Analysis of laser-amplifier noise: A review of the method and its application to optical communications

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Abstract

This study summarizes the theory of laser-amplifier noise based on the master equation, the quantum-mechanical Langevin equation, and the rate equation. The three approaches are used differently depending on the application. Laser-amplifier noise in the direct-detection receiver is described by the master equation. Meanwhile, the quantum-mechanical Langevin equation is most effective for analyzing noise appearing in coherent optical communication systems. The rate equation approach provides a clear understanding of the intensity-noise characteristics of laser oscillators, including the amplitude squeezing.

1 Introduction

Laser amplifiers, such as erbium-doped fiber amplifiers (EDFAs) and semiconductor optical amplifiers (SOAs), have widely been employed in optical communication systems and devices. Therefore, the noise characteristics of laser amplifiers are critical for improving the performance of systems and devices. Theoretically, the laser-amplifier noise has been analyzed using the master equation [1], which focuses on fluctuations in the number of photons. The performance of the pre-amplified direct-detection (DD) receiver can be evaluated using the derived noise formulae. As an alternative approach, the quantum-mechanical Langevin equation is introduced into the analysis of noise in laser amplifiers [2]. In this approach, the evolution of the electric field along the amplifier is calculated, and thus we can evaluate the performance of coherent optical communication systems using laser amplifiers [3]. The rate-equation approach can also analyze the laser-amplifier noise, although it has been most commonly used to analyze the semiconductor-laser characteristics including noise [4].

The purpose of this study is to review the three approaches described above. By bridging physics and engineering, particular attention is paid to the application of the theory to the optical communication system. Consequently, the derived noise formulae are finally expressed in terms of system parameters, such as photocurent, optical bandwidth, and receiver bandwidth.

The organization of this paper is as follows. Section 2 describes emission and absorption of photons, optical modes in free space, and coherent states of light in preparation for discussions in the remaining sections. Sections 3, 4, and 5 deal with the master equation, Langevin equation, and rate equation, respectively. In Sec. 6, we examine the physical meaning of the noise formulae derived in Sections 3, 4, and 5. Sections 7 and 8 discuss the sensitivity of the pre-amplified DD receiver and the coherent receiver, respectively. In Sec. 9, we discuss the noise figure (NF) of optical amplifiers based on the concept of the equivalent input noise. Section 10 analyzes accumulation of the noise in a laser-amplifier chain, where laser amplifiers act as optical repeaters. In Sec. 11, we describe the intensity noise and FM noise of semiconductor lasers, using the rate-equation approach. Finally, Sec. 12 concludes this paper.

2 Preparation

In this section, we summarize the fundamentals necessary for noise analyses of laser amplifiers. Namely, photon emission and absorption in the two-level atomic system, optical modes in free space, and coherent states of light are explained.

2.1 Emission and absorption of photons in the two-level atomic system

We consider the two-level atomic system. Lower state b has the energy of E_b, whereas upper state a has the energy of E_a.

As shown by Fig. 1, when the electronic state of an atom is a at $t = 0$, it makes a transition from state a to b, emitting a photon with an energy of $hf = E_a - E_b$, where h denotes Planck's constant, and f is the frequency of light. This phenomenon is called the spontaneous emission. Let the probability of the transition from state a to b per unit time be A_{ab}. The time variation of the number of atoms N_a in state a is given by

$$-\frac{dN_a}{dt} = A_{ab} N_a \ , \tag{1}$$

where A_{ab} is called Einstein's A coefficient.

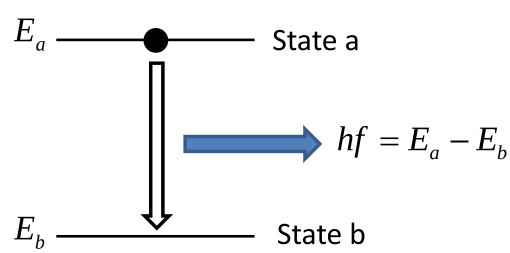

Figure 1: Spontaneous emission process in a two-level atomic system.

On the other hand, when a photon having an energy of $hf = E_a - E_b$ interacts with a two-level atomic system, two types of transitions occur. The transition in Fig. 2 shows the light absorption process. By absorbing a photon, transition of the atom takes place from state b to state a. In contrast, Fig. 3 represents the stimulated emission process[1] in which the atom in state a makes a transition to state b under the influence of an incident photon and emits an additional photon. Let the energy density per unit frequency of light be $\rho(f)$. Then, the probabilities of these transitions are given by

$$W_{ba} = B_{ba}\rho(f) \ , \tag{2}$$
$$W_{ab} = B_{ab}\rho(f) \ , \tag{3}$$

where B_{ba} and B_{ab} are called Einstein's B coefficients.

Therefore, the overall transition probability per unit time from state a to b consists of the probabilities of spontaneous emission and stimulated emission and it is given by

$$W_{ab} = B_{ab}\rho(f) + A_{ab} \ . \tag{4}$$

On the other hand, the transition probability from state b to a is caused by light absorption and it is given by

$$W_{ba} = B_{ba}\rho(f) \ . \tag{5}$$

[1]This process is also called induced emission.

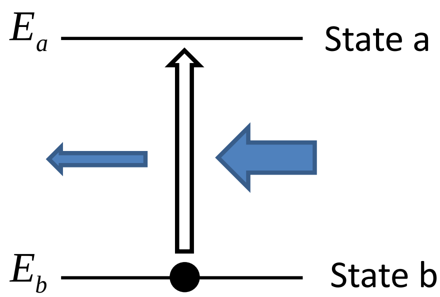

Figure 2: Light absorption process in a a two-level atomic system.

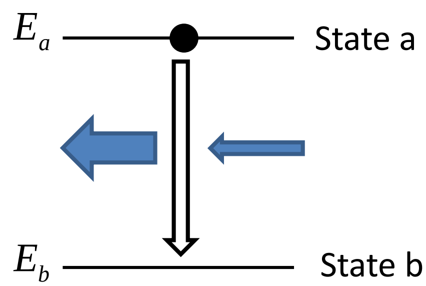

Figure 3: Stimulated emission process in a two-level atomic system.

Meanwhile, it is well known that blackbody radiation at temperature T has the spectrum given by

$$\rho(f) = \frac{8\pi h f^3}{c^3} \frac{1}{\exp\left(\frac{hf}{kT}\right) - 1} \ . \tag{6}$$

This is called Planck distribution.

Since the relation

$$N_a W_{ab} = N_b W_{ba} \tag{7}$$

holds in thermal equilibrium, we have

$$N_a \left[B_{ab}\rho(f) + A_{ab} \right] = N_b B_{ba} \rho(f) \ . \tag{8}$$

Using

$$\frac{N_a}{N_b} = \exp\left(-\frac{hf}{kT}\right) \ , \tag{9}$$

we can derive

$$\frac{8\pi h f^3}{c^3} \frac{1}{\exp\left(\frac{hf}{kT}\right) - 1} = \frac{A_{ab}}{B_{ba}\exp\left(\frac{hf}{kT}\right) - B_{ab}} \ . \tag{10}$$

Therefore, we obtain

$$B_{ab} = B_{ba} \ , \tag{11}$$

$$\frac{A_{ab}}{B_{ab}} = \frac{8\pi h f^3}{c^3} \ . \tag{12}$$

We consider the transition probability from state a to b again. Noting that the mode density of the electromagnetic field in the three-dimensional space is given by

$$g_m(f) = \frac{8\pi f^2}{c^3} \ , \tag{13}$$

we have the photon emission probability as

$$\begin{aligned}[A_{ab} + B_{ab}\rho(f)]\, N_a &= \left[A_{ab}\left(1 + \frac{\rho(f)}{hf g_m(f)}\right)\right] N_a \\ &= \left[A_{ab}\left(1 + \overline{n}(f)\right)\right] N_a \ . \end{aligned} \tag{14}$$

In this equation, $\overline{n}(f)$ denotes the average number of photons per optical mode at the frequency f and represents the ratio of the probabilities of induced emission and spontaneous emission. Thus, the induced emission occurs in proportion to the photon number but the spontaneous emission does not depend on the photon number. On the other hand, the transition probability from state b to a is given by

$$\begin{aligned}B_{ab}\rho(f)N_b &= A_{ab}\frac{\rho(f)}{hf g_m(f)}N_b \\ &= A_{ab}\overline{n}(f)N_b \ , \end{aligned} \tag{15}$$

which shows that photon absorption takes place in proportion to the number of photons.

