Inhomogeneous Wave Equation, Liénard-Wiechert Potentials, and Hertzian Dipole in Weber Electrodynamics

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***Abstract*—Aiming to bypass the Lorentz force, this study analyzes Maxwell's equations from the perspective of a receiver at rest. This approach is necessary because experimental results suggest that the general validity of the Lorentz force might be questionable in non-stationary cases. Calculations in the receiver's rest frame are complicated and, thus, are rarely performed. In particular, the most important case is missing: namely, the solution of a Hertzian dipole moving in the rest frame of the receiver. The present article addresses this knowledge gap. First, this work demonstrates how the inhomogeneous wave equation can be derived and generically solved in the rest frame of the receiver. Subsequently, the solution for two uniformly moving point charges is derived, and the close connection between Maxwell's equations and Weber electrodynamics is highlighted. The gained insights are then applied to compute the far-field solution of a moving Hertzian dipole in the receiver's rest frame. The resulting solution is analyzed, and an explanation is presented regarding why an invariant and symmetric wave equation is possible for Weber electrodynamics and why the invariance could be the consequence of a quantum effect.**

***Index Terms*—Maxwell equations, Vector wave equation, Electromagnetic forces, Electromagnetic propagation, Lorentz force, Weber force, Weber electrodynamics**

I. Introduction

The four Maxwell equations have been the basis of electrodynamics for more than 150 years. In addition to Maxwell's equations, however, there is a fifth equation that often receives far less attention, but is of great importance: the Lorentz force equation.

In contrast to Maxwell's equations, the Lorentz force is not a differential equation. It does not depend on time or location, which is most likely why it is overshadowed by the Maxwell equations. The Lorentz force combines the electric and magnetic fields into a total electromagnetic force on a point-like test charge, which serves as the receiver of the force. However, because only forces or voltages can be measured, the Lorentz force is the mediator between the calculated fields obtained from Maxwell's equations and the measured experimental result, highlighting the decisive importance of the Lorentz force. Thus, it is surprising that the literature has focused almost exclusively on electric and magnetic fields and has rarely addressed the field of the force directly.

At the time of the origin of Maxwell's equations, a number of different formulas existed for the force between current elements, which are all equivalent if the current elements are connected to a closed conductor loop. J. C. Maxwell wrote on this subject in 1873 [1, p. 161], stating that there are considerable degrees of freedom for these formulas and that four parameters can be chosen independently. He concluded that the original formula of A. M. Ampère from 1822 was the most reasonable formula, but he also mentioned other formulas, such as that of H. Graßmann, which is used in contemporary electrodynamics, because Ampère's formula is not compatible with the Lorentz force.

It has been clearly established that the Lorentz force is correct for electro- and magnetostatic problems under non-relativistic conditions. Electro- and magnetostatic problems are problems in which the displacement current in Maxwell's equations can be neglected. This case corresponds to closed circuits with direct current because open conductors change their net charge if current flows, which leads to a time-varying electric field. However, a time-varying electric field implies the presence of a displacement current, which in turn means that the full set of Maxwell's equations is needed.

When the displacement current is not neglected, the complete set of Maxwell's equations is remarkably powerful because it describes how electromagnetic fields propagate in space and time. As one can assert, the addition of the displacement current by J. C. Maxwell ushered in the age of technological modernity. However, thought experiments and actual experiments indicate that the Lorentz force might be invalid in the presence of a displacement current [2]. In other words, by adding the displacement current, degrees of freedom are lost in the force law, and it is not self-evident that the Graßmann formula, which is perfectly appropriate for electro- and magnetostatics, retains its validity.

Fortunately, Maxwell's equations can also be used without the Lorentz force because, due to the principle of relativity, a uniformly moving test charge, i.e., a measuring probe or the receiver of a force, can be considered to be at rest, and instead, the field-generating charge, i.e., the transmitter of the force, can be interpreted as moving. This is the method applied in this article. It follows the approach of H. Dodig, who recently demonstrated that Maxwell's equations are a direct consequence of the validity of Coulomb's law and the fact that the force in the rest frame of the receiver always propagates at speed c [3].

The primary objective of this article is to determine whether the approach of solving Maxwell's equations in the receiver's rest frame and then transforming the force to the transmitter's rest frame gives the same result obtained by solving Maxwell's equations in the transmitter's rest frame and then applying the Lorentz force. For this purpose, the inhomogeneous wave equation is initially derived and solved in the receiver's rest frame. On the basis of this solution, it becomes clear that the solution for two uniformly moving point charges seems to contain – in formal terms – a hidden Galilean transformation. Using the framework of special relativity also seems to be consistent, but only at the price of greatly increased complexity and intricacy.

By applying the Galilean transformation with a weak additional assumption, we obtain the force formula of W. Weber, derived in 1846 from the force formula of A. M. Ampère, which was preferred by J. C. Maxwell and has proven its value in numerous experiments and applications [4]. This result shows that Weber electrodynamics, which has received increasing attention over the past decades due to the work of A. K. T. Assis and others [5]–[15], is closely related to Maxwell's equations and can be interpreted with the resulting inhomogeneous wave equation. Furthermore, this feature allows us to generalize Weber electrodynamics, which is usually interpreted as action-at-a-distance theory, to a field theory, which is de facto equivalent to standard electrodynamics, provided that one strictly considers Maxwell's equations as valid only in the receiver's rest frame.

The last section of the article takes advantage of the gained insights to study the most important special case of electrodynamics – the Hertzian dipole – in the rest frame of the receiver. The solution calculated herein is valid in terms of both standard electrodynamics and Weber electrodynamics. However, the solution suggests that special relativity may be superfluous or insufficient, because it uses purely mathematical terms to explain an experimental phenomenon that may originate from an actual physical mechanism.

II. Nomenclature notes

In this article, electric charges are called *transmitters* (of the force) when their property of generating an electromagnetic field is the focus of attention. Conversely, electric charges are referred to as *receivers* (of the force) when their reaction to an existing electromagnetic field is the focus. These terms are also used when electric charges move uniformly with respect to each other and, thus, no waves are present. Furthermore, in this article, standard electrodynamics is referred to as Lorentz-Einstein electrodynamics in order to be distinguished from Weber electrodynamics, which, as will become evident, can be interpreted as a special case of Maxwellian electrodynamics. However, because the field theory resulting from the considerations presented in this article goes beyond the scope of Weber electrodynamics, which is often but not always interpreted as an action-at-a-distance or direct-action theory [6], [16], [17], the field theory is denoted hereafter as Weber-Maxwell electrodynamics.

III. Inhomogeneous wave equation

The Maxwell equations in vacuum

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \, \boldsymbol{j} + \frac{1}{c^2} \, \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

are the starting point of this article. Together with the Lorentz force

$$\boldsymbol{F} = q_d \, \boldsymbol{E} + q_d \, \boldsymbol{v}_d \times \boldsymbol{B}, \tag{5}$$

Maxwell's equations form the basic set of equations and mathematical framework of what is currently considered valid electrodynamics, denoted as Lorentz-Einstein electrodynamics in this article.

The Maxwell equations can be combined into a single partial differential equation – the inhomogeneous wave equation – which reveals the essence of Maxwell's equations in a clear form. To obtain this equation, we exploit the fact that the laboratory system is allowed to move at the same velocity $\boldsymbol{v}_d$ as the receiver in Lorentz-Einstein electrodynamics, due to the principle of relativity. In this case, the Lorentz force simplifies to

$$\boldsymbol{F} = q_d \, \boldsymbol{E}, \tag{6}$$

and the term with the cross product is omitted. Thus, in Lorentz-Einstein electrodynamics, one does not actually need formula (5). Instead, one uses this equation only because it is convenient.

To derive the inhomogeneous wave equation, we calculate the derivative of the fourth Maxwell equation (4) with respect to time t. We obtain

$$\nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = \mu_0 \, \frac{\partial \boldsymbol{j}}{\partial t} + \frac{1}{c^2} \, \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{7}$$

By inserting the third Maxwell equation (3) and rearranging the terms, one obtains

$$\frac{1}{c^2} \, \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \nabla \times (\nabla \times \boldsymbol{E}) = -\mu_0 \, \frac{\partial \boldsymbol{j}}{\partial t}, \tag{8}$$

which, because of $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \, (\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$, corresponds to

$$\Box \, \boldsymbol{E} + \nabla \, (\nabla \cdot \boldsymbol{E}) = -\mu_0 \, \frac{\partial \boldsymbol{j}}{\partial t}. \tag{9}$$

The operator $\Box$ denotes the d'Alembert operator:

$$\Box := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{10}$$

Please note that the sign of the d'Alembert operator is not consistent in the literature. By substituting equation (1) into equation (9) and applying equation (6), one finally arrives at the inhomogeneous wave equation [18, p. 246, eq. (6.49)]:

$$\Box \, \boldsymbol{F} = -\frac{q_d}{\varepsilon_0} \left(\frac{1}{c^2} \, \frac{\partial \boldsymbol{j}}{\partial t} + \nabla \, \rho \right). \tag{11}$$

Equation (11) is valid for arbitrary charge and current distributions, as long as they satisfy the continuity equation.

