mathbb{N}$. The proof is as follows:

$[1]_5 = \{a | a = 5 * K + 1, K \in \mathbb{Z}\}$. $\forall x \in [1]_5 \Rightarrow \exists K \in \mathbb{Z}$ such that $x = 5 * K + 1 \Rightarrow x \in [5I+1]_{6I+1} \cup \{1\}$, since $\min([5I+1]_{6I+1}) = 5I + 1 = x$ when $I = K$ and $x > 1$ ($x = 1$ when $K = 0$ is trivial due to $x \in \{1\}$.)

Therefore,

$$k \in [5I+1]_{6I+1} \cup [I]_{6I-1} \backslash \{I\}, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}.$$

(3) Summarizing (1) and (2), therefore, we have following result that looks more symmetrical.

$k \in [5I+1]_{6I+1} \cup [I]_{6I-1} \backslash \{I\}, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}$.

$k \in [I]_{6I+1} \backslash \{I\} \cup [5I-1]_{6I-1}, I \in \mathbb{N} \Rightarrow 6k+1 \notin \mathbb{P}$.

That is,

$$\begin{cases} k \in [I]_{6I-1} \backslash \{I\} \cup [5I+1]_{6I+1}, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}, \\ k \in [5I-1]_{6I-1} \cup [I]_{6I+1} \backslash \{I\}, I \in \mathbb{N} \Rightarrow 6k+1 \notin \mathbb{P}. \end{cases} \tag{18}$$

Or,

$$\begin{cases} k \in [I]_{6I-1} \backslash \min([I]_{6I-1}) \cup [5I+1]_{6I+1}, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}, \\ k \in [5I-1]_{6I-1} \cup [I]_{6I+1} \backslash \min([I]_{6I+1}), I \in \mathbb{N} \Rightarrow 6k+1 \notin \mathbb{P}. \end{cases} \tag{19}$$

Or,

$$\begin{cases} k \in [I]_{6I-1} \backslash \min([I]_{6I-1}) \cup [-I]_{6I+1}, I \in \mathbb{N} \Rightarrow 6k-1 \notin \mathbb{P}, \\ k \in [-I]_{6I-1} \cup [I]_{6I+1} \backslash \min([I]_{6I+1}), I \in \mathbb{N} \Rightarrow 6k+1 \notin \mathbb{P}. \end{cases} \tag{20}$$

Lemma 1. $\bigcup_{I\in\mathbb{N}}[-I]_{6I+1} \subseteq \bigcup_{I\in\mathbb{N}}[I]_{6I-1} \backslash \min([I]_{6I-1})$.

Proof. $\forall k \in \bigcup_{I\in\mathbb{N}}[-I]_{6I+1}$, $\exists J \in \mathbb{N}$ such that $k \in [-J]_{6J+1}$, thus $k = (6J+1)*W - J, W \in \mathbb{N}$. Note that, $k = (6J+1)*W - J = (6J-1)*W + J + (2W-2J)$. When $W = J$, then

$k \in [J]_{6J-1} \backslash \min([I]_{6I-1}) \quad \because W \in \mathbb{N} \Rightarrow k = (6J-1)*W + J > J$
$\subseteq \bigcup_{I\in\mathbb{N}}[I]_{6I-1} \backslash \min([I]_{6I-1})$.

Obviously, W is determined by J. □

Lemma 2. $\bigcup_{I\in\mathbb{N}}[I]_{6I-1} \backslash \min([I]_{6I-1}) \subseteq \bigcup_{I\in\mathbb{N}}[-I]_{6I+1}$.