2.2 Modes in free space

Let us consider a traveling wave propagating in free space along the z-direction. To define a mode for the electromagnetic field propagating in free space, we assume a virtual boundary as shown in Fig. 4, where L is the length in the traveling direction (time interval $T = L/c$) and that the periodic boundary condition is satisfied[2]. In other words, we consider the electromagnetic field at the right edge to be continuously connected to the left edge. A box with the length L is traveling in the z-direction at a speed of c. Taking the wave number of the traveling wave as k, the periodic boundary condition is given by $kL = 2\pi n$ where n is an integer; then, the frequency of the eigenmode is given by $f = nc/L = n/T$. The eigen-frequencies are aligned at equal intervals of $1/T$ on the frequency axis and each mode is represented with a line spectrum, as shown in Fig. 5, where f_c denotes the carrier frequency. Each mode can handle the bandwidth of $1/T$, surrounded by broken lines; and thus, the whole bandwidth is covered. Consequently, the optical filtering bandwidth, Δf, is written in units of $1/T$ as

$$\Delta f = \frac{M}{T} \ , \tag{16}$$

where M is an integer showing the number of modes included in the bandwidth[3].

2.3 Coherent state of light

The quantized electric field of light is expressed in terms of the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$, which correspond to the classical complex amplitude of the electric field and its complex conjugate, respectively, and satisfy the following commutation relation:

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \ . \tag{17}$$

[2] Since the time interval T is a virtual value, it can be set arbitrarily. The actually measured amount of noise is expressed in terms of the optical and electrical bandwidths instead of T.

[3] In the following, we assume that $\Delta f \gg 1/T$.

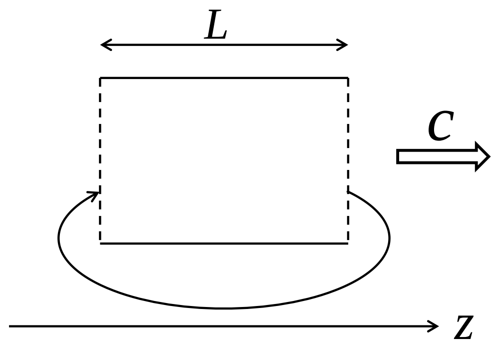

Figure 4: Definition of the eigenmode of traveling waves. A segment of length L is propagating towards the right side at the speed of light c. We assume virtually that the electromagnetic field in the right edge is connected to the left edge.

The operator of the real part of the complex amplitude is given by

$$\hat{a}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2} \ , \tag{18}$$

and the operator of the imaginary part of the complex amplitude is given by

$$\hat{a}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i} \ . \tag{19}$$

The operators, $\hat{a}_1$ and $\hat{a}_2$, satisfy

$$[\hat{a}_1, \hat{a}_2] = \frac{i}{2} \ . \tag{20}$$

Moreover, the photon number operator is expressed as

$$\hat{n} = \hat{a}^\dagger \hat{a} \ . \tag{21}$$

From Eqs. (18), (19), and (21), we obtain the following relation:

$$\hat{a}_1^2 + \hat{a}_2^2 = \hat{n} + \frac{1}{2} \ , \tag{22}$$

and hence, the energy of the electric field (Hamiltonian) is expressed as

$$\begin{aligned} \hat{H} &= hf_c \left(\hat{a}_1^2 + \hat{a}_2^2 \right) \\ &= hf_c \left(\hat{n} + \frac{1}{2} \right) \ . \end{aligned} \tag{23}$$

Equation (23) demonstrates that the energy of light includes the n-photon energy and the half-photon vacuum-fluctuation energy[4].

The coherent state of light, $|\alpha\rangle$, satisfies

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{24}$$

where α is a complex number. We find that $|\alpha\rangle$ is an eigenstate of $\hat{a}$, and the eigenvalue is α. Then, α means a parameter corresponding the classical complex amplitude. Since α is the eigenvalue of $\hat{a}$,

[4]The DD receiver measures the number of photons, and is insensitive to the vacuum field; on the other hand, the coherent receiver can detect the vacuum fluctuations.

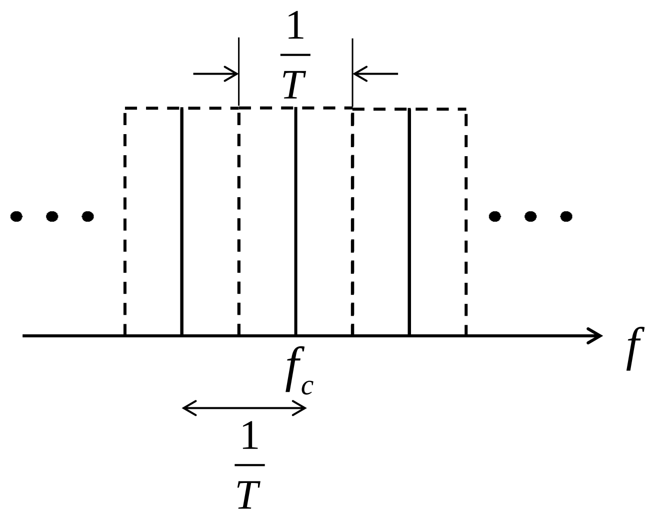

Figure 5: Relationship between the signal spectrum and the eigenmode. One mode can handle the bandwidth $1/T$.

it seems as if the complex amplitude of the coherent state has no fluctuation. However, this idea is incorrect. Since the operator $\hat{a}$ is not Hermitian, there are no means to measure it directly. On the other hand, it is possible to measure the real and imaginary parts of the complex amplitude separately, because the corresponding operators, $\hat{a}_1$ and $\hat{a}_2$, are Hermitian. However, in such a case, fluctuations are present in the measured values.

In the coherent state, the average of the real part is given as[5]

$$\langle \hat{a}_1 \rangle = \frac{\alpha + \alpha^\star}{2} \ . \tag{25}$$

In addition, the average of the square of the real part is given as

$$\left\langle \hat{a}_1^2 \right\rangle = \frac{1}{4}\left(\alpha^2 + \alpha^{\star 2} + 2|\alpha|^2 + 1\right) \ . \tag{26}$$

Therefore, the variance of $\hat{a}_1$ is calculated as

$$\langle \Delta \hat{a}_1^2 \rangle = \frac{1}{4} \ . \tag{27}$$

Similarly, we have

$$\langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4} \ . \tag{28}$$

Equations (27) and (28) show that the uncertainty product of the coherent state, $\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle$, is 1/16, which is the minimum value allowed in the Heisenberg uncertainty principle [3]. Moreover, it can be seen that the fluctuations are equally distributed in the real and imaginary parts of the complex amplitude. Therefore, if the complex amplitude is illustrated on a complex plane, it will be shown as Fig. 6. The average complex amplitude is α, which is associated with the noise having the isotropic distribution. Since this noise is present even when the state of light is $|0\rangle$, it called the "vacuum fluctuations." Thus, light in the coherent state can be interpreted as the superposition of the fixed complex amplitude α and the vacuum fluctuations. Meanwhile, the average and variance of the photon

[5] We express the average as $\langle * \rangle$ instead of $\langle \alpha | * | \alpha \rangle$ for short.

number for the coherent state are given by

$$\langle \hat{n} \rangle = |\alpha|^2 \ , \tag{29}$$

$$\left\langle \Delta \hat{n}^2 \right\rangle = |\alpha|^2 \ . \tag{30}$$

From the classical point of view, the photon-number fluctuations are considered to be the Poisson process, because the average and variance of the number of photons are equal as seen from Eqs. (29) and (30). In this sense, the photon number fluctuations in the coherent state are called the "shot noise" in the DD system.

