The Schott Energy and the Reactive Energy in Electromagnetic Radiation and Mutual Coupling

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Abstract

In the proposed theory for analyzing the electromagnetic radiation and electromagnetic mutual couplings in vacuum, the electromagnetic energy associated with a source is separated into three parts: a Coulomb-velocity energy, a radiative energy and a macroscopic Schott energy. At the instant that the sources disappear, the Coulomb-velocity energy disappers simultaneously; a short time later, the total macroscopic Schott energy becomes zero, while the radiative energy keeps propagating and the total radiative energy becomes constant. By applying the Lienard-Wiechert potentials to a moving charge, this paper illustrates that the macroscopic Schott energy is corresponding to the Schott energy in the charged particle theory.

Keywords: Schott energy, reactive energy, radiative energy, Lienard-Wiechert potential, electromagnetic coupling

1. Introduction

As observed by many researchers, although the classical electromagnetic theory has achieved tremendous success, some issues are still not solved satisfactorily, or have to be explained with quantum electromagnetic theory [1]-[6], such as the 4/3 factor problem [2], the physical meaning of the vector potential, the violation of equivalence between mass and energy of an electron [1][5], and the electromagnetic radiation and coupling problem. They become century old issues not only because they are complicated but may also because they are probably considered as small flaws to the classical electromagnetic theory and have limited impact on engineering applications so far. However, it is still of great significance to find proper solutions or interpretations to them within the frame of the classical electromagnetic theory, especially the issue concerning with the electromagnetic radiation and the electromagnetic mutual coupling. There is still no widely accepted formulation for evaluating the stored reactive energy and the Q factor of a radiator [7]-[15], evern in free space. The main difficulty comes from the fact that there is no explicit definition in macroscopic electromagnetic theory for the reactive electromagnetic energy.

We have tried to provide a theory for the interpretation of the electromagnetic radiation and coupling based on the macroscopic Maxwell theory [16]-[20], in which the energy associated with a source in vacuum is separated into three parts, a Coulomb-velocity energy carried by the Coulomb fields and the velocity fields, a radiative energy carried by the radiative fields, and a macroscopic Schott energy. For a pulse radiator in free space, the nonzero periods of the three energies are strictly derived, which show that the Coulomb-velocity energy disappears at the instant that the sources disappear. A short time later, the total macroscopic Schott energy becomes zero, while the radiative energy keeps propagating and the total radiative energy remians to be constant unless the radiative fields interact with other sources in the space.

The macroscopi Schott energy is borrowed from the charged particle theory [21]-[25], and has been introduced in [20] to account for the energy exchanging between the Coulomb-velocity energy and the radiative energy. It is explicitly defined as an integral over the whole space,

$$W_S(t) = \int_{V_\infty} \frac{1}{2}\frac{\partial}{\partial t}[\mathbf{D}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)]d\mathbf{r} \qquad (1)$$

where $\mathbf{D}(\mathbf{r},t)$ is the electric flux density and $\mathbf{A}(\mathbf{r},t)$ the vector potential. They are from the same source in radiation problems, and may from different sources in mutual coupling problems.

The Coulomb-velocity energy is denoted by $W_{\rho J}(t)$. It is the volume integral of the source-potential products over the source region V_s,

$$W_{\rho J}(t) = \int_{V_S} \left[\frac{1}{2}\mathbf{J}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{2}\rho(\mathbf{r},t)\phi(\mathbf{r},t)\right] d\mathbf{r} \quad (2)$$

The Coulomb-velocity energy includes the contribution from the Coulomb fields and the velocity fields. It is considered to be attached to the charge distribution $\rho(\mathbf{r},t)$ and the current distribution $\mathbf{J}(\mathbf{r},t)$. The scalar potential $\phi(\mathbf{r},t)$ and the vector potential $\mathbf{A}(\mathbf{r},t)$ evaluated at the observation point $\mathbf{r}$ and the time t are defined in their usual way. They are subject to the Lorentz Gauge, and their reference zero points are put at the infinity. Majumdar *et al* [26] have shown that the potentials under these conditions are Gauge-invariant. As clearly indicated in equation (2), when the sources vanish, $W_{\rho J}(t)$ becomes zero even if the potentials are not zero.