Proof. $\forall k \in \bigcup_{I\in\mathbb{N}}[I]_{6I-1} \backslash \min([I]_{6I-1})$, $\exists J \in \mathbb{N}$ such that $k \in [J]_{6J-1} \backslash \min([J]_{6J-1})$, thus $k = (6J-1)*W + J, W \in \mathbb{N}$ (instead of $W \in \mathbb{Z}$ since $k > J$ due to $\backslash \min([J]_{6J-1})$). Note that, $k = (6J-1)*W + J = (6J+1)*W - J + (2J-2W)$. When $W = J$, then

$k \in [-J]_{6J+1} \subseteq \bigcup_{I\in\mathbb{N}}[-I]_{6I+1}$. Obviously, W is determined by J. □

Theorem 1. $\bigcup_{I\in\mathbb{N}}[I]_{6I-1} \backslash \min([I]_{6I-1}) = \bigcup_{I\in\mathbb{N}}[-I]_{6I+1}$.

Proof. It is straightforward due to Lemma 1 and Lemma 2. □

More specifically, we discover a mapping between the j-th element in all residue classes in $[I]_{6I-1} \backslash \min([I]_{6I-1})$ and a residue class in $[-I]_{6I+1}$. In other words or roughly speaking, a column exactly equals a row, if those two residue classes are looked as a matrix. Next, we explain this result in the following.

Definition 2. *Function $\pi(\cdot,\cdot)$: $s \times i \in \mathbb{N}$ takes as input a set of sets s whose elements are ordered increasingly and a sequence number i, outputs the i-th element (minimal) in s.*

Recall that $[I]_{6I-1} \backslash \min([I]_{6I-1}), I \in \mathbb{N}$ is a set of sets consisting of $[i]_{6i-1} \backslash \min([i]_{6i-1}), i = 1, 2, 3, ...$

Theorem 2. $\pi(S = [I]_{6I-1} \backslash \min([I]_{6I-1}), j) = [-j]_{6j+1}$.

Proof. $\forall s_i \in S = [I]_{6I-1} \backslash \min([I]_{6I-1})$. W.o.l.g., let $s_i = [i]_{6i-1} \backslash \min([i]_{6i-1}) = \{k | k = (6i-1)*W + i, W \in \mathbb{N}\}$. Thus, the j-th minimal element in s_i is $(6i-1)j + i$. $\forall i \in \mathbb{N}$, $(6i-1)j + i = 6ji - j + i = (6j+1)i - j$. That is, $\{k | k = (6i-1)j + i, i \in \mathbb{N}\} = [-j]_{6j+1}$. □

Corollary 1. *$A^t = B$, where A is a matrix generated by $\bigcup_{I\in\mathbb{N}}[I]_{6I-1}\setminus\min([I]_{6I-1})$, B is a matrix generated by $\bigcup_{I\in\mathbb{N}}[-I]_{6I+1}$, t means matrix transposition, the i-th row of A is an increasingly ordered set $[i]_{6i-1}\setminus\{i\} = \{k|k = (6i-1)*W+i, W \in \mathbb{N}\}$, the i-th row of B is an increasingly ordered set $[-i]_{6i+1} = \{k|k = (6i+1)*W-i, W \in \mathbb{N}\}$.*

In other words, $A[x,y] = x+(6x-1)*y$ and $B[x,y] = -x+(6x+1)*y, x, y \in \mathbb{N}$.

Proof. It is straightforward due to Theorem 2. Alternatively, $A[x,y]^t = A[y,x] = y + (6y-1)*x = y + 6yx - x = -x + (6x+1)*y = B[x,y]$. □

Corollary 2. $[I]_{6I-1}\setminus\min([I]_{6I-1}) = [-I]_{6I+1} =$
$\{k|k = (6I-1)*W+I, W \in \mathbb{N}, I \le W, I \in \mathbb{N}\} \cup$
$\{k|k = (6I+1)*W-I, W \in \mathbb{N}, I \le W, I \in \mathbb{N}\}$.

Proof. It is straightforward due to Theorem 2 or Corollary 1. Simply speaking, since $A^t = B$, $A(B)$'s upper triangle is $B(A)$'s lower triangle. Thus, B's upper triangle combines A's upper triangle equals total A or B. □

Besides, we discover that $[-I]_{6I-1} \cup [I]_{6I+1}\setminus\min([I]_{6I+1}), I \in \mathbb{N}$ itself is an symmetric matrix, if the sets are listed row by row as a matrix and each row is $[-I]_{6I-1}$ or $[I]_{6I+1}\setminus\min([I]_{6I+1})$.