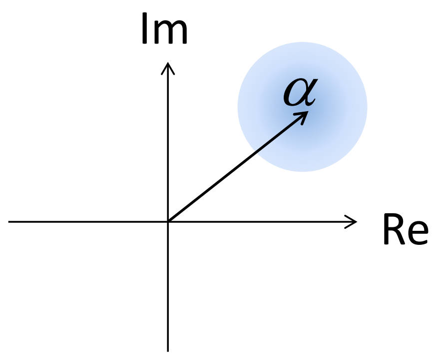

Figure 6: Complex amplitude of the coherent state. The vacuum field fluctuations are associated with the fixed complex amplitude α.

Let the fluctuation of the in-phase component be $\Delta\hat{a}_1$, and the fluctuation of the quadrature component be $\Delta\hat{a}_2$[6]. In such a case, we have

$$\hat{a}_1 = |\alpha| + \Delta\hat{a}_1 \ , \tag{31}$$

$$\hat{a}_2 = \Delta\hat{a}_2 \ . \tag{32}$$

The photon number operator is also represented by using the average photon number $\langle \hat{n} \rangle$ and the fluctuation part $\Delta\hat{n}$ as

$$\hat{n} = \langle \hat{n} \rangle + \Delta\hat{n} \ . \tag{33}$$

Equations (22), (31), (32), and (33) yield the relation between the amplitude fluctuation and the photon-number fluctuation as follows:

$$\Delta\hat{n} = 2\,|\alpha|\,\Delta\hat{a}_1 \ . \tag{34}$$

From Eqs. (27) and (34), we have

$$\left\langle \Delta\hat{n}^2 \right\rangle = 4\,|\alpha|^2 \left\langle \Delta\hat{a}_1^2 \right\rangle = \langle \hat{n} \rangle \ , \tag{35}$$

which demonstrates that the shot noise is interpreted as the noise generated from the beat between the signal and vacuum fluctuations. Note that the vacuum fluctuations alone cannot be detected by the direct detection receiver because the number of photons is zero, but generate the photon number fluctuations called the shot noise.

Meanwhile, the fluctuation of the phase can be approximately represented as

$$\Delta\hat{\phi} = \frac{\Delta\hat{a}_2}{|\alpha|} \ . \tag{36}$$

Hence, using Eqs. (27), (28), (35), and (36), the uncertainty product of the photon number and phase is found to be

$$\langle \Delta\hat{n}^2 \rangle \langle \Delta\hat{\phi}^2 \rangle = \frac{1}{4} \ . \tag{37}$$

[6]The orthogonal coordinate system is rotated such that α is located on the real axis.

3 Master Equation Approach

The master-equation approach to analyze the noise of optical amplifiers was presented by Shimoda et al. in 1957 [1], and is the simplest way to understand the intensity-noise characteristics of laser amplifiers. The noise characteristics of EDFAs used in the DD system have been analyzed by this approach.

3.1 Derivation of the master equation

When the box passes through the amplifier, the photon number inside the box is amplified (see Fig.4). The emission and absorption probabilities of photons per unit time are given by Eqs. (14) and (15). Consequently, we define the gain coefficient, g, and loss coefficient, γ, as

$$g = AN_a \ , \tag{38}$$

$$\gamma = AN_b \ , \tag{39}$$

where $A = A_{ab}$. In the following, we ignore changes in N_a and N_b along the traveling direction and regard them as constants[7].

The spontaneous emission of photons occurs in one mode with a probability per unit time g. Thus, since the number of modes that pass through the filter bandwidth Δf is M (see Eq. (16)), the total spontaneous emission probability is given as Mg.

We represent the probability that the number of photons in the box is m at t as $P_m(t)$. The transition of the atomic state induced by absorption and emission of a photon is shown by Fig. 7. State A having the number of photons m undergoes absorption and makes a transition to state B having the number of photons $m-1$. The probability of this transition is γm per unit time. Further, by stimulated and spontaneous emissions, state A makes a transition to state C having the number of photons $m+1$. The probability of this transition is $gm + Mg$ per unit time. In the same way, state B makes a transition to state A by stimulated and spontaneous emissions with a probability of $g(m-1) + Mg$. In addition, state C makes a transition to state A by absorption with a probability of $\gamma(m+1)$. We can express all of the above processes as a differential equation in terms of $P_m(t)$:

$$\frac{dP_m}{dt} = -\left[(g+\gamma)m + Mg\right] P_m + \left[g(m-1) + Mg\right] P_{m-1} + \gamma(m+1)P_{m+1} \ . \tag{40}$$

This differential equation is called the master equation, and enables the calculation of n-th moment of the number of photons [1].

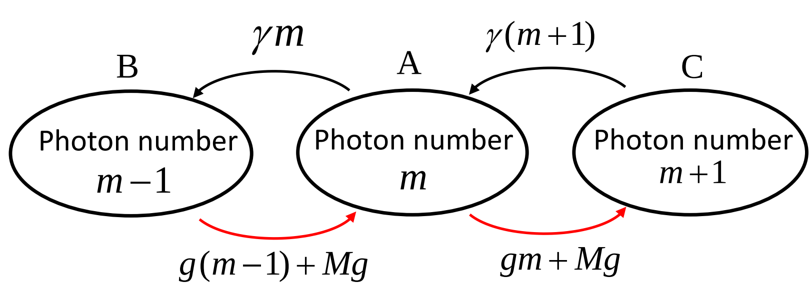

Figure 7: Diagram of atomic state transition accompanied with absorption and emission of a photon.

[7]See 4.5 as to the analysis of distributed-constant amplifiers.

3.2 Average number of photons

The average number of photons is given by[8]

$$\overline{m} = \sum_{m=0}^{\infty} m P_m \ . \tag{41}$$

Using the relation

$$\frac{d\overline{m}}{dt} = \sum_{m=0}^{\infty} m \frac{dP_m}{dt} \tag{42}$$

and Eq. (40), we have

$$\begin{aligned} \frac{d\overline{m}}{dt} &= -(g+\gamma) \sum_{m=0}^{\infty} m^2 P_m - Mg \sum_{m=0}^{\infty} m P_m \\ &+ g \sum_{m=1}^{\infty} m(m-1) P_{m-1} + Mg \sum_{m=1}^{\infty} m P_{m-1} + \gamma \sum_{m=0}^{\infty} m(m+1) P_{m+1} \ . \end{aligned} \tag{43}$$

Further, using the following relations

$$\sum_{m=0}^{\infty} m^2 P_m = \overline{m^2} \ , \tag{44}$$

$$\sum_{m=1}^{\infty} m(m-1) P_{m-1} = \overline{m^2} + \overline{m} \ , \tag{45}$$

$$\sum_{m=1}^{\infty} m P_{m-1} = \overline{m} + 1 \ , \tag{46}$$

$$\sum_{m=0}^{\infty} m(m+1) P_{m+1} = \overline{m^2} - \overline{m} \ , \tag{47}$$

we can derive the differential equation for the average number of photons as

$$\frac{d\overline{m}}{dt} = (g-\gamma)\overline{m} + Mg \ . \tag{48}$$

When we solve Eq. (48) under the initial condition that the average number of photons is n_0 at $t = 0$, the output number of photons is obtained as

$$\overline{m\,(t)} = G(t) n_0 + M\left(G(t) - 1\right) n_{sp} \ , \tag{49}$$

where

$$G(t) = \exp\left[(g-\gamma)t\right] \ , \tag{50}$$

$$n_{sp} = \frac{g}{g-\gamma} (\geq 1) \ . \tag{51}$$

Since the first term of Eq. (49) indicates that the incident number of photons has become G times, it is clear that G represents the amplifier gain. Since the second term is independent of the input, it represents a process in which the spontaneously emitted photon is amplified and output. Hence, it is called " amplified spontaneous emission (ASE)." The parameter n_{sp} is called the spontaneous emission factor. When all the atoms are excited ($N_b = 0$, $N_a = 1$), n_{sp} has the minimum value of 1. This state is the ideal operating condition of the amplifier.