Particularly important, however, are those cases in which the field-generating charge distribution is point-like. Here, the charge density is given by

$$\rho = q_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s), \tag{12}$$

where $\boldsymbol{r}_s$ represents the trajectory of the charge q_s and $\boldsymbol{r}$ is the location of the receiver at rest q_d. The corresponding current density is

$$\boldsymbol{j} = \boldsymbol{v}_s\, \rho, \tag{13}$$

where

$$\boldsymbol{v}_s := \dot{\boldsymbol{r}}_s \tag{14}$$

is the velocity of the charge q_s from the perspective of the charge q_d at rest.

Thus, the wave equation

$$\square\, \boldsymbol{F} = -\frac{q_d}{\varepsilon_0}\left(\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{v}_s\, q_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s) + \nabla\, q_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s)\right) \tag{15}$$

follows from equation (11). This equation represents the force exerted by a transmitter q_s with trajectory $\boldsymbol{r}_s$ on a receiver q_d at rest at location $\boldsymbol{r}$. As mentioned above, calculating the force in the receiver's rest frame does not lead to a restriction of generality in Lorentz-Einstein electrodynamics, as it should always be possible to perform a Lorentz transformation into the rest frame of the transmitter.

This article will demonstrate that the same wave equation (15) is compatible with the Weber force, provided that we define the charge q_s as a function of velocity $\boldsymbol{v}_s$ by

$$q_s(\boldsymbol{v}_s) := \gamma(\boldsymbol{v}_s)\, q_{s0}, \tag{16}$$

with q_{s0} being the rest charge, i.e., the charge obtained by measuring the force between a known resting test charge using Coulomb's law. As usual, γ is the Lorentz factor, which is defined by

$$\gamma(\boldsymbol{v}) := \frac{1}{\sqrt{1 - \|\boldsymbol{v}\|^2/c^2}}. \tag{17}$$

To calculate the acceleration $\ddot{\boldsymbol{r}}$ for a receiver of force $\boldsymbol{F}$ in classical mechanics, one needs only Newton's second law:

$$\boldsymbol{F} = m\, \ddot{\boldsymbol{r}}. \tag{18}$$

In Lorentz-Einstein electrodynamics, however, equation (18) does not apply. Instead, one needs the following equation:

$$\boldsymbol{F} = \gamma(\dot{\boldsymbol{r}})\, m\, \ddot{\boldsymbol{r}} + \frac{m}{c^2}\,\gamma(\dot{\boldsymbol{r}})^3\, \left(\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}\right)\, \dot{\boldsymbol{r}}, \tag{19}$$

which gives

$$\ddot{\boldsymbol{r}} = \frac{1}{\gamma(\dot{\boldsymbol{r}})\, m}\left(\boldsymbol{F} - \frac{1}{c^2}\,\dot{\boldsymbol{r}}\, \left(\boldsymbol{F} \cdot \dot{\boldsymbol{r}}\right)\right) \tag{20}$$

when solved for the acceleration.

For Weber-Maxwell electrodynamics, this approach is not necessary. As will be shown below, Newton's second law remains valid in the original form (18). It is remarkable that this results in an electrodynamics that can describe the propagation of electromagnetic waves and satisfy Einstein's postulates without requiring a Lorentz transformation. At the same time, magnetism as a dual force of electromagnetism disappears and becomes a residual effect, due to the inability to simultaneously shield the electric force in all reference frames for a multi-particle system with varying relative velocities.

IV. Solution of the wave equation

A. General solution in the rest frame of the receiver

Because of its importance, the solution of the inhomogeneous wave equation (15) is demonstrated in this section. Special emphasis is placed on ensuring that the calculation is as simple and comprehensible as possible.

First, we change to the equivalent potential notation [18, p. 239]

$$\begin{aligned} \boldsymbol{E} &= -\nabla\, \Phi - \frac{\partial}{\partial t}\boldsymbol{A} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} \end{aligned} \tag{21}$$

and insert these potentials into equation (6), which holds in the rest frame of the receiver in both Lorentz-Einstein electrodynamics and Weber-Maxwell electrodynamics. From this, we obtain the Lorentz force in potential notation in the rest frame of the receiver:

$$\boldsymbol{F} = -q_d\left(\frac{\partial}{\partial t}\boldsymbol{A} + \nabla\, \Phi\right). \tag{22}$$

Equation (22) can now be substituted into the wave equation (15). After rearranging the terms, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\square\, \boldsymbol{A} - \frac{q_s}{\varepsilon_0\, c^2}\,\boldsymbol{v}_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s)\right) = \\ &-\nabla\left(\square\, \Phi - \frac{q_s}{\varepsilon_0}\,\delta(\boldsymbol{r} - \boldsymbol{r}_s)\right). \end{aligned} \tag{23}$$

Equation (23) shows that we can find the solution of the differential equation (15) by separating it into two equations:

$$\frac{\partial}{\partial t}\left(\square\, \boldsymbol{A} - \frac{q_s}{\varepsilon_0\, c^2}\,\boldsymbol{v}_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s)\right) = \nabla\lambda \tag{24}$$

and

$$-\nabla\left(\square\, \Phi - \frac{q_s}{\varepsilon_0}\,\delta(\boldsymbol{r} - \boldsymbol{r}_s)\right) = \nabla\lambda, \tag{25}$$

with λ being any arbitrary scalar field depending on $\boldsymbol{r}$ and t. However, because λ may be chosen freely, it is possible to set $\lambda = -\partial/\partial t\, \square\, \Lambda$, with Λ being another arbitrary scalar field. Substituting this into equations (24) and (25) gives us

$$\square\, \boldsymbol{A} - \frac{q_s}{\varepsilon_0\, c^2}\,\boldsymbol{v}_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s) = -\square\, \nabla\, \Lambda \tag{26}$$

and

$$\square\, \Phi - \frac{q_s}{\varepsilon_0}\,\delta(\boldsymbol{r} - \boldsymbol{r}_s) = \square\, \frac{\partial}{\partial t}\,\Lambda. \tag{27}$$

For each specific choice of Λ, we have different equations (26) and (27) with different solutions $\boldsymbol{A}$ and Φ. At first glance, this seems problematic because we must insert these ostensibly arbitrary solutions $(\Phi, \boldsymbol{A})$ into equation (22) in order to obtain the force $\boldsymbol{F}$. In fact, however, each different pair of potentials $(\Phi, \boldsymbol{A})$ leads to the same force, independent of the choice of Λ. To demonstrate this, we rearrange equations (26) and (27) as

$$\square\, \left[\boldsymbol{A} + \nabla\, \Lambda\right] - \frac{q_s}{\varepsilon_0\, c^2}\,\boldsymbol{v}_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s) = \boldsymbol{0} \tag{28}$$

and

$$\square\, \left[\Phi - \partial/\partial t\, \Lambda\right] - \frac{q_s}{\varepsilon_0}\,\delta(\boldsymbol{r} - \boldsymbol{r}_s) = 0. \tag{29}$$

Now, it is obvious that the function Λ is exactly the gauge function [18, p. 240], which we can freely choose and therefore set to zero. We can easily see that the specific choice of Λ is irrelevant by applying the substitutions $\boldsymbol{A} \to \boldsymbol{A} + \nabla\Lambda$ and $\Phi \to \Phi - \partial/\partial t\,\Lambda$ for the force (22), which leaves equation (22) unchanged.

Thus, without loss of generality, we ultimately obtain the two following differential equations:

$$\Box \boldsymbol{A} = \frac{q_s}{\varepsilon_0\, c^2}\, \boldsymbol{v}_s\, \delta(\boldsymbol{r} - \boldsymbol{r}_s) \tag{30}$$

and

$$\Box \Phi = \frac{q_s}{\varepsilon_0}\, \delta(\boldsymbol{r} - \boldsymbol{r}_s), \tag{31}$$

which can be solved separately[1].