Theorem 3. *$\pi(S = [-I]_{6I-1}, j) = [-j]_{6j-1}$.*

Proof. $\forall s_i \in S = [-I]_{6I-1}$. W.o.l.g., let $s_i = [-i]_{6i-1} = \{k|k = (6i-1)*W-i, W \in \mathbb{N}\}$. Thus, the j-th minimal element in s_i is $(6i-1)j-i$. $\forall i \in \mathbb{N}$, $(6i-1)j-i = 6ji-j-i = (6j-1)i-j$. That is, $\{k|k = (6i-1)j+i, i \in \mathbb{N}\} = [-j]_{6j-1}$. □

Theorem 4. *$\pi(S = [I]_{6I+1}\setminus\min([I]_{6I+1}), j) = [j]_{6j+1}$.*

Proof. $\forall s_i \in S = [I]_{6I+1}\setminus\min([I]_{6I+1})$. W.o.l.g., let $s_i = [i]_{6i+1}\setminus\min([i]_{6i+1}) = \{k|k = (6i+1)*W+i, W \in \mathbb{N}\}$. Thus, the j-th minimal element in s_i is $(6i+1)j+i$. $\forall i \in \mathbb{N}$, $(6i+1)j+i = 6ji+j+i = (6j+1)i+j$. That is, $\{k|k = (6i+1)j+i, i \in \mathbb{N}\} = [j]_{6j+1}$. □

Corollary 3. *$A^t = A$, where A is a matrix generated by $[-I]_{6I-1}$, $I \in \mathbb{N}$; t means matrix transposition; the i-th row of A is an increasingly ordered set $[-i]_{6i-1}$. In other words, $A[x,y] = -x+(6x-1)*y$.*

Proof. It is straightforward due to Theorem 3. Alternatively, $A[x,y]^t = A[y,x] = -y + (6y-1)*x = -y + 6xy - x = -x + (6x-1)*y = A[x,y]$. □

Corollary 4. *$A^t = A$, where A is a matrix generated by $[I]_{6I+1}\setminus\min([I]_{6I+1})$, $I \in \mathbb{N}$; t means matrix transposition; the i-th row of A is an increasingly ordered set $[i]_{6i+1}\setminus\{i\}$. In other words, $A[x,y] = x+(6x+1)*y$.*

Proof. It is straightforward due to Theorem 4. Alternatively, $A[x,y]^t = A[y,x] = y + (6y+1)*x = y + 6xy + x = x + (6x+1)*y = A[x,y]$. □

Corollary 5. $[-I]_{6I-1} = \{k|k = (6I-1)*W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}.\}$

Proof. It is straightforward due to Theorem 3. $\square$

Corollary 6. $[I]_{6I+1} \setminus \min([I]_{6I+1}), I \in \mathbb{N} = \{k|k = (6I+1)*W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$.

Proof. It is straightforward due to Theorem 4. $\square$

5 6k index Conjecture

Let $S_l = [-I]_{6I+1} = [I]_{6I-1} \setminus \min([I]_{6I-1}), I \in \mathbb{N}$.
Let $S_r = [-I]_{6I-1} \cup [I]_{6I+1} \setminus \min([I]_{6I+1}), I \in \mathbb{N}$.
Recall that, $k \in S_l \Rightarrow 6k-1 \notin \mathbb{P}$, $k \in S_r \Rightarrow 6k+1 \notin \mathbb{P}$.

Proposition 13. $\forall k \in \mathbb{N}$, $k \notin S_l \Rightarrow 6k-1 \in \mathbb{P}$.
$\forall k \in \mathbb{N}$, $k \notin S_r \Rightarrow 6k+1 \in \mathbb{P}$.

Proof. Straightforward. $\square$

Proposition 14. *Given* $\forall K_t \in \mathbb{N}, \exists k \geq K_t, k \notin S_l$.