[8] The statistical average is written as $\overline{*}$ differently from $\langle * \rangle$.

3.3 Photon number fluctuations

The second moment of m is given by

$$\overline{m^2} = \sum_{m=0}^{\infty} m^2 P_m \ . \tag{52}$$

From Eq. (40), the differential equation for $\overline{m^2}$ is obtained as

$$\frac{d\overline{m^2}}{dt} = 2(g-\gamma)\overline{m^2} + (g+\gamma+2Mg)\overline{m} + Mg \ . \tag{53}$$

Equation (53) can be solved using Eq. (49). If we assume that the input light is in the coherent state and the variance of the number of photons is n_0 at $t=0$, the variance at the output is calculated as

$$\overline{\Delta m\,(t)^2} = G(t)n_0 + (G(t)-1)Mn_{sp} + 2G(t)(G(t)-1)n_0 n_{sp} + (G(t)-1)^2 M n_{sp}^2 \ . \tag{54}$$

This photon-number fluctuation appears in the baseband below the frequency of $1/2T$. In other words, Eq. (54) demonstrates the electrical single-mode effect[9]. The noise of direct-detection (DD) receivers can be derived from Eq. (54) by including the electrical multi-mode effect, which will be discussed in Sec. 7.

4 Langevin Equation Approach

The Langevin-equation approach can describe the evolution of the electric field. Therefore, this approach is effective for analyzing the noise characteristics of coherent optical communication systems, where laser amplifiers are employed as repeaters and pre-amplifiers.

4.1 Quantum-mechanical Langevin equation

When the annihilation operator for a signal electric field, $\hat{a}$, passes through a medium with gain and loss, it obeys the following Langevin equation [2][5][10]:

$$\frac{d\hat{a}\,(t)}{dt} = \frac{1}{2}\,(g-\gamma)\,\hat{a}\,(t) + \hat{f}_g\,(t) + \hat{f}_\gamma\,(t) \ . \tag{55}$$

The fluctuation operators, $\hat{f}_\gamma$ and $\hat{f}_g$, result from the loss and gain of the medium, respectively. The solution to Eq. (55) is given as

$$\hat{a}\,(t) = \sqrt{G\,(t)}\left[\hat{a}\,(0) + \int_0^t \frac{\left(\hat{f}_g\,(t') + \hat{f}_\gamma\,(t')\right)}{\sqrt{G\,(t')}} dt'\right] \ . \tag{56}$$

4.2 Properties of the fluctuation operators

The annihilation operator, $\hat{a}\,(t)$, and the creation operator, $\hat{a}^\dagger\,(t)$, must satisfy the commutation relation given by Eq. (17). Consequently, Eq. (56) demonstrates that the commutation relation between $\hat{f}_\gamma\,(t)$ and $\hat{f}_\gamma^\dagger\,(t')$, and that between $\hat{f}_g\,(t)$ and $\hat{f}_g^\dagger\,(t')$, should be given by [6]

$$\left[f_\gamma\,(t)\,, f_\gamma^\dagger\,(t')\right] = \gamma\delta\,(t-t') \ , \tag{57}$$

$$\left[f_g\,(t)\,, f_g^\dagger\,(t')\right] = -g_\ell\delta\,(t-t') \ . \tag{58}$$

[9] The optical bandwidth Δf includes M optical modes. Therefore, Eq. (54) deals with the optical multi-mode effect.
[10] This is an optically single-mode equation. When $M \geq 2$, we need to include multi-mode effects.

By contrast, according to the quantum damping theory [5], the correlation properties between the above-mentioned fluctuation operators are expressed, when the photon energy is much larger than the thermal energy, as

$$\left\langle \hat{f}_\gamma{}^\dagger(t)\,\hat{f}_\gamma(t')\right\rangle = 0 \ , \tag{59}$$

$$\left\langle \hat{f}_\gamma(t)\,\hat{f}_\gamma{}^\dagger(t')\right\rangle = \gamma\delta(t-t') \ , \tag{60}$$

$$\left\langle \hat{f}_g{}^\dagger(t)\,\hat{f}_g(t')\right\rangle = g\delta(t-t') \ , \tag{61}$$

$$\left\langle \hat{f}_g(t)\,\hat{f}_g{}^\dagger(t')\right\rangle = 0 \ . \tag{62}$$

In addition, $\hat{f}_g$ and $\hat{f}_\gamma$ have no correlation.

4.3 Closed-form expression of the amplified electric field

We convert Eq. (56) into a closed-form expression of the amplified electric field. Using the gain, G, and n_{sp}, we define the noise operator, $\hat{c}^\dagger$, as

$$\sqrt{(G(t)-1)\,n_{sp}}\hat{c}^\dagger = \sqrt{G(t)}\int_0^t \frac{\hat{f}_g(t')}{\sqrt{G(t')}}dt' \ . \tag{63}$$

Equations (58) and (63) demonstrate that the noise operators, $\hat{c}$ and $\hat{c}^\dagger$, satisfy the following commutation relation:

$$\left[\hat{c},\hat{c}^\dagger\right] = 1 \ . \tag{64}$$

Equation (64) suggests that $\hat{c}$ and $\hat{c}^\dagger$ are the annihilation and creation operators operating on the vacuum field, respectively. Moreover, we define the noise operator, $\hat{d}$, as

$$\sqrt{(G(t)-1)\,(n_{sp}-1)}\hat{d} = \sqrt{G(t)}\int_0^t \frac{\hat{f}_\gamma(t')}{\sqrt{G(t')}}dt' \ . \tag{65}$$

Equations (57) and (65) yield the commutation relation between $\hat{d}$ and $\hat{d}^\dagger$ as

$$\left[\hat{d},\hat{d}^\dagger\right] = 1 \ , \tag{66}$$

which suggests that $\hat{d}$ and $\hat{d}^\dagger$ are the annihilation and creation operators operating on the vacuum field, respectively. The noise operators, $\hat{c}$ and $\hat{d}$, are not correlated with each other, and have no correlation with the input signal, $\hat{a}(0)$.

Consequently, by using Eqs. (63) and (65), Eq. (56) is reduced to the closed-form expression of the amplified electric field, given by

$$\hat{a}(t) = \sqrt{G(t)}\hat{a}(0) + \sqrt{(G(t)-1)\,n_{sp}}\hat{c}^\dagger + \sqrt{(G(t)-1)\,(n_{sp}-1)}\hat{d} \ . \tag{67}$$

The first term represents the electric field of the amplified signal, the second term represents noise associated with gain, and the third term represents noise associated with loss. Equation (67) enables us to calculate the noise characteristics of optical amplifiers through simple manipulations of the operators.

4.4 Noise formulae

The photon-number operator of the output signal is given by $\hat{n}(t) = \hat{a}^\dagger(t)\,\hat{a}(t)$. Using Eq. (67), we can obtain its average as

$$\langle \hat{n}(t) \rangle = G(t)\,\langle \hat{n}(0) \rangle + (G(t)-1)\,n_{sp} \ , \tag{68}$$

which coincides with Eq. (49) when $M = 1$. When $M \geq 2$, there are modes that only include ASE; thus, the second term in Eq. (68) should be multiplied by M.