The general solution of a wave equation of form

$$\Box \boldsymbol{F}(\boldsymbol{r}, t) = \boldsymbol{f}(\boldsymbol{r}, t) \tag{32}$$

is [18, p. 243-245]

$$\boldsymbol{F}(\boldsymbol{r}, t) = \iiint_V \frac{\boldsymbol{f}\,(\boldsymbol{r}', t')}{4\,\pi\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\, \mathrm{d}\boldsymbol{r}', \tag{33}$$

where t' is defined by

$$t' := t - \frac{1}{c}\|\boldsymbol{r} - \boldsymbol{r}'\|. \tag{34}$$

Applying this result to the differential equations (30) and (31), we obtain

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{\varepsilon_0\, c^2} \iiint_V \frac{q_s(\boldsymbol{v}_s(t'))\, \boldsymbol{v}_s(t')\, \delta(\boldsymbol{r}' - \boldsymbol{r}_s(t'))}{4\,\pi\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\, \mathrm{d}\boldsymbol{r}' \tag{35}$$

and

$$\Phi(\boldsymbol{r}, t) = \frac{1}{\varepsilon_0} \iiint_V \frac{q_s(\boldsymbol{v}_s(t'))\, \delta(\boldsymbol{r}' - \boldsymbol{r}_s(t'))}{4\,\pi\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\, \mathrm{d}\boldsymbol{r}'. \tag{36}$$

Despite the presence of the Dirac function, the integrals in equations (35) and (36) cannot be solved directly, because t' is a function of $\boldsymbol{r}'$ according to formula (34). However, by using the Jacobian determinant D, each equation can be converted into a formally solvable form [19, p. 616-617] by using the definitions

$$\boldsymbol{s} := \boldsymbol{r}' - \boldsymbol{r}_s(t') \tag{37}$$

and

$$D := \det \begin{vmatrix} \frac{\partial s_x}{\partial r'_x} & \frac{\partial s_x}{\partial r'_y} & \frac{\partial s_x}{\partial r'_z} \\ \frac{\partial s_y}{\partial r'_x} & \frac{\partial s_y}{\partial r'_y} & \frac{\partial s_y}{\partial r'_z} \\ \frac{\partial s_z}{\partial r'_x} & \frac{\partial s_z}{\partial r'_y} & \frac{\partial s_z}{\partial r'_z} \end{vmatrix}. \tag{38}$$

We obtain

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{\varepsilon_0\, c^2} \iiint_V \frac{q_s(\boldsymbol{v}_s(t'))\, \boldsymbol{v}_s(t')\, \delta(\boldsymbol{s})}{4\,\pi\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\, \frac{1}{D}\, \mathrm{d}\boldsymbol{s} \tag{39}$$

and

$$\Phi(\boldsymbol{r}, t) = \frac{1}{\varepsilon_0} \iiint_V \frac{q_s(\boldsymbol{v}_s(t'))\, \delta(\boldsymbol{s})}{4\,\pi\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\, \frac{1}{D}\, \mathrm{d}\boldsymbol{s}, \tag{40}$$

with t' and $\boldsymbol{r}'$ now being implicitly dependent on $\boldsymbol{s}$.

[1]These equations are equivalent to equations (6.15) and (6.16) in [18, p. 240].

The integrations in equations (35) and (36) can now be performed, and we obtain

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{q_s(\boldsymbol{v}_s(t'))\, \boldsymbol{v}_s(t')}{4\,\pi\,\varepsilon_0\, c^2\, \|\boldsymbol{r} - \boldsymbol{r}'\|}\, \frac{1}{D} \tag{41}$$

and

$$\Phi(\boldsymbol{r}, t) = \frac{q_s(\boldsymbol{v}_s(t'))}{4\,\pi\,\varepsilon_0\, \|\boldsymbol{r} - \boldsymbol{r}'\|}\, \frac{1}{D} \tag{42}$$

under the constraint $\boldsymbol{s} = \boldsymbol{0}$. This result can be further simplified, as the calculation of the Jacobian determinant (38) yields

$$D = 1 - \frac{\boldsymbol{v}_s\,(t') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{c\,\|\boldsymbol{r} - \boldsymbol{r}'\|}. \tag{43}$$

Substituting this result into equation (42) gives

$$\Phi(\boldsymbol{r}, t) = \frac{q_s(\boldsymbol{v}_s(t'))}{4\,\pi\,\varepsilon_0\, \|\boldsymbol{r} - \boldsymbol{r}'\| \left(1 - \frac{\boldsymbol{v}_s(t') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{c\,\|\boldsymbol{r} - \boldsymbol{r}'\|}\right)}, \tag{44}$$

and for the vector potential, we find

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c^2}\, \boldsymbol{v}_s(t')\, \Phi(\boldsymbol{r}, t). \tag{45}$$

Equation (37) must be zero because of the constraint; hence, it follows that $\boldsymbol{r}' = \boldsymbol{r}_s(t')$. Moreover, because of equation (34),

$$\|\boldsymbol{r} - \boldsymbol{r}'\| = c\,(t - t') \tag{46}$$

holds. If we insert both relations into the scalar potential and rearrange slightly, we finally obtain

$$\Phi(\boldsymbol{r}, t) = \frac{q_s(\boldsymbol{v}_s(t'))\, c}{4\,\pi\,\varepsilon_0\, \left(c^2\,(t - t') - \boldsymbol{v}_s(t') \cdot (\boldsymbol{r} - \boldsymbol{r}_s(t'))\right)} \tag{47}$$

with

$$t' := t - \frac{1}{c}\|\boldsymbol{r} - \boldsymbol{r}_s(t')\|. \tag{48}$$

Here, t' is the time at which the field has left the transmitter q_s at location $\boldsymbol{r}_s(t')$ to meet the receiver q_d at location $\boldsymbol{r}$ at time t.

B. *Uniformly moving point charges*

With the potential (47), it is now possible to calculate retarded electromagnetic forces between arbitrarily moving point charges. In this section, we discuss the simplest case and assume that there is no acceleration or that the acceleration can be neglected. For this purpose, we consider a uniformly moving point charge q_s with trajectory $\boldsymbol{r}_s(t) = \boldsymbol{v}\,t$, which exerts a force on a stationary point charge at location $\boldsymbol{r}$.

To calculate the field of the force according to equation (22), the potential (47) must be known. Thus, one must solve equation (48), which has two solutions in this particular case, with only one solution satisfying causality $t \geq t'$:

$$t' = \frac{c^2\,t - \boldsymbol{r} \cdot \boldsymbol{v} - \sqrt{c^2\,\|\boldsymbol{r} - \boldsymbol{v}\,t\|^2 - \|\boldsymbol{r} \times \boldsymbol{v}\|^2}}{c^2 - v^2}. \tag{49}$$

This solution can be substituted into equation (47) to obtain [19, p. 618]

$$\Phi(\boldsymbol{r}, t) = \frac{q_s(\boldsymbol{v})\, c}{4\,\pi\,\varepsilon_0\, \sqrt{\|\boldsymbol{r} - \boldsymbol{v}\,t\|^2\,(c^2 - v^2) + ((\boldsymbol{r} - \boldsymbol{v}\,t) \cdot \boldsymbol{v})^2}}, \tag{50}$$

using the relations $\boldsymbol{r}_s(t') = \boldsymbol{v}\, t'$ and $\boldsymbol{v}_s(t') = \boldsymbol{v}$. The potential can then be inserted into equation (45) and thereafter into equation (22). After calculating the derivatives and summing all terms, we obtain [19, p. 618]

$$\boxed{\boldsymbol{F} = \frac{c^3\, q_d\, q_s(\boldsymbol{v})\, (\boldsymbol{r} - \boldsymbol{v}\, t)\, \gamma(\boldsymbol{v})^{-2}}{4\,\pi\,\varepsilon_0\, \left(\|\boldsymbol{r} - \boldsymbol{v}\, t\|^2\, (c^2 - v^2) + ((\boldsymbol{r} - \boldsymbol{v}\, t)\cdot \boldsymbol{v})^2\right)^{3/2}}.} \tag{51}$$

With the following definition

$$q_s(\boldsymbol{v}) := q_{s0} \tag{52}$$

in Lorentz-Einstein electrodynamics, formula (51) gives the force from the perspective of the receiver at location $\boldsymbol{r}$. The same equation (51) applies in Weber-Maxwell electrodynamics, but with formula (16) instead of formula (52).

In equation (51), it is remarkable that the term $\boldsymbol{r}-\boldsymbol{v}\, t$ appears several times. This term corresponds to the location of the receiver at time t from the perspective of the transmitter. The appearance of this term indicates that the center of the field is moving in conjunction with the transmitter. If one was not aware of the special theory of relativity, it would be natural to apply the (inverse) *Galilean transformation*:

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + \boldsymbol{v}\, t \\ t &\to t \end{aligned} \tag{53}$$

in order to transform the force (51) into the rest frame of the transmitter. In this way, one would obtain the following equation:

$$\boldsymbol{F} = \frac{c^3\, q_d\, q_s(\boldsymbol{v})\, \boldsymbol{r}\, \gamma(\boldsymbol{v})^{-2}}{4\,\pi\,\varepsilon_0\, \left(r^2\, (c^2 - v^2) + (\boldsymbol{r}\cdot \boldsymbol{v})^2\right)^{3/2}}, \tag{54}$$

which is time-independent but, unlike the Coulomb force, depends on the velocity of the receiver.