For the sake of convenience, a principal radiative energy is introduced in [20] as

$$W_{rad}^{pri}(t) = \int_{V_\infty} \frac{1}{2}\left(\frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{A} - \mathbf{D}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) d\mathbf{r} \quad (3)$$

It has been derived in [20] that the total electromagnetic energy associated with an electromagnetic source is the sum of the Coulomb-velocity energy and the principal radiative energy, i.e.,

$$W_{tot}(t) = W_{rad}^{pri}(t) + W_{\rho J}(t) \quad (4)$$

Consequently, the total reactive energy and the total radiative energy are expressed by,

$$W_{react}(t) = W_{\rho J}(t) + W_S(t) \quad (5)$$

$$W_{rad}(t) = W_{rad}^{pri}(t) - W_S(t) \quad (6)$$

Therefore, the Schott energy plays the role of a bridge of energy exchange between the reactive energy and the radiative energy.

2. Schott Energy for a Charged Particle

Consider a nonrelativistic charged particle with charge e and velocity $\mathbf{v}(t)$ in free space. The field generated by the charge can be derived from the following Lienard-Wiechert potentials [2][3],

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{ce}{Rc-\mathbf{R}\cdot\mathbf{v}}\right]_{t'} = \frac{e}{4\pi\varepsilon_0}\left[\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right]_{t'} \quad (7)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\left[\frac{ec\mathbf{v}}{Rc-\mathbf{R}\cdot\mathbf{v}}\right]_{t'} = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\boldsymbol{\beta}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right]_{t'} \quad (8)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{x}(t')$, $R = |\mathbf{r} - \mathbf{x}(t')|$, and $\mathbf{x}(t')$ is the trajectory of the moving charge. $t' = t - R/c$ is the retarded time, and c is the light velocity in vacuum. Note that the quantities at the righthand side of equation (7) and equation (8) are evaluated at t'. $\mathbf{v}(t') = d\mathbf{x}(t')dt'$, $\boldsymbol{\beta} = \mathbf{v}/c$, and $\mathbf{n} = \mathbf{R}/R$.

The non-relativistic Schott energy of the moving charge is found to be (equation (16) in [23], equation (4) in [24], equation (3.24) in [25])

$$E_S(t) = -\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}\mathbf{a}\cdot\mathbf{v} \quad (9)$$

where $\mathbf{a}$ is the acceleration of the charge. Note that there is a factor $1/(4\pi\varepsilon_0)$ because SI unit system is used in this paper while the results in some references adopted Gaussian (cgs) system of units.

We are now to show that $E_S(t)$ can be derived by substituting the Lienard-Wiechert potentials and the corresponding fields into $W_S(t)$. Recalling that $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$, and making use of the Lorentz Gauge $\nabla\cdot\mathbf{A} + c^{-2}\,\partial\phi/\partial t = 0$, we can transform $W_S(t)$ to an integral of the potentials,

$$W_S(t) = -\frac{\varepsilon_0}{4}\frac{\partial^2}{\partial t^2}\int_{V_\infty}[c^{-2}\phi^2(\mathbf{r},t) + |\mathbf{A}(\mathbf{r},t)|^2]\, d\mathbf{r} \quad (10)$$

Since (10) is a full time derivative, we can approximately evaluate the integration in a simple way. For $v = |\mathbf{v}| \ll c$, the potential terms can be expanded with power series of $(\mathbf{n}\cdot\boldsymbol{\beta})$. Making use of $(1-x)^{-2} \approx 1 + 2x + 3x^2$ for small x and keeping the terms to the second order of $(\mathbf{n}\cdot\boldsymbol{\beta})$ we have

$$\phi^2(\mathbf{r},t) \approx \left(\frac{e}{4\pi\varepsilon_0}\right)^2\left[\frac{1}{R^2}(1 + 2\mathbf{n}\cdot\boldsymbol{\beta} + 3(\mathbf{n}\cdot\boldsymbol{\beta})^2)\right]_{t'} \quad (11)$$

$$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A} \approx \left(\frac{e}{4\pi\varepsilon_0 c}\right)^2\left[\frac{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}{R^2}\right]_{t'} \quad (12)$$