The noise characteristics of laser amplifiers can be calculated by simple algebraic manipulation of the creation and annihilation operators in Eq. (67). For example, the variance of the photon number is obtained from Eq. (67) after straightforward calculations as

$$\left\langle \Delta\hat{n}(t)^2 \right\rangle = G(t)\,\langle \hat{n}(0) \rangle + (G(t)-1)\,n_{sp} + 2G(t)\,(G(t)-1)\,\langle \hat{n}(0) \rangle\,n_{sp} + (G(t)-1)^2\,n_{sp}^2 \ , \tag{69}$$

which is the same as Eq. (54) when $M = 1$. When $M \geq 2$, we can easily derive Eq. (54) from Eq. (69), taking the modes that only include ASE into account.

Moreover, the average and variance of the real part of the amplified electric field are given from Eq. (67) as

$$\langle \hat{a}_1(t) \rangle = \sqrt{G(t)}\,\langle \hat{a}_1(0) \rangle \ , \tag{70}$$

$$\left\langle \Delta\hat{a}_1(t)^2 \right\rangle = G(t)\left\langle \Delta\hat{a}_1(0)^2 \right\rangle + (G(t)-1)\,n_{sp}\,\langle \hat{c}_1^2 \rangle + (G(t)-1)\,(n_{sp}-1)\left\langle \hat{d}_1^2 \right\rangle \ , \tag{71}$$

where the suffix "1" of the operators indicates the real part. Noting that $\left\langle \Delta\hat{a}_1(0)^2 \right\rangle = 1/4$, and $\langle \hat{c}_1^2 \rangle = \left\langle \hat{d}_1^2 \right\rangle = 1/4$, we find from Eq. (71) that

$$\left\langle \Delta\hat{a}_1(t)^2 \right\rangle = \frac{1}{4} + \frac{1}{2}\,(G(t)-1)\,n_{sp} \ . \tag{72}$$

The first term on the right-hand side of Eq. (72) originates from vacuum fluctuations, whereas the second term originates from ASE. The noise characteristics of the imaginary part of the signal electric field, $\hat{a}_2(t)$, are the same as those of the real part, because laser amplifiers can amplify the real and imaginary parts of the signal electric field by the same gain, as expressed by Eq. (67)[11]. Equation (72) is useful for analyzing coherent optical communication systems[12].

4.5 Distributed-constant laser amplifiers

We generalize our noise theory, developed in 4.3 and 4.4, to include the distributed gain and loss coefficients [7]. In other words, the gain coefficient, g, and loss coefficient, γ, are dependent on t. In Erbium-doped fiber amplifiers (EDFAs), the pump wave attenuates along the fiber length, and hence, g and γ usually have t-dependence.

First, the gain of the amplifier is rewritten as

$$G(t) = \exp\left\{ \int_0^t \left[g(t') - \gamma(t') \right] dt' \right\} \ . \tag{73}$$

Next, we define R and S as

$$R(t) = G(t) \int_0^t \frac{g(t')}{G(t')} dt' \ , \tag{74}$$

$$S(t) = G(t) \int_0^t \frac{\gamma(t')}{G(t')} dt' \ , \tag{75}$$

[11] The real and imaginary parts are simultaneously measured by coherent receivers, the noise, originating from the vacuum fluctuations, is doubled, as discussed in Sec. 8.

[12] The noise of coherent receivers is derived from Eq,(72) and discussed in Sec 8. The multi-mode effect when $M \geq 2$ and the difference in the impact of optical filtering between DD and coherent receivers are elucidated there.

which satisfy

$$R(t) - S(t) = G(t) - 1 \ . \tag{76}$$

Then, Eqs. (63) and (65) are expressed as

$$\sqrt{R\,(t)}\hat{c}^{\dagger} = \sqrt{G\,(t)} \int_0^t \frac{\hat{f}_g\,(t')}{\sqrt{G\,(t')}} dt' \ . \tag{77}$$

$$\sqrt{S\,(t)}\hat{d} = \sqrt{G\,(t)} \int_0^t \frac{\hat{f}_\gamma\,(t')}{\sqrt{G\,(t')}} dt' \ , \tag{78}$$

and Eq. (67) is modified to

$$\hat{a}\,(t) = \sqrt{G(t)}\hat{a}\,(0) + \sqrt{R(t)}\hat{c}^{\dagger} + \sqrt{S(t)}\hat{d} \ . \tag{79}$$

Equation (79), in turn, yields the following revised noise formulae, corresponding to Eqs. (68), (69), and (72):

$$\langle \hat{n}\,(t) \rangle = G(t) \langle \hat{n}\,(0) \rangle + R(t) \ , \tag{80}$$

$$\left\langle \Delta \hat{n}\,(t)^2 \right\rangle = G(t) \langle \hat{n}\,(0) \rangle + R(t) + 2G(t)R(t) \langle \hat{n}\,(0) \rangle + R(t)^2 \ , \tag{81}$$

$$\left\langle \Delta \hat{a}_1\,(t)^2 \right\rangle = \frac{1}{4} + \frac{1}{2}R(t) \ . \tag{82}$$

Note that the term, $(G-1)n_{sp}$, in Eqs. (68), (69), and (72) should be replaced with $R(t)$ in the revised noise formulae.

5 Rate Equation Approach

This section describes the rate-equation approach for analyzing the laser-amplifier noise. Moreover, the laser noise can be analyzed by the rate equations very effectively, because the photon-electron interaction in the lasing system can clearly be introduced to the rate equations. Such analyses will be given in Sec. 11.

5.1 Derivation of the rate equation

The Langevin equation, given by Eq. (55), can be transformed into the rate equation for $\hat{n}(t) = \hat{a}^{\dagger}(t)\hat{a}(t)$ as

$$\begin{aligned} \frac{d\hat{n}(t)}{dt} &= \hat{a}^{\dagger}(t)\frac{d\hat{a}(t)}{dt} + \frac{d\hat{a}^{\dagger}(t)}{dt}\hat{a}(t) \\ &= (g-\gamma)\,\hat{n}(t) + \hat{a}(t)^{\dagger}\hat{f}(t) + \hat{f}^{\dagger}(t)\hat{a}(t) \ , \end{aligned} \tag{83}$$

where

$$\hat{f}(t) = \hat{f}_g(t) + \hat{f}_\gamma(t) \ . \tag{84}$$

The Langevin force, $\hat{f}(t)$, satisfies the following correlation relations:

$$\left\langle \hat{f}^{\dagger}(t)\hat{f}(t+\tau) \right\rangle = g\delta(\tau) \ , \tag{85}$$

$$\left\langle \hat{f}(t)\hat{f}^{\dagger}(t+\tau) \right\rangle = \gamma\delta(\tau) \ . \tag{86}$$

Equations (56), (85), and (86) yield

$$\left\langle \hat{a}(t)^{\dagger}\hat{f}(t) + \hat{f}^{\dagger}(t)\hat{a}(t) \right\rangle = g \ . \tag{87}$$

Next, using t_c that satisfies $|t - t_c| \simeq 0$, we define $\hat{f}_c(t)$ as

$$\hat{f}_c(t) = \hat{a}^{\dagger}(t_c)\hat{f}(t) + \hat{f}(t)^{\dagger}\hat{a}(t_c) \ . \tag{88}$$

Given that the correlation between $\hat{a}^{\dagger}(t_c)$ and $\hat{f}(t)$ disappears, we have

$$\left\langle \hat{f}_c(t) \right\rangle = 0 \ , \tag{89}$$

$$\left\langle \hat{f}_c(t)\hat{f}_c(t+\tau) \right\rangle = \{g\left[\langle \hat{n}(t)\rangle + 1\right] + \gamma \langle \hat{n}(t)\rangle\}\, \delta(\tau) \ . \tag{90}$$

The first, second, and third terms in the right-hand side of Eq. (90) represent noise associated with induced emission, spontaneous emission, and loss, respectively. Consequently, the final form of the rate equation is derived from Eq. (83) as

$$\frac{d\hat{n}(t)}{dt} = (g - \gamma)\, \hat{n}(t) + g + \hat{f}_c(t) \ , \tag{91}$$

which can be solved as

$$\hat{n}(t) = G\,(t)\, \hat{n}\,(0) + (G\,(t) - 1)\, n_{sp} + G\,(t) \int_0^t \frac{\hat{f}_c\,(t')}{G\,(t')} dt' \ . \tag{92}$$

Using Eqs. (89), (90) and (92), we can derive the noise formula such as Eqs. (68) and (69).