However, in Lorentz-Einstein electrodynamics, the Galilean transformation cannot be applied to transform the force (51) into the rest frame of the transmitter. Instead, we must use the (inverse) *Lorentz transformation*. In classical vector notation [19, p. 635], we have

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + (\gamma(v) - 1)\, \boldsymbol{r}\cdot \boldsymbol{v} \frac{\boldsymbol{v}}{v^2} + \gamma(v)\, \boldsymbol{v}\, t \\ t &\to \gamma(v) \left(t + \frac{1}{c^2} \boldsymbol{r}\cdot \boldsymbol{v} \right). \end{aligned} \tag{55}$$

We can verify that this expression is indeed the Lorentz transformation by applying (55) to equation $\|\boldsymbol{r}\|^2 = c^2\, t^2$ and verifying that it remains unchanged.

In Lorentz-Einstein electrodynamics, a point charge at rest has no magnetic field. Instead, it has only an electric field. For this reason, the force exerted by a point charge at rest on a moving point charge is equal to the Coulomb force, because the term $\boldsymbol{v} \times \boldsymbol{B}$ in the Lorentz force cancels out due to $\boldsymbol{B} = \boldsymbol{0}$. Thus, the force is velocity-independent. However, the acceleration effect caused by the force is not velocity-independent, because the framework of relativistic dynamics must be applied, i.e., we must insert the Coulomb force into equation (20). Consequently, even in Lorentz-Einstein electrodynamics, the acceleration ultimately depends on the velocity of the receiver.

However, a problem arises because applying the Lorentz transformation (55) to the force (51) does not generally give the Coulomb force. Instead, one obtains the Coulomb force only if the condition $\boldsymbol{r} \parallel \boldsymbol{v}$ is satisfied, which corresponds to a one-dimensional Lorentz transformation. We demonstrate this for a special case:

$$\boldsymbol{r}\cdot \boldsymbol{v} = 0. \tag{56}$$

For this purpose, we apply equation (56) to the Lorentz transformation (55) and obtain

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + \gamma(v)\, \boldsymbol{v}\, t \\ t &\to \gamma(v) t. \end{aligned} \tag{57}$$

Then, we apply this specific Lorentz transformation (57) to the solution of the wave equation (51). Because

$$\boldsymbol{r} - \boldsymbol{v}\, t \;\to\; [\boldsymbol{r} + \gamma(v)\, \boldsymbol{v}\, t] - \boldsymbol{v}\, [\gamma(v)\, t] = \boldsymbol{r}, \tag{58}$$

we obtain a formula that formally corresponds to equation (54). This result can be further simplified for the special case (56). Thus, we insert equation (56) and definition (52) and obtain the following expression:

$$\boldsymbol{F}(\boldsymbol{r}, t) = \gamma(\boldsymbol{v})\, \frac{q_d\, q_{s0}\, \boldsymbol{r}}{4\,\pi\,\varepsilon_0\, r^3} \neq \frac{q_d\, q_{s0}\, \boldsymbol{r}}{4\,\pi\,\varepsilon_0\, r^3}, \tag{59}$$

which does not agree with the Coulomb force. For this reason, in special relativity, one must distinguish between forces that act parallel or perpendicular to the relative velocity, which again demonstrates the utmost complexity of special relativity. Further difficulties of Lorentz-Einstein electrodynamics become apparent in section IV-D, in which force fields that transport information are analyzed.

Because of these difficulties with the Lorentz transformation, let us return to equation (54) and hypothesize that it represents the true Newtonian force perceived by a uniformly moving receiver due to a point charge at rest. It is obvious that this assumption cannot be invalid for velocities that are small compared with the speed of light. Therefore, we can calculate the non-relativistic approximation. For this purpose, we apply definition (16) and expand equation (54) into a Taylor series for v. We obtain the second-order approximation:

$$\boldsymbol{F} \approx \frac{q_{s0}\, q_d}{4\,\pi\,\varepsilon_0} \left(1 + \frac{v^2}{c^2} - \frac{3}{2} \left(\frac{\boldsymbol{r}}{r} \cdot \frac{\boldsymbol{v}}{c} \right)^2 \right) \frac{\boldsymbol{r}}{r^3}. \tag{60}$$

This formula was first documented in 1846 by W. Weber. Obviously, the Weber formula is an approximation of the solution of the inhomogeneous wave equation (15), provided that the weak assumption given by equation (16) is correct. Thus, the Weber force is closely connected to Maxwell's equations and represents more than just a fortuitous fit to empirical data [21].

It should be mentioned that formula (54) can be brought into a form that is more geometrically readable [19, p. 619] by applying the angle α between $\boldsymbol{r}$ and $\boldsymbol{v}$. For this angle,

$$(\boldsymbol{r}\cdot \boldsymbol{v})^2 = r^2\, v^2\, \cos(\alpha)^2 \tag{61}$$

holds. Substituting this expression into formula (54), we obtain

$$\boldsymbol{F} = \frac{q_d\, q_s(\boldsymbol{v})}{4\,\pi\,\varepsilon_0}\, \frac{\boldsymbol{r}}{r^3}\, \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\, \sin(\alpha)^2\right)^{3/2}} \tag{62}$$

after some rearrangement of the terms [22, p. 68]. Clearly, the field (54) corresponds to a central force because the field lines are always straight lines starting at the transmitter q_s. The strength of the force, however, depends on the angle.

If charge q_s is moving directly toward or away from charge q_d, the term $\sin(\alpha)$ is zero and the force is weaker than the Coulomb force. However, if the charges are moving exactly parallel past each other, then $\sin(\alpha) = 1$ and the force is stronger than the Coulomb force. This angular dependence becomes more pronounced for high velocities.

Moreover, the field (54) is symmetrical, similar to the Weber force (60), because the transmitter exerts a force on the receiver that is equal in magnitude and opposite in direction to the force produced by the receiver on the transmitter. Therefore, Newton's third law is satisfied for charges with uniform motion. Consequently, the laws of conservation for momentum and angular momentum are fulfilled here in the original Newtonian sense.

C. Galilean transformation for convenience

The time-independent force (54) was obtained by applying a Galilean transformation (53) to solution (51). Application of the Lorentz transformation (55) to equation (51) leads to another formula that does not depend on time but is quite complicated. Therefore, it is useful to subject the potentials (45) and (50) to *any* coordinate transformation that simplifies the calculation of the derivatives in equation (22) and to revert this transformation after having solved the problem. The Galilean transformation (53) seems most suitable for this purpose.

In equation (22), two differential operators appear. For the gradient, we have

$$\Big(\nabla\Phi(\boldsymbol{r},t)\Big)\Big|_{\boldsymbol{r}\to\boldsymbol{r}+\boldsymbol{v}\,t} = \nabla\Phi(\boldsymbol{r}+\boldsymbol{v}\,t,t). \tag{63}$$

In this case, the order in which one performs the Galilean transformation and calculates the derivative is irrelevant. For the time derivative, however, this does not apply. Here, we have

$$\Big(\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t)\Big)\Big|_{\boldsymbol{r}\to\boldsymbol{r}+\boldsymbol{v}\,t} = \left(\frac{\partial}{\partial t} - (\boldsymbol{v}\cdot\nabla)\right)\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{v}\,t,t), \tag{64}$$

where the differential operator $\boldsymbol{v}\cdot\nabla$ is defined by

$$\boldsymbol{v}\cdot\nabla := u_x\,\frac{\partial}{\partial x} + u_y\,\frac{\partial}{\partial y} + u_z\,\frac{\partial}{\partial z}. \tag{65}$$

By applying this result to equation (22), we obtain

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}+\boldsymbol{v}\,t,t) = -q_d\left(\frac{\partial}{\partial t} - (\boldsymbol{v}\cdot\nabla)\right)\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{v}\,t,t) - \\ q_d\,\nabla\Phi(\boldsymbol{r}+\boldsymbol{v}\,t,t),\end{aligned} \tag{66}$$

where $\Phi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ are the potentials in the rest frame of the receiver. In section IV-D it will become apparent that the equation (66) is almost identically with the formula of the Lorentz force.