Proof. Straightforward. It is due to Proposition 6. $\square$

Proposition 15. *Given* $\forall K_t \in \mathbb{N}, \exists k \geq K_t, k \notin S_r$.

Proof. Straightforward. It is due to Proposition 7. $\square$

Conjecture 1. (6k index Conjecture.) Given $\forall K_t \in \mathbb{N}$, $\exists k \geq K_t, k \in \mathbb{N}$, such that $k \notin S_l \wedge k \notin S_r$.

Proposition 16. *6k index Conjecture is equivalent to Twin Prime Conjecture.*

Proof. If Trap Conjecture is true, that is, $\forall K_t \in \mathbb{N}$, $\exists k > K_t$, $k \notin S_l \wedge k \notin S_r$. Thus, $6k-1 \in \mathbb{P}$ and $6k+1 \in \mathbb{P}$. Let $x = 6k-1, y = 6k+1$; those are Twin Prime. Thus, Twin Prime Conjecture is true.

Similarly, if Twin Prime Conjecture is true, the Trap Conjecture is true. $\square$

Proposition 17. *If* $\exists K_t \in \mathbb{N}$, $\forall k \geq K_t$, $k \in S_l \vee k \in S_r$, *then Twin Prime Conjecture is false.*

Proof. Straightforward. $\square$

Proposition 18. *If given* $\forall K_t \in \mathbb{N}$, $\exists k \geq K_t, k \in \mathbb{N}$, $k \notin S_l \wedge k \notin S_r$, *then Twin Prime Conjecture is True.*

Proof. Straightforward. $\square$

Trap Conjecture and Proposition 18 provides sufficient and necessary condition for the proof of soundness and completeness of Twin Prime Conjecture. We depict a graph to show the rationale for better understanding in Fig. 1.

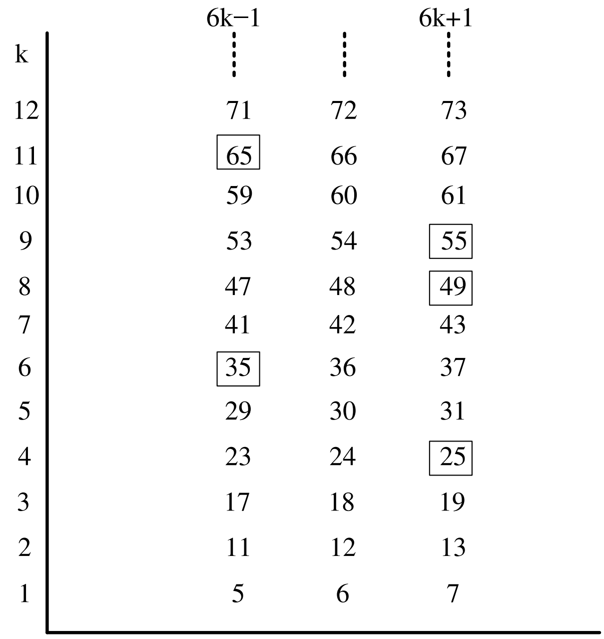

Fig. 1. The sieve of non-prime numbers in $6k-1, 6k+1, k \in \mathbb{N}$. If there exist either trap (denoted as a box) in column $(6k-1)$ and column $(6k+1)$ for any $k > K_t$ (k is the row number), then Twin Prime Conjecture is false. Otherwise, when and only when $\forall K_t, \exists k > K_t$ such that at k row there exists no box at either column, then Twin Prime Conjecture is true.

6 Applications

Let $S_l = S_{l1} \cup S_{l2}$, $S_r = S_{r1} \cup S_{r2}$.