Moreover, taking multi-mode ASE into account, we can modify Eqs. (91) and (90) as

$$\frac{d\hat{n}(t)}{dt} = (g - \gamma)\, \hat{n}(t) + Mg + \hat{f}_c(t) \ , \tag{93}$$

$$\left\langle \hat{f}_c(t)\hat{f}_c(t+\tau) \right\rangle = \{g\left[\langle \hat{n}(t)\rangle + M\right] + \gamma \langle \hat{n}(t)\rangle\}\, \delta(\tau) \ . \tag{94}$$

The rate equation, given by Eqs. (93) and (94), is commonly used for analyses of the laser noise. This is because the photon-electron interaction in the lasing system can clearly be described by this approach. Details are given in Sec. 11[13].

5.2 Classical interpretation of Langevin force in the rate equation

We discuss the classical interpretation of the Langevin force in the rate equation, which is associated with gain/loss of photons. Let the number of photons obey the following rate equation:

$$\frac{dn(t)}{dt} = A(t) + f_n(t) \ , \tag{95}$$

where $A(t)$ generally describes the photon emission/absorption process, and $f_n(t)$ is the noise term associated with $A(t)$. The noise term, $f_n(t)$, satisfies the following correlation relation[14]:

$$\overline{f_n(t)f_n(t+\tau)} = a\delta(\tau) \ . \tag{96}$$

[13] In Sec. 11, the rate equations are treated classically. See also 5.2.

[14] From this correlation function, we find that $f_n(t)$ is a white noise, which means that its spectral density is constant. The two-sided spectral density defined in $(-\infty, +\infty)$ is given by a.

The number of emitted photons within Δt is given from Eq. (95) as

$$n = \int_0^{\Delta t} A(t)dt + \int_0^{\Delta t} f_n(t)dt \ . \tag{97}$$

The average and variance of n are written as

$$\overline{n} = A\Delta t \ , \tag{98}$$

$$\begin{aligned} \overline{\Delta n^2} &= \int_0^{\Delta t}\int_0^{\Delta t} \overline{f_n(t)f_n(t')}dtdt' \\ &= a\Delta t \ . \end{aligned} \tag{99}$$

Note that $\overline{n} = \overline{\Delta n^2}$, when the photon emission/absorption process is regarded as the Poisson process. In such a case, we have $A = a$, and actually, Eqs. (93) and (94) satisfy this relation. Thus, we can interpret the rate equation classically, assuming that the photon emission/absorption process obeys the Poisson statistics.

Finally, we find that Eq. (93) can be regarded as a c-number equation, and the Langevin force, Eq. (94), can be assumed to be a white Gaussian noise[15]. Actually, in Sec. 11, the rate equations are treated in such a way.

5.3 Relation between the rate equation and master equation

The rate equation can be transformed into the differential equation for the moments of photon number fluctuations, which is derived from the master equation in Sec. 3. For example, the differential equation for the second moment, $\langle \hat{n}^2 \rangle$, is obtained from Eq. (93) as follows. First, we have

$$\begin{aligned} \frac{d\hat{n}(t)^2}{dt} &= 2\hat{n}(t)\frac{d\hat{n}(t)}{dt} \\ &= 2(g-\gamma)\hat{n}(t)^2 + 2Mg\hat{n}(t) + 2\hat{f}_c(t)\hat{n}(t) \ . \end{aligned} \tag{100}$$

Next, the average of Eq. (100) is given by

$$\frac{d\left\langle \hat{n}(t)^2 \right\rangle}{dt} = 2(g-\gamma)\left\langle \hat{n}(t)^2 \right\rangle + 2Mg\left\langle \hat{n}(t) \right\rangle + 2\left\langle \hat{f}_c(t)\hat{n}(t) \right\rangle \ . \tag{101}$$

Since Eqs. (92) and (94) yield

$$\left\langle \hat{f}_c(t)\hat{n}(t) \right\rangle = \frac{1}{2}\left[g\left(\langle \hat{n}(t) \rangle + M\right) + \gamma \langle \hat{n}(t) \rangle\right] \ , \tag{102}$$

we have

$$\frac{d\left\langle \hat{n}(t)^2 \right\rangle}{dt} = 2(g-\gamma)\left\langle \hat{n}(t)^2 \right\rangle + (g+\gamma+2Mg)\left\langle \hat{n}(t) \right\rangle + Mg \ . \tag{103}$$

Equation (103) coincides with Eq. (53), which is derived from the master equation approach.

6 Physical Interpretation of the Laser-Amplifier Noise

In this section, we examine the physical meaning of the noise formulae for laser amplifiers derived in Sections 3, 4, and 5.

[15] The Poisson distribution can be approximated by the Gaussian distribution, when the number of occurrences is large.

Equation (49) shows that the output number of photons can be written as

$$\overline{m} = n_0 + (G-1)(n_0 + Mn_{sp}) \ . \tag{104}$$

The first term represents the incident number of photons and the second term represents the number of photons added by the amplifier. This situation is illustrated in Fig. 8(a). The amplified signal light is composed of the photons $(G-1)n_0$ generated by the stimulated emission and the incident photons n_0 themselves.

Meanwhile, the noise is generated as follows. Equation (72) demonstrates that the amplified electric field consists of the vacuum fluctuation and ASE. When the vacuum field (where the number of photons is zero) is incident on the amplifier, it "induces spontaneous emissions." Consequently, with the spontaneously emitted photons as the seed, stimulated emissions occur and ASE is generated, as shown Fig. 8(b). Thus, we find that the amplifier output includes the ASE and vacuum field in addition to the amplified signal.

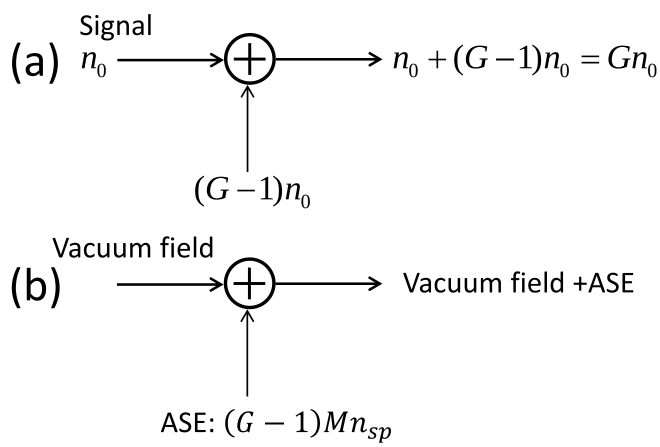

Figure 8: The concept of optical amplification. (a): Signal amplification process and (b): noise generation process.