To demonstrate the advantage of this formula (66) by an example, we repeat the calculation in section IV-B. First, the potential (50) is translated using the Galilean transformation $\boldsymbol{r}\to\boldsymbol{r}+\boldsymbol{v}\,t$, which gives

$$\Phi = \frac{q_s(\boldsymbol{v})\,c}{4\,\pi\,\varepsilon_0\,\sqrt{r^2\,(c^2-v^2)+(\boldsymbol{r}\cdot\boldsymbol{v})^2}}. \tag{67}$$

The vector potential is calculated from the scalar potential (67) using equation (45) as

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi. \tag{68}$$

Both potentials (67) and (68) can now be substituted into equation (66). The time derivative is zero because of the time independence of the potentials, and thus, it follows that

$$\begin{aligned}\boldsymbol{F} &= q_d\,(\boldsymbol{v}\cdot\nabla)\,\frac{\boldsymbol{v}}{c^2}\,\Phi - q_d\,\nabla\Phi \\ &= q_d\,\frac{\boldsymbol{v}}{c^2}\,(\boldsymbol{v}\cdot\nabla)\,\Phi - q_d\,\nabla\Phi \\ &= q_d\,\frac{\boldsymbol{v}}{c^2}\,(\boldsymbol{v}\cdot\nabla\,\Phi) - q_d\,\nabla\Phi.\end{aligned} \tag{69}$$

Now, only the gradient of the scalar potential must be calculated and inserted. After combining all terms, we obtain the force (54) and, after reversion of the Galilean transformation, the original time-dependent solution (51) in the rest frame of the receiver.

We have explicitly demonstrated that this approach of using the Galilean transformation is also *mathematically* correct in Lorentz-Einstein electrodynamics. We will apply this fact in the following section.

D. Hertzian dipole

Section IV-B addressed two point charges moving uniformly with respect to each other. Another special case of utmost importance is a bound particle that oscillates within itself. Such a bound particle may be electrically neutral, but inside, it consists of two charge quantities, $+q_s$ and $-q_s$. These charge quantities can oscillate with respect to each other so that the point charge $+q_s$ moves upward while the other point charge $-q_s$ moves downward. Of course, the strong attractive electric force between the two charge quantities prevents them from separating too far and causes the direction of motion to periodically reverse. Thus, an oscillation of the form $\boldsymbol{p}_0\,\sin(\omega\,t)$ arises. Here, $\boldsymbol{p}_0$ is the polarization vector, which specifies the spatial direction of the oscillation and the maximum distance between the two charge quantities that we want to model as point charges. ω is the angular frequency of the oscillation.

In electrodynamics, such an object is called a Hertzian dipole. This dipole is a standard study object for understanding electromagnetic waves and has the same importance in classical electrodynamics as the hydrogen atom in atomic physics, as it represents the simplest possible antenna. Hence, corresponding calculations can be found in numerous textbooks on classical electrodynamics. However, the solutions found in textbooks *always* assume that the center of the bound particle does not move and is located at the coordinate origin. The solution then consists of two fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, which clearly indicate that the waves move at speed c only for a resting receiver.

To calculate the force on a moving receiver, one applies the same fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with the same time dependencies and wave velocities by substituting them into the Lorentz force (5). However, the Lorentz force is a very simple formula, and the time dependencies are not affected in $\boldsymbol{E}$ or $\boldsymbol{B}$. For example, if the receiver is moving away from the transmitter along a straight line at velocity v, it would observe a wave velocity of $c-v$ in its own rest frame. Because this contradicts experimental findings, it seemed necessary to introduce the Lorentz transformation.

However, if we solve the Maxwell equations in the rest frame of the receiver, this problem does not arise because the moving receiver perceives the same field as if it were moving *in its own* rest frame at c. Remarkably, this is true for *any* receiver. This characteristic sounds like a logical impossibility because, at first glance, it seems impossible for a physical entity such as a force field to move at the same speed c for two receivers moving at different speeds in relation to the source of the field. In other words, how can the transmitter know the speed at which it must emit the wave so that it has speed c in the receiver's rest frame? What if there are two or more receivers moving at different speeds? Would not the transmitter then have to emit a suitable and different wave for each receiver?

However, there is not necessarily a logical contradiction because this phenomenon might be caused by a simple physical mechanism that does not require any concepts beyond Newtonian mechanics, such as the spacetime continuum or luminiferous ether, for its explanation. Further details can be found in [20]. At this point, we simply describe the basic idea: a transmitter emits a field of electromagnetic force carriers with random emission velocities. The paradox is resolved because each receiver can perceive only those force carriers whose propagation velocities in its own rest frame do not exceed c. One can then show that, although the waves propagate at all wave velocities, only the portion with a speed of exactly c remains in each rest frame. All other wave components either interfere destructively or are not perceptible because they are too fast.

Another hypothesis would be to assume that the electromagnetic field is a quantum mechanical superposition of all possible velocities and that the velocity is fixed to the value c only when it interacts with the receiver in a type of quantum collapse. Although there might be physical and logical reasons for electromagnetic waves to have the same speed c in all frames for all receivers, these reasons have not been considered by physicists. The problem with both hypotheses is that they satisfy Einstein's postulates but are not equivalent to special relativity because they are both nonlinear, whereas the Lorentz transformation assumes linearity [23]. However, we will postpone such considerations to a later point and continue our calculations.

To obtain the field of a Hertzian dipole, we consider a point charge $q_s = +q$ moving along the following trajectory:

$$\boldsymbol{r}_s(t) = \boldsymbol{v}\, t + \boldsymbol{s}(t), \tag{70}$$

where $\boldsymbol{s}(t)$ is any arbitrary but small spatial oscillation. The solution of the Hertzian dipole is obtained by considering a second point charge $-q$ following the trajectory $\boldsymbol{v}\, t - \boldsymbol{s}(t)$. For $\boldsymbol{s}(t) = \boldsymbol{p}_0 \sin(\omega\, t)$, the sum of the fields of the two charges gives the field of the Hertzian dipole, with its center moving along trajectory $\boldsymbol{v}\, t$. Hence, to calculate the force of the Hertzian dipole, it is sufficient to initially calculate only the force exerted by a transmitter $+q$ with trajectory (70) on a receiver resting at location $\boldsymbol{r}$.

We obtain this field by first calculating the potential (47). In this case, we have

$$\boldsymbol{r}_s(t') = \boldsymbol{v}\, t' + \boldsymbol{s}(t') \approx \boldsymbol{v}\, t' \tag{71}$$

and

$$\boldsymbol{v}_s(t') = \dot{\boldsymbol{r}}_s(t') = \boldsymbol{v} + \dot{\boldsymbol{s}}(t'). \tag{72}$$

The approximation in equation (71) is valid only if the spatial displacement of the oscillation $\boldsymbol{s}(t)$ is sufficiently small at all times t to be negligible. Furthermore, we will assume in the following calculation that the approximation $q = \gamma(\boldsymbol{v}_s(t'))\, q_0 \approx \gamma(\boldsymbol{v})\, q_0$ holds.

Equations (71) and (72) can now be substituted into the potential (47), and we arrive at the following scalar potential:

$$\Phi_+ = \frac{q\, c}{4\,\pi\,\varepsilon_0\, \left(c^2\,(t-t') - (\boldsymbol{v} + \dot{\boldsymbol{s}}(t'))\cdot(\boldsymbol{r} - \boldsymbol{v}\, t')\right)}, \tag{73}$$

which is caused by the positive charge. The vector potential is obtained by substituting the scalar potential (73) into equation (45). In this case, we obtain

$$\boldsymbol{A}_+ = \frac{q\, (\boldsymbol{v} + \dot{\boldsymbol{s}}(t'))}{4\,\pi\,\varepsilon_0\, c\, \left(c^2\,(t-t') - (\boldsymbol{v} + \dot{\boldsymbol{s}}(t'))\cdot(\boldsymbol{r} - \boldsymbol{v}\, t')\right)}. \tag{74}$$

To further simplify the calculation, we can again exploit the fact that the displacement caused by the oscillation $\boldsymbol{s}(t)$ is small and that the scalar potential and vector potential can therefore be approximated with respect to amplitude via a Taylor series. For the scalar potential, the first-order approximation is

$$\begin{aligned}\Phi_+ \approx &\frac{q\, c}{4\,\pi\,\varepsilon_0\, \left(c^2\,(t-t') - \boldsymbol{r}\cdot\boldsymbol{v} + t'\, v^2\right)} + \\ &\frac{q\, c\, \left((\boldsymbol{r} - \boldsymbol{v}\, t')\cdot\dot{\boldsymbol{s}}(t')\right)}{4\,\pi\,\varepsilon_0\, \left(c^2\,(t-t') - \boldsymbol{r}\cdot\boldsymbol{v} + t'\, v^2\right)^2}.\end{aligned} \tag{75}$$