Proposition 19. *$k \notin (S_{l1} \cup S_{l2}) \Rightarrow 6k-1 \in \mathbb{P}$, where $S_{l1} = \{k|k = (6I-1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$ and $S_{l2} = \{k|k = (6I+1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$.*

Proof. It is straightforward due to Corollary 2. □

Proposition 20. *$k \notin (S_{r1} \cup S_{r2}) \Rightarrow 6k+1 \in \mathbb{P}$, where $S_{r1} = \{k|k = (6I-1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$ and $S_{r2} = \{k|k = (6I+1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$.*

Proof. It is straightforward due to Corollary 5 and Corollary 6. □

If we can obtain the concrete set of S_l and S_r, then we will be able to generate primes directly. Recall that, $S_l = S_{l1} \cup S_{l2}$, $S_r = S_{r1} \cup S_{r2}$.

$S_{l1} = \{k|k = (6I-1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW - W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy + (x-y), x, y \in \mathbb{N}, x \leq y\}$.

$S_{l2} = \{k|k = (6I+1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW + W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy - (x-y), x, y \in \mathbb{N}, x \leq y\}$.

$S_{r1} = \{k|k = (6I-1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW - W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy - (x+y), x, y \in \mathbb{N}, x \leq y\}$.

$S_{r2} = \{k|k = (6I+1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW + W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy + (x+y), x, y \in \mathbb{N}, x \leq y\}$.

Proposition 21. *If $\forall K_t \in \mathbb{N}, \exists k \in \mathbb{N}$, $k > K_t, k \notin (S_{l1} \cup S_{l2} \cup S_{r1} \cup S_{r2})$, then Twin Prime Conjecture is True.*

Proof. Straightforward. □

Proposition 22. $\forall K_t \in \mathbb{N}$, $\exists k > K_t, k \notin (S_{l1} \cup S_{l2})$.

Proof. Straightforward. The number of prime with form $6k-1$ is infinite. □

Proposition 23. $\forall K_t \in \mathbb{N}$, $\exists k > K_t, k \notin (S_{r1} \cup S_{r2})$.

Proof. Straightforward. The number of prime with form $6k+1$ is infinite. □

Proposition 19 and Proposition 20 provide a method (or algorithm) to generate primes.

Following proposition provides a method (or algorithm) to generate twin-primes.

Proposition 24. $\forall k \in \mathbb{N}$, $k \notin S_l \wedge k \notin S_r \Rightarrow \mathsf{TwinPrim}(6k-1, 6k+1)$.

Proof. Straightforward. □

7 Conclusion

In this paper, we derive $\{k|6k-1 \notin \mathbb{P}, k \in \mathbb{N}\}$ and $\{k|6k+1 \notin \mathbb{P}, k \in \mathbb{N}\}$ to approach Twin Prime conjecture. We find that

$k \in S_l \Rightarrow 6k-1 \notin \mathbb{P}$, where $S_l = [-I]_{6I+1} = [I]_{6I-1} \setminus \min([I]_{6I-1}), I \in \mathbb{N}$.

$k \in S_r \Rightarrow 6k+1 \notin \mathbb{P}$, where $S_r = [-I]_{6I-1} \cup ([I]_{6I+1} \setminus \min([I]_{6I+1})), I \in \mathbb{N}$.

That is,

$k \notin (S_{l1} \cup S_{l2}) \Rightarrow 6k-1 \in \mathbb{P}$ where

$S_{l1} = \{k|k = (6I-1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW - W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy + (x-y), x, y \in \mathbb{N}, x \leq y\}$;

$S_{l2} = \{k|k = (6I+1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW + W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy - (x-y), x, y \in \mathbb{N}, x \leq y\}$.

$k \notin (S_{r1} \cup S_{r2}) \Rightarrow 6k+1 \in \mathbb{P}$, where

$S_{r1} = \{k|k = (6I-1) * W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW - W - I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy - (x+y), x, y \in \mathbb{N}, x \leq y\}$;

$S_{r2} = \{k|k = (6I+1) * W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6IW + W + I, W \in \mathbb{N}, I \leq W, I \in \mathbb{N}\}$
$= \{k|k = 6xy + (x+y), x, y \in \mathbb{N}, x \leq y\}$.

We also propose 6k index conjecture that is equivalent to Twin Prime Conjecture.

The source codes and outputting data by computer programmers used to support the findings of this study can be downloaded from IEEE Dataport [7].

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