Next, we consider the physical meaning of the photon-number fluctuations of the amplifier output given by Eq. (54) in a classical manner, using the beat-noise concept. Let the output electric field, E_s, be written as

$$E_s = a_s + (\Delta a_{s1} + \Delta a_{v1}) + i\,(\Delta a_{s2} + \Delta a_{v2}) \ , \tag{105}$$

where a_s is the signal electric field, Δa_{s1} is the in-phase ASE noise, Δa_{v1} is the in-phase vacuum noise, Δa_{s2} is the quadrature ASE noise, and Δa_{v1} is the quadrature vacuum noise. All of them are classical parameters, and the noise parameters are regarded as Gaussian random variables. The energy of the output light is given by

$$\begin{aligned} |E_s|^2 = |a_s|^2 &+ 2a_s\,(\Delta a_{v1} + \Delta a_{s1}) + (\Delta a_{s1}^2 + \Delta a_{s2}^2) + (\Delta a_{v1}^2 + \Delta a_{v2}^2) \\ &+ 2(\Delta a_{s1}\Delta a_{v1} + \Delta a_{s2}\Delta a_{v2}) \ . \end{aligned} \tag{106}$$

The first term is the output photon number, Gn_0. The second term represents the beat noise between the signal and vacuum field, and the beat noise between the signal and ASE. Their variances are given by

$$\left(\overline{\Delta n^2}\right)_{sig-vac} = 4Gn_0\overline{\Delta a_{v1}^2}$$
$$= Gn_0 \ , \tag{107}$$
$$\left(\overline{\Delta n^2}\right)_{sig-ASE} = 4Gn_0\overline{\Delta a_{s1}^2}$$
$$= 2G(G-1)n_0 n_{sp} \ . \tag{108}$$

The third term shows the ASE energy. The real and imaginary parts of the ASE obey the zero-mean Gaussian distribution, and their variance is given as

$$\overline{\Delta a_{s1,s2}^2} = \sigma^2 = \frac{1}{2}(G-1)n_{sp} \ . \tag{109}$$

Therefore, the average of the ASE energy is given by

$$(\overline{n})_{ASE} = \overline{\Delta a_{s1}^2} + \overline{\Delta a_{s2}^2} = (G-1)n_{sp} \ . \tag{110}$$

Meanwhile, the variance of the ASE energy fluctuations, which is called the ASE-ASE beat noise, is expressed as

$$\left(\overline{\Delta n^2}\right)_{ASE-ASE} = \overline{\left\{(\Delta a_{s1}^2 + \Delta a_{s2}^2) - \left(\overline{\Delta a_{s1}^2} + \overline{\Delta a_{s2}^2}\right)\right\}^2}$$
$$= \left(\overline{\Delta a_{s1}^4} - \overline{\Delta a_{s1}^2}^2\right) + \left(\overline{\Delta a_{s2}^4} - \overline{\Delta a_{s2}^2}^2\right) . \tag{111}$$

The fourth moment of the Gaussian noise, $\Delta a_{s1,s2}$, is given as

$$\overline{\Delta a_{s1,s2}^4} = 3\sigma^4 = \frac{3}{4}(G-1)^2 n_{sp}^2 \ . \tag{112}$$

Then, from Eqs. (111) and (112), we have

$$\left(\overline{\Delta n^2}\right)_{ASE-ASE} = (G-1)^2 n_{sp}^2 \ . \tag{113}$$

In the multi-mode operation, Eq. (113) is multiplied by M. The fourth term represents the vacuum fluctuation energy, which is not measured by DD receivers and is omitted[16]. The sixth term demonstrates the beat noise between the ASE and vacuum field. Its variance is given by

$$\left(\overline{\Delta n^2}\right)_{ASE-vac} = 4\left(\overline{\Delta a_{s1}^2}\ \overline{\Delta a_{v1}^2} + \overline{\Delta a_{s2}^2}\ \overline{\Delta a_{v2}^2}\right)$$
$$= (G-1)n_{sp} \ . \tag{114}$$

In the multi mode operation, Eqs. (110) and (114) are multiplied by M. Thus, in total, the photon number fluctuation is given by Eq. (54).

7 Pre-amplified DD Receivers

The DD receiver measures the incoming rate of photons, $\hat{N}(t)$, usually by using the photocurrent of photodiodes, $\hat{I}(t)$. When ideal photodiodes having 100% quantum efficiency are employed as detectors, we have

$$\hat{I}(t) = q\hat{N}(t) \ , \tag{115}$$

[16]The vacuum fluctuation energy must be treated quantum mechanically

where q denotes the electron charge. On the other hand, the relation between $\hat{n}$ and $\hat{N}$ is given by

$$\hat{N}(t) = \frac{\hat{n}(t)}{T} \ . \tag{116}$$

Let a pre-amplifier be placed in front of the DD receiver. Since Eq. (16) demonstrates that the mode number within the filtering bandwidth Δf of ASE is given by $M = \Delta f \cdot T$, Eq. (49) yields

$$\left\langle \hat{N}(t) \right\rangle = G(t) N_0 + (G(t) - 1) n_{sp} \Delta f \ , \tag{117}$$

where N_0 represents the photon rate of the input signal given by

$$N_0 = \frac{n_0}{T} \ . \tag{118}$$

From Eq. (117), we find that the signal power, P_{sig}, and ASE power, P_{ASE}, are written as

$$P_{sig} = hf_c G(t) N_0 \ , \tag{119}$$

$$P_{ASE} = hf_c (G(t) - 1) n_{sp} \Delta f \ . \tag{120}$$

Equation (120) demonstrates that ASE is white noise[17], and its spectral density is given by

$$S_{ASE} = hf_c (G(t) - 1) n_{sp} \ , \tag{121}$$

which is commonly used for computer simulations[18].

Next, we express $\left\langle \Delta\hat{N}(t)^2 \right\rangle$, using Δf and B, both of which are determined from an experimental setup[19]. As discussed in Sec. 6, Eq. (54) is valid when the electrical bandwidth is $B = 1/2T$, because the beat noise appears only at $f = 0$ (see Fig. 9). Expanding the electrical bandwidth such that $B \geq 1/2T$, beat noise components appear at $f = k/T$ (k: integer), as shown by Fig. 9. Note that the beat noise, which is generated from the beat between different optical modes in Fig. 5, appear in the both sidebands, whose width is $2B$[20]. Considering that the number of modes in the both sidebands is given by $2BT$ and summing up all of the beat noise components, we obtain[21]

$$\left\langle \Delta\hat{n}(t)^2 \right\rangle = 2BT \left[G(t) n_0 + (G(t)-1) M n_{sp} + 2G(t)(G(t)-1) n_0 n_{sp} + (G(t)-1)^2 M n_{sp}^2 \right] \ . \tag{122}$$

On the other hand, the variance of $\left\langle \hat{N}(t) \right\rangle$ is given by

$$\left\langle \Delta\hat{N}(t)^2 \right\rangle = \frac{\left\langle \Delta\hat{n}(t)^2 \right\rangle}{T^2} \ . \tag{123}$$

Consequently, Eqs. (122) and (123) yield the following final expression of $\left\langle \Delta\hat{N}(t)^2 \right\rangle$:

$$\left\langle \Delta\hat{N}(t)^2 \right\rangle = 2B \left[G(t) N_0 + (G(t)-1) n_{sp} \Delta f + 2G(t)(G(t)-1) N_0 n_{sp} + (G(t)-1)^2 n_{sp}^2 \Delta f \right] \ , \tag{124}$$

[17] We assume that $fc \gg \Delta f$.

[18] In the simulations, where the sampling time interval is T_s, the ASE power is given by S_{ASE}/T_s. The electric field of ASE is regarded as a Gaussian random variable, as discussed in Sec. 6.

[19] Since T is a virtual parameter, it must be eliminated from noise formulae.

[20] Provided that we consider the single sideband spectrum, its spectral density must be doubled.

[21] When the number of optical modes is M, $M - k$ optical beat notes contribute to the k-th electrical beat-noise component. Therefore, strictly speaking, the ASE-ASE beat noise is not a white noise, because its spectral density depends on k. However, when $B \ll \Delta f$, the spectral density of the ASE-ASE beat noise is approximately constant within B.

and the variance of the photocurrent, $\left\langle \Delta\hat{I}(t)^2 \right\rangle$, is then given by

$$\begin{aligned}\left\langle \Delta\hat{I}(t)^2 \right\rangle &= q^2 \left\langle \Delta\hat{N}(t)^2 \right\rangle \\ &= 2qB\left[G(t)I_0 + q(G(t)-1)n_{sp}\Delta f + 2(G(t)-1)I_0 n_{sp} + q(G(t)-1)^2 n_{sp}^2 \Delta f\right] \ ,\end{aligned} \tag{125}$$

where I_0 represents the photocurrent measured when $G = 1$.