For the vector potential, we obtain

$$\boldsymbol{A}_+ \approx \frac{\boldsymbol{v}}{c^2}\,\Phi_+ + \frac{q\, \dot{\boldsymbol{s}}(t')}{4\,\pi\,\varepsilon_0\, c\, \left(c^2\,(t-t') - \boldsymbol{r}\cdot\boldsymbol{v} + t'\, v^2\right)}. \tag{76}$$

Now, the potentials generated by the positive charge are known. However, the Hertzian dipole consists of two charges, where the oscillation of the negative charge is exactly inverse to that of the positive charge. The potentials Φ_- and $\boldsymbol{A}_-$ are obtained by substituting $q \to -q$, $\boldsymbol{s}(t') \to -\boldsymbol{s}(t')$, and $\dot{\boldsymbol{s}}(t') \to -\dot{\boldsymbol{s}}(t')$ in equations (75) and (76). Therefore, the total potentials are the sums of the partial potentials, i.e., $\Phi = \Phi_+ + \Phi_-$ and $\boldsymbol{A} = \boldsymbol{A}_+ + \boldsymbol{A}_-$. Consequently, the scalar potential of the Hertzian dipole is

$$\Phi = \frac{q\, c\, \left((\boldsymbol{r} - \boldsymbol{v}\, t')\cdot\dot{\boldsymbol{s}}(t')\right)}{2\,\pi\,\varepsilon_0\, \left(c^2\,(t-t') - \boldsymbol{r}\cdot\boldsymbol{v} + t'\, v^2\right)^2}. \tag{77}$$

For the vector potential, we find

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi + \frac{q\, \dot{\boldsymbol{s}}(t')}{2\,\pi\,\varepsilon_0\, c\, \left(c^2\,(t-t') - \boldsymbol{r}\cdot\boldsymbol{v} + t'\, v^2\right)}. \tag{78}$$

To use the potentials (77) and (78), we still need the retarded time t' as a function of $\boldsymbol{r}$ and t. This term can be obtained by solving equation (48) and by assuming that the amplitude of the oscillation $\boldsymbol{s}(t')$ is very small compared with the distance $\boldsymbol{r}$ and thus can be neglected. Therefore, the solution t' for the Hertzian dipole is given by equation (49).

Now, to determine the field in the receiver's rest frame, we could substitute the retarded time (49) into the potentials (77) and (78) and then apply equation (22). However, it is easier to make use of equation (66), which is also valid in Lorentz-Einstein electrodynamics even if it applies a Galilean transformation. For this step, we need to transform the potentials (77) and (78) into the rest frame of the transmitter by replacing all occurrences of $\boldsymbol{r}$ in equations (49), (77), and (78) with $\boldsymbol{r}+\boldsymbol{v}\,t$. Equation (49) then becomes

$$t' = t - \tau \tag{79}$$

with

$$\tau := \frac{\boldsymbol{r}\cdot\boldsymbol{v} + \sqrt{c^2\,r^2 - \|\boldsymbol{r}\times\boldsymbol{v}\|^2}}{c^2 - v^2}. \tag{80}$$

For the potentials (77) and (78), we obtain

$$\Phi = \frac{q\,c\;(\boldsymbol{r}+\boldsymbol{v}\,\tau)\cdot\dot{\boldsymbol{s}}(t-\tau)}{2\,\pi\,\varepsilon_0\;(c^2\,r^2 - \|\boldsymbol{r}\times\boldsymbol{v}\|^2)} \tag{81}$$

and

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi + \frac{q\,\dot{\boldsymbol{s}}(t-\tau)}{2\,\pi\,\varepsilon_0\,c\;\sqrt{c^2\,r^2 - \|\boldsymbol{r}\times\boldsymbol{v}\|^2}} \tag{82}$$

by applying the Galilean transformation $\boldsymbol{r}\rightarrow\boldsymbol{r}+\boldsymbol{v}\,t$ and using equations (79) and (80).

The potentials (81) and (82) can now be substituted into equation (66). First, from equation (66), we obtain

$$\boldsymbol{F} = -q_d\,\frac{\partial}{\partial t}\boldsymbol{A} + q_d\;(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A} - q_d\,\nabla\Phi. \tag{83}$$

The calculations of the spatial derivatives can be greatly simplified by considering that the oscillation $\boldsymbol{s}(t)$ is a generic function without a given direction of oscillation. For this reason, $\boldsymbol{r} = r_x\,\boldsymbol{e}_x$ with $r_x > 0$ can be assumed without a loss of generality. This assumption simplifies equation (83) to

$$\boldsymbol{F} = -q_d\,\frac{\partial}{\partial t}\boldsymbol{A} + q_d\,v_x\,\frac{\partial}{\partial r_x}\boldsymbol{A} - q_d\,\boldsymbol{e}_x\,\frac{\partial}{\partial r_x}\Phi. \tag{84}$$

For equation (80), we have

$$\tau = \frac{r_x}{v_\tau} \tag{85}$$

with

$$v_\tau := -v_x + \sqrt{c^2 - (v^2 - v_x^2)} \tag{86}$$

as the velocity, which is independent of r_x. The potentials become much simpler as well, and we obtain

$$\Phi = \frac{q\,c\,\left(\boldsymbol{e}_x + \frac{\boldsymbol{v}}{v_\tau}\right)\cdot\dot{\boldsymbol{s}}\left(t-\frac{r_x}{v_\tau}\right)}{2\,\pi\,\varepsilon_0\;(c^2 - (v^2 - v_x^2))\;r_x} \tag{87}$$

for the scalar potential (81) by using equation (85). For the vector potential (82), we find

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi + \frac{q\,\dot{\boldsymbol{s}}\left(t-\frac{r_x}{v_\tau}\right)}{2\,\pi\,\varepsilon_0\,c\,r_x\;\sqrt{c^2 - (v^2 - v_x^2)}}. \tag{88}$$

As can be seen, the spatial derivatives of the potentials are now as simple to calculate as the temporal derivatives. In particular, we obtain the following equations:

$$\frac{\partial}{\partial r_x}\boldsymbol{A} = -\left(\frac{\boldsymbol{A}}{r_x} + \frac{1}{v_\tau}\frac{\partial}{\partial t}\boldsymbol{A}\right) \tag{89}$$

and

$$\frac{\partial}{\partial r_x}\Phi = -\left(\frac{\Phi}{r_x} + \frac{1}{v_\tau}\frac{\partial}{\partial t}\Phi\right). \tag{90}$$

Equations (89) and (90) can be further simplified if one is interested only in the far-field. The terms $\boldsymbol{A}/r_x$ and Φ/r_x decrease with $1/r_x^2$ and therefore do not affect the far-field. Thus, the following approximations apply:

$$\frac{\partial}{\partial r_x}\boldsymbol{A} \approx -\frac{1}{v_\tau}\frac{\partial}{\partial t}\boldsymbol{A} \tag{91}$$

and

$$\frac{\partial}{\partial r_x}\Phi \approx -\frac{1}{v_\tau}\frac{\partial}{\partial t}\Phi. \tag{92}$$

The far-field approximations (91) and (92) can now be substituted into equation (84), which gives

$$\boldsymbol{F} = -q_d\left(\left(1+\frac{v_x}{v_\tau}\right)\frac{\partial\boldsymbol{A}}{\partial t} - \frac{\boldsymbol{e}_x}{v_\tau}\frac{\partial\Phi}{\partial t}\right). \tag{93}$$

This equation contains only time derivatives, which are easy to calculate.

Remarkably, from equation (93), it becomes apparent that the restriction to $\boldsymbol{r} = r_x\,\boldsymbol{e}_x$ is actually not necessary. This conclusion arises from the fact that equation (93) depends only on the direction-independent quantity v, the direction vector $\boldsymbol{e}_x = \boldsymbol{r}/r_x$, and the projection $v_x = \boldsymbol{v}\cdot\boldsymbol{r}/r_x$. At the same time, however, the oscillation $\boldsymbol{s}(t)$ is generic and consequently independent of direction. For this reason, r_x can be replaced by the distance r, and we obtain

$$\boldsymbol{F} = -q_d\left(\left(1+\tau\,\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^2}\right)\frac{\partial\boldsymbol{A}}{\partial t} - \tau\,\frac{\boldsymbol{r}}{r^2}\frac{\partial\Phi}{\partial t}\right) \tag{94}$$

because $v_\tau = r/\tau$. For the potentials, we can apply the general functions (81) and (82), and for the time τ, we can apply equation (80).