Assume that the source exists in a very short time period of $[t'-dt, t']$, and $t' = t - r_o/c$. Here r_o is a very small distance from the charged particle at $\mathbf{x}(t')$. In can be checked that, except the first term in the righthand side of equation (11), all other terms in the integrand are approximately nonzero only in a thin shell with radius ranging from r_o to $r_o + c(1-\mathbf{n}\cdot\boldsymbol{\beta})dt$. Note that the thickness of the shell is not uniform due to the moving of the charge. In the spherical coordinate system with origin locating at $\mathbf{x}(t')$, we have $R = r$, as shown in Fig.1. The integral of equation (10) can be cast into

$$dI_{A\phi}(t) \approx \left(\frac{e}{4\pi\varepsilon_0 c}\right)^2 \int_0^{2\pi}\int_0^{\pi}\int_{r_o}^{r_o+c(1-\mathbf{n}\cdot\boldsymbol{\beta})dt}\frac{1}{r^2}$$
$$\times [(1 + 2\mathbf{n}\cdot\boldsymbol{\beta} + 3(\mathbf{n}\cdot\boldsymbol{\beta})^2 + \boldsymbol{\beta}\cdot\boldsymbol{\beta})] r^2 \sin\theta\, dr d\theta d\varphi$$
$$\approx \left(\frac{e}{4\pi\varepsilon_0 c}\right)^2 \int_0^{2\pi}\int_0^{\pi}[(\mathbf{1} + \mathbf{n}\cdot\boldsymbol{\beta} + 3(\mathbf{n}\cdot\boldsymbol{\beta})^2 + \boldsymbol{\beta}\cdot\boldsymbol{\beta})]$$
$$\times c(1-\mathbf{n}\cdot\boldsymbol{\beta})dt \sin\theta\, d\theta d\varphi$$
$$\approx \left(\frac{e}{4\pi\varepsilon_0 c}\right)^2 \int_0^{2\pi}\int_0^{\pi}[(\mathbf{1} + \mathbf{n}\cdot\boldsymbol{\beta} + (\mathbf{n}\cdot\boldsymbol{\beta})^2 + \boldsymbol{\beta}\cdot\boldsymbol{\beta})]$$
$$\times \sin\theta\, d\theta d\varphi (cdt)$$

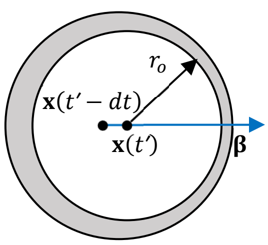

Fig.1 Integration region

A convenient choice is to put $\boldsymbol{\beta}$ on the z-axis, then $\mathbf{n}\cdot\boldsymbol{\beta} = |\boldsymbol{\beta}|\cos\theta$ and the integrand of $dI_{A\phi}(t)$ is symmetrical with z-axis. Note that $\int_0^\pi \sin\theta\, d\theta = 2$, $\int_0^\pi \cos\theta \sin\theta\, d\theta = 0$, and $\int_0^\pi \cos^2\theta \sin\theta\, d\theta = 2/3$. Carrying out the integration $dI_{A\phi}(t)$ and putting dt to the left side of the equation we have

$$\frac{dI_{A\phi}(t)}{dt} = \frac{e^2}{4\pi\varepsilon_0^2 c} + \frac{1}{4\pi\varepsilon_0}\frac{4e^2}{3\varepsilon_0 c^3}\mathbf{v}\cdot\mathbf{v} \tag{13}$$

Equation (13) holds true for $r_o = 0$, i.e., $t = t'$. Therefore, it can be deduced from equation (13) that

$$W_S(t) = -\frac{\varepsilon_0}{4}\frac{d}{dt}\left(\frac{dI_{A\phi}(t)}{dt}\right) \approx -\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}\mathbf{a}\cdot\mathbf{v} \tag{14}$$

which is in agreement with equation (9).

3. Conclusions

The electromagnetic radiation and mutual couplings are fundamental classical electromagnetic problems. The author believe that it is meaningful to handle the issue in the frame of the classical electromagnetic theory. The theory proposed in [20] is aimed to provide a more compatible interpretation for the electromagnetic radiation and mutual couplings in the language of macroscopic Maxwell theory so as that those commonly used quantities, such as the reactive electro-magnetic energies, radiative energies, Q-factors, and the electromagnetic coupling energies can all be evaluated consistently in time domain and in frequency domian. The introduction of the Schott energy perhaps can make it more intuitive to understand the process of building the energy balance in a typical electromagnetic radiation or mutual coupling problem. In the case of the Hertzian dipole, the reactive energies obtained with the proposed theory exactly agree with those predicted by using the equivalence circuit model [27]. To the best of my knowledge, no other related formulation can get such exact verification. This note shows that the macroscopic Schott energy derived in this way is indeed in agreement with the one introduced in charged particle theory one century ago. Although the significance of the Schott energy itself still has not been fully clarified in charged particle theory, and the Schott term may differ with a factor, the author still wants to try to show that the role of the Schott energy in macroscopic electromagnetic radiation and mutual couplings has to be taken into account.

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