We now analyze formula (94). In the simplest case, $\boldsymbol{v} = \boldsymbol{0}$ and $\tau = r/c$. Then, equation (94) becomes

$$\boldsymbol{F} = -q_d\left(\frac{\partial\boldsymbol{A}}{\partial t} - \frac{\boldsymbol{r}}{r\,c}\frac{\partial\Phi}{\partial t}\right). \tag{95}$$

For $v = 0$, equations (81) and (82) simplify to

$$\Phi = \frac{q\,\boldsymbol{r}\cdot\dot{\boldsymbol{s}}\left(t-\frac{r}{c}\right)}{2\,\pi\,\varepsilon_0\,c\,r^2} \tag{96}$$

and

$$\boldsymbol{A} = \frac{q\,\dot{\boldsymbol{s}}\left(t-\frac{r}{c}\right)}{2\,\pi\,\varepsilon_0\,c^2\,r}. \tag{97}$$

Substituting the potentials (96) and (97) into equation (95) gives

$$\begin{aligned}\boldsymbol{F} &= -\frac{q_d\,q}{2\,\pi\,\varepsilon_0\,c^2\,r}\left(\ddot{\boldsymbol{s}}\,(t-\tau) - \frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\;\frac{\boldsymbol{r}}{r}\right)\\ &= \frac{q_d\,q}{2\,\pi\,\varepsilon_0\,c^2\,r}\left(\frac{\boldsymbol{r}}{r}\times\left(\frac{\boldsymbol{r}}{r}\times\ddot{\boldsymbol{s}}\,(t-\tau)\right)\right).\end{aligned} \tag{98}$$

A reader familiar with electrodynamics will certainly recognize that this is the type of field one would expect for a Hertzian dipole. For $\boldsymbol{s}(t) = \boldsymbol{e}_z\, p_0/(2\,q)\, \sin(\omega\, t)$, this expression corresponds exactly to the field one can usually find in textbooks (e.g., [19, p. 470]).

Equation (98) provides valuable insight into the essential properties of a point-like transmitter. In particular, the force vanishes for $\boldsymbol{r} \parallel \boldsymbol{s}(t)$, indicating that there is no radiation perpendicular to the direction of oscillation. Furthermore, it becomes obvious that the field in the far-field region is always aligned parallel to the direction of the dipole oscillation and propagates in the shape of a ring perpendicular to the axis of oscillation. This ring-wave propagation explains why the wave amplitude decreases with $1/r$ and not with $1/r^2$, as would be the case for a spherical wave. Interestingly, the term $\ddot{\boldsymbol{s}}\,(t-\tau)$ shows that the information contained in the oscillation $\boldsymbol{s}(t)$ propagates at the speed of light c.

However, this is only true if the receiver is at rest relative to the center of gravity of the transmitter. For $\boldsymbol{v} \neq \boldsymbol{0}$ and $v \ll c$, one obtains the first-order approximations as

$$\tau \approx \frac{r}{c} + \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{c^2}, \tag{99}$$

$$\Phi \approx \frac{q}{2\,\pi\,\varepsilon_0\, c\, r}\left(\frac{\boldsymbol{r}}{r} + \frac{\boldsymbol{v}}{c}\right)\cdot \dot{\boldsymbol{s}}\,(t-\tau)\,, \tag{100}$$

and

$$\boldsymbol{A} \approx \frac{q}{2\,\pi\,\varepsilon_0\, c^2\, r}\left(\dot{\boldsymbol{s}}\,(t-\tau) + \frac{\boldsymbol{r}}{r}\cdot \dot{\boldsymbol{s}}\,(t-\tau)\;\frac{\boldsymbol{v}}{c}\right) \tag{101}$$

by performing a Taylor series expansion of equations (80), (81), and (82). Substituting equation (99) into equation (94) yields the following relation:

$$\boldsymbol{F} = -q_d\left(\frac{\partial \boldsymbol{A}}{\partial t} - \frac{\boldsymbol{r}}{r\,c}\,\frac{\partial \Phi}{\partial t}\right)\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right) - q_d\left(\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)^2\frac{\partial \boldsymbol{A}}{\partial t}. \tag{102}$$

The quadratic term $((\boldsymbol{r}\cdot\boldsymbol{v})/(r\,c))^2$ can be neglected for very small relative speeds v, and we arrive at

$$\boldsymbol{F} \approx -q_d\left(\frac{\partial \boldsymbol{A}}{\partial t} - \frac{\boldsymbol{r}}{r\,c}\,\frac{\partial \Phi}{\partial t}\right)\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right). \tag{103}$$

Now, the two potentials (100) and (101) can be substituted, and we obtain

$$\begin{aligned}\boldsymbol{F} = &-\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\,\pi\,\varepsilon_0\, c^2\, r}\left(\ddot{\boldsymbol{s}}\,(t-\tau) - \frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\;\frac{\boldsymbol{r}}{r}\right) - \\ &\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\,\pi\,\varepsilon_0\, c^2\, r}\left(\frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\;\frac{\boldsymbol{v}}{c} - \frac{\boldsymbol{v}}{c}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\;\frac{\boldsymbol{r}}{r}\right).\end{aligned} \tag{104}$$

This expression can be simplified by using Graßmann's identity, and we finally obtain

$$\boxed{\begin{aligned}\boldsymbol{F} = \;&\frac{q_d\, q\left(1+\frac{\boldsymbol{v}}{c}\cdot\frac{\boldsymbol{r}-\boldsymbol{v}\,t}{\|\boldsymbol{r}-\boldsymbol{v}\,t\|}\right)}{2\,\pi\,\varepsilon_0\, c^2\,\|\boldsymbol{r}-\boldsymbol{v}\,t\|}\Bigg\{\\ &\left(\frac{\boldsymbol{r}-\boldsymbol{v}\,t}{\|\boldsymbol{r}-\boldsymbol{v}\,t\|}\times\left(\frac{\boldsymbol{r}-\boldsymbol{v}\,t}{\|\boldsymbol{r}-\boldsymbol{v}\,t\|}\times\ddot{\boldsymbol{s}}\,(t-\tau)\right)\right) + \\ &\left(\left(\frac{\boldsymbol{v}}{c}\times\frac{\boldsymbol{r}-\boldsymbol{v}\,t}{\|\boldsymbol{r}-\boldsymbol{v}\,t\|}\right)\times\ddot{\boldsymbol{s}}\,(t-\tau)\right)\Bigg\},\end{aligned}} \tag{105}$$

having reverted the initial Galilean transformation by applying the substitution $\boldsymbol{r} \to \boldsymbol{r} - \boldsymbol{v}\,t$.

V. Interpretations and conclusions

Equation (105) describes the force or electromagnetic wave as perceived by the receiver in its own rest frame in both Lorentz-Einstein electrodynamics and Weber-Maxwell electrodynamics. Differences exist only in determining how to transform into the rest frame of the transmitter. Lorentz-Einstein electrodynamics states that one must transform this force into the rest frame of any other receiver or into the rest frame of the transmitter by using the Lorentz transformation. In doing so, one must be careful to properly decompose the force into perpendicular and parallel components. In addition, one must apply the laws of relativistic dynamics. These calculations are likely very complicated.

In contrast, in Weber-Maxwell electrodynamics, we know that $\boldsymbol{v}$ is the velocity of the transmitter from the perspective of the resting receiver. However, the solution (105) has the same form for all receivers. Consequently, there is no reason to not interpret $\boldsymbol{v}$ as the relative velocity between the transmitter and receiver, just as in the original Weber electrodynamics. Thus, in the wave equation (15), we interpret and redefine $\boldsymbol{v}_s - \boldsymbol{0}$ as the relative velocity $\dot{\boldsymbol{r}}$ between transmitter and receiver and $\boldsymbol{r}_s - \boldsymbol{r}$ as the difference $\boldsymbol{r}$ of the position vectors of transmitter and reciever. Then, using the equation (16) and the identity $\delta(-\boldsymbol{r}) = \delta(\boldsymbol{r})$, we finally obtain the *invariant* and *symmetric* wave equation

$$\boxed{\Box\,\boldsymbol{F} = -\frac{q_d\, q_{s0}}{\varepsilon_0}\left(\frac{1}{c^2}\,\frac{\partial}{\partial t}\,\dot{\boldsymbol{r}}\,\gamma(\dot{\boldsymbol{r}})\,\delta(\boldsymbol{r}) + \nabla\,\gamma(\dot{\boldsymbol{r}})\,\delta(\boldsymbol{r})\right)}, \tag{106}$$

which might be interpreted as the wave equation and quintessence of Weber-Maxwell electrodynamics, with equations (51), (105), Weber force and Coulomb force being specific solutions.

In standard electrodynamics, Maxwell's equations are usually solved in the rest frame of the transmitter, while implicitly assuming that the receiver is also at rest, because the receiver's velocity in the Lorentz force (5) is not included anywhere in the calculation of the potentials. The first time when the velocity of the receiver is considered is at the end of the calculation, when the rest frame potentials (96) and (97) are inserted into the formula for the Lorentz force. Notably, this makes Lorentz-Einstein electrodynamics asymmetric.

To make this even more obvious, let us now repeat the calculation of the solution of the Hertzian dipole using the methods of Lorentz-Einstein electrodynamics. First, when we use potentials instead of fields, the Lorentz force (5) reads

$$\boldsymbol{F} = -q_d\,\frac{\partial}{\partial t}\boldsymbol{A} + q_d\,\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) - q_d\,\nabla\Phi. \tag{107}$$

Note that for $v^2/c^2 \approx 0$ this equation is equivalent to equation (83), which we used previously to simplify the calculation[2]. We have seen that only the full potentials (81) and (82) may be used in this equation when the receiver has a velocity relative to the transmitter. However, in standard electrodynamics one nevertheless inserts the stripped-down and velocity

[2]It holds that $\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) = \nabla\,(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A}$. If we use relation (68), we find $\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) = \frac{v^2}{c^2}\,\nabla\,\Phi - (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A}$. By substituting this result into the Lorentz force (107) and adjusting the sign of $\boldsymbol{v}$, we find that equation (83) and the Lorentz force (107) are identical for $v^2/c^2 \approx 0$.

independent potentials (96) and (97) and obtains in the far-field region, i.e., by neglecting all terms of order $1/r^2$, the following approximation:

$$\boldsymbol{F} \approx \frac{q_d\, q}{2\,\pi\,\varepsilon_0\, c^2\, r} \left(\left(\frac{\boldsymbol{r}}{r} - \frac{\boldsymbol{v}}{c} \right) \times \left(\frac{\boldsymbol{r}}{r} \times \ddot{\boldsymbol{s}} \left(t - \frac{r}{c} \right) \right) \right), \tag{108}$$

which can be found in numerous textbooks, usually split into magnetic and electric fields and often given in spherical coordinates.

The result (108) can now be compared with the force (105). As can be seen, the two forces are identical only for $v = 0$. For non-zero velocities, it must be noted that the meaning of $\boldsymbol{v}$ in the two formulas differs in its sign. In the Lorentz force, $\boldsymbol{v}$ is the velocity of the receiver from the perspective of the transmitter; in contrast, in formula (105), $\boldsymbol{v}$ is the velocity of the transmitter from the perspective of the receiver. However, even when the sign is adjusted, the two equations differ significantly for $v \neq 0$.

The most essential difference between equations (105) and (108) is, however, not the structure of the terms with the cross products, but the argument of the function $\ddot{\boldsymbol{s}}(.)$. In equation (108), the argument is $t - r/c$. In equation (105), the argument is $t - \tau$, where τ is given by equation (80). Here, τ has just the value necessary for the wave to have exactly velocity c in the receiver's rest frame.

Thus, if we solve Maxwell's equations for a receiver and transmitter that are both simultaneously at rest and then use the Lorentz force, a problem arises in that the wave moves at c only for a resting receiver, but not for a receiver in motion. Indeed, if one were to perform a Galilean transformation into the receiver's rest frame, the wave would have a propagation velocity different from c in most cases. For this reason, the Lorentz transformation is indispensable in Lorentz-Einstein electrodynamics. In contrast, if one applies Maxwell's equations rigorously in the rest frame of the receiver without the Lorentz force, as described in this article, the wave velocity is always exactly c, independent of the velocity relative to the transmitter. This result has also been recently shown by H. Dodig [3].

It is important to highlight that these findings are perfectly sufficient to satisfy Einstein's postulates:

1) There is no absolute velocity, only relative velocities. A velocity, like a voltage, always needs a reference point.
2) The propagation velocity of an electromagnetic wave is equal to c for every observer in vacuum.

The first principle of relativity (1) is clearly satisfied in Weber-Maxwell electrodynamics because $\boldsymbol{v} = \dot{\boldsymbol{r}}$ can be interpreted as the relative velocity between the transmitter and receiver and does not depend on any uninvolved observer. This can also be clearly seen in the wave equation (106). The second postulate (2) is also satisfied without the Lorentz transformation because an observer can *principally* measure the wave velocity only in his own rest frame. Thus, an observer is always a receiver, because only receivers can receive and measure an electromagnetic wave. If the observer moves with respect to the transmitter, this motion is explicitly considered in the wave equation (15), and the calculated force for the receiver propagates at speed c. If the receiver does not move with respect to the transmitter, then this lack of motion is taken into account by the wave equation (15), and the propagation speed of the wave is again exactly c. Thus, the speed of the receiver with respect to the transmitter is irrelevant because, after the wave equation (15) is solved, the force travels for the receiver or observer at speed c. This can also be clearly seen in the wave equation (106), because the d'Alembert operator determines the wave velocity as exactly c, independent of the relative velocity $\dot{\boldsymbol{r}}$ between receiver and transmitter.

The above explanation is based on a mathematical viewpoint. Logically, however, this wave equation (106) seems to be deeply counterintuitive. How can it be that the equation has the same form in every frame of reference? Do we really need a redefinition of space and time? This could be the case, but it does not have to be. In fact, the phenomenon can be also explained physically by at least two hypotheses, which both contradict Einstein's geometrical and implicitly linear [23] approach.

The first hypothesis is that an electromagnetic wave is a quantum mechanical superposition of all wave velocities and that the matching velocity at the receiver is the result of a quantum collapse when the field quanta or photons hit the receiver. In the second hypothesis, we assume that the electric force is mediated by force carriers emitted by the transmitter with stochastic velocities. A sparkler can serve as an illustrative analogy: the force carriers correspond to the radiated sparks, which do not have a uniform emission velocity, but are radially symmetric with respect to the transmitter. If one moves the sparkler, the center of radiation moves as well. This explains the first of Einstein's postulates. The second postulate can be explained by assuming that each receiver can only perceive force carriers that do not move faster than c relative to itself. If a receiver is moving away from the transmitter, it can perceive force carriers that are faster than c relative to the transmitter. Yet, for the receiver, the carriers move at exactly c. If the observer or receiver moves toward the source, the opposite effect occurs.

Notably, not only is this force-carrier model compatible with the two postulates of special relativity, Maxwell's equations and wave equation (106), it can also explain the specific elliptic form of the Weber force (62). Furthermore, this model offers new opportunities for interpreting quantum effects due to its field quantization. Unfortunately, little investigation has been conducted on the consequences of the mechanism postulated here for phenomena outside of classical electrodynamics. However, it is known that one can derive from the Weber force not only magnetism, but likely gravitation and inertial effects as well[3]. Therefore, further study of these new ideas may be promising.

VI. Summary

This article has shown that the inhomogeneous wave equation (15), which follows from Maxwell's equations for the rest frame of a receiver, is not only valid for Lorentz-Einstein electrodynamics but also for Weber-Maxwell electrodynamics. This finding is supported by the fact that the solution of the

[3]For an first overview, see [4].

wave equation for transmitters and receivers moving uniformly with respect to each other corresponds to the Weber force (60) for relative velocities much smaller than c.

Until now, Weber electrodynamics seemed to be isolated from modern electrodynamics and to be a direct-action theory, which cannot explain electromagnetic waves. However, this article shows that the Weber force has a close connection to Maxwell's equations and that the difference between Lorentz-Einstein electrodynamics and generalized Weber electrodynamics – denoted here as Weber-Maxwell electrodynamics – primarily lies in how one interprets and applies the field equations and the resulting wave equation.

In Lorentz-Einstein electrodynamics, one usually solves Maxwell's equations in the reference frame of the transmitter, as this seems to correspond to the natural viewpoint and allows one to apply the established methods of electro- and magnetostatics. However, this approach implies that the wave generated by the transmitter moves at velocity c only with respect to the transmitter. Yet, numerous experiments have shown that the wave velocity is c in the reference frame of every receiver, whereby receiver can be used synonymously with observer. This contradiction finally led to the development of the Lorentz transformation.

However, if one solves the wave equation rigorously in the rest frame of the receiver or observer, this contradiction does not exist and one obtains solutions showing that the wave indeed propagates at velocity c simultaneously for every observer, even when the observers do not have the same velocities. This result can be recognized specifically in the far-field approximation (105) of a moving Hertzian dipole, as reported in this article for the first time.

Furthermore, the article has shown that it is possible to represent Weber-Maxwell electrodynamics with a single symmetric wave equation (106), which has the same form in all inertial frames of reference. It has been suggested that this counterintuitive property can be interpreted as a kind of quantum effect. Moreover, the article has shown that it might be too simplistic to solve Maxwell's equations for resting transmitters and receivers and to expect that the obtained solutions could be generalized to moving reference frames by just applying the Lorentz force formula.

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