Meanwhile, the thermal noise of the receiver circuit is given by [8]

$$\overline{I_{th}^2} = \frac{4k_B T_c B}{R} \ , \tag{126}$$

where k_B is the Boltzmann constant, T_c is the temperature of the circuit, and R is the load resistance of the circuit. The thermal noise is usually the main cause determining the SN ratio of DD receivers without optical pre-amplifiers. However, when the pre-amplifier gain is sufficiently high, the photocurrent noise can overwhelm the thermal noise of the circuit, that is, $\left\langle \Delta\hat{I}(t)^2 \right\rangle \gg \overline{I_{th}^2}$, and we cam improve the SN ratio. As far as $G \gg 1$ and $N_0 \gg n_{sp}\Delta f$, the signal-ASE beat noise (the third term of the left-hand side of Eq. (124)) predominates as the cause of noise. Thus, Eqs. (117) and (124) yield the SN ratio of the amplifier output, given by

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{out} = \frac{N_0}{4Bn_{sp}} \ . \tag{127}$$

On the other hand, the SN ratio of the input signal is given by[22]

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{in} = \frac{N_0}{2B} \ . \tag{128}$$

Although the pre-amplifier improves the SN ratio through the relative reduction in the circuit-noise effect, the SN ratio is still degraded by $2n_{sp}(\geq 2)$, compared with the SN ratio of the input signal. The SN-ratio degradation by optical amplification is called the NF of the amplifier. Hence, the NF of laser amplifiers is $2n_{sp}$, the minimum value of which is 2 (3 dB).

8 Pre-amplified Coherent Receivers

We consider the phase-diversity homodyne receiver [3] to measure the in-phase (I) and quadrature (Q) components of the signal. In the pre-amplified homodyne receiver, the signal is pre-amplified and split into two branches. Each of the split signals is homodyne-detected, and the IQ components of the signal are measured simultaneously. The photocurrent for the in-phase signal electric field with respect to the local electric field, A_ℓ, is given by [3]

$$\hat{I}_1(t) = \frac{qA_\ell}{T}\left(\hat{a}_{s1}(t) + \Delta\hat{a}_{n1}\right) \ , \tag{129}$$

where $\hat{a}_{s1}$ is the in-phase component of the signal electric field, and $\Delta\hat{a}_{n1}$ is the in-phase component of the vacuum electric field. The vacuum electric field is merged into the signal through the 3-dB beam splitter in the receiver. On the other hand, the photocurrent for the quadrature signal electric field is written as

$$\hat{I}_2(t) = \frac{qA_\ell}{T}\left(\hat{a}_{s2}(t) + \Delta\hat{a}_{n2}\right) \ , \tag{130}$$

where $\hat{a}_{s2}$ is the quadrature component of the signal electric field, and $\Delta\hat{a}_{n2}$ is the quadrature component of the vacuum electric field.

[22] This is the shot-noise-limited SN ratio obtained when $G = 1$ and the thermal noise is neglected.

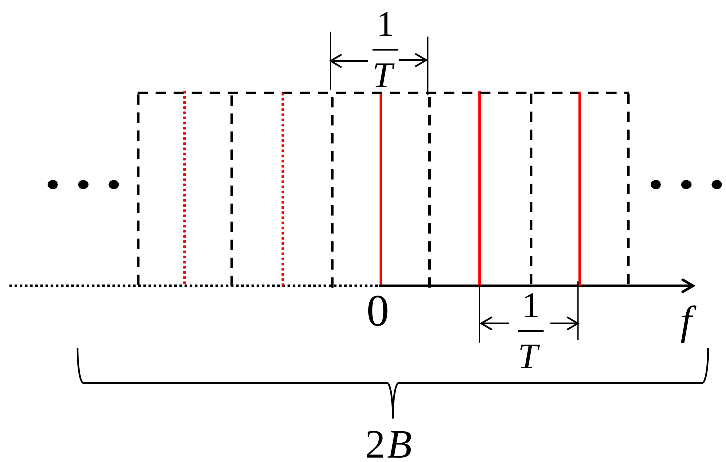

Figure 9: Beat noise spectrum. The red solid lines show beat-noise components appearing in the single sideband ($f \geq 0$). Meanwhile, the dotted lines show the spectra in both sidebands. The number of modes within $2B$ is $2BT$, where B represents the electrical bandwidth.

Let us consider the case that the incoming rate of photons of the signal, N_0, is measured by the phase-diversity coherent receiver[23]. Firstly, we measure the IQ components separately. The averages of $\hat{I}_1(t)$ and $\hat{I}_2(t)$ are given by

$$\left\langle \hat{I}_1(t) \right\rangle = \frac{qA_\ell}{T} \langle \hat{a}_{s1}(t) \rangle \ , \tag{131}$$

$$\left\langle \hat{I}_2(t) \right\rangle = \frac{qA_\ell}{T} \langle \hat{a}_{s2}(t) \rangle \ . \tag{132}$$

Squaring and adding them, we obtain

$$\begin{aligned} \overline{I(t)}^2 &= \left\langle \hat{I}_1(t) \right\rangle^2 + \left\langle \hat{I}_2(t) \right\rangle^2 \\ &= qI_\ell G(t) N_0 \ , \end{aligned} \tag{133}$$

where I_ℓ is the photocurrent, generated by the local light alone, and is given by

$$I_\ell = \frac{qA_\ell^2}{T} \ . \tag{134}$$

Using Eq. (133), we can determine N_0 from the measured photocurrents[24].

Meanwhile, the variance of the photocurrent, $\hat{I}_1$, is derived from Eq. (129) as

$$\left\langle \Delta\hat{I}_1(t)^2 \right\rangle = \frac{qI_\ell}{T} \left(\left\langle \Delta\hat{a}_{s1}(t)^2 \right\rangle + \left\langle \Delta\hat{a}_{n1}^2 \right\rangle \right) \ . \tag{135}$$

Moreover, the number of modes included in the baseband width, B, is $2BT$, as seen in Fig. 9. Then, using Eq. (72), Eq. (135) is transformed into

$$\begin{aligned} \left\langle \Delta\hat{I}_1(t)^2 \right\rangle &= 2qI_\ell B \left(\left\langle \Delta\hat{a}_{s1}(t)^2 \right\rangle + \left\langle \Delta\hat{a}_{n1}^2 \right\rangle \right) \\ &= 2qI_\ell B \left[\frac{1}{2} + \frac{1}{2} \left(G(t) - 1 \right) n_{sp} \right] \ . \end{aligned} \tag{136}$$

[23] The measurement of the in-phase/quadrature components can be analyzed by the similar method.

[24] N_0 is determined independent of the carrier phase. This fact is the origin of the name of "phase diversity."

The variance of $\hat{I}_2$ is the same as Eq. (136). Thus, the total variance is given by

$$\begin{aligned}\overline{\Delta I(t)^2} &= \left\langle \Delta \hat{I}_1(t)^2 \right\rangle + \left\langle \Delta \hat{I}_2(t)^2 \right\rangle \\ &= 2qI_\ell B \left[1 + (G(t)-1)\, n_{sp}\right] \ . \end{aligned} \tag{137}$$

The SN ratio is given from Eqs. (133) and (137) by

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{out1} = \frac{N_0}{2Bn_{sp}} \ , \tag{138}$$

when $G \gg 1$. If we do not use the pre-amplifier and the circuit-noise effect is ignored, the SN ratio is given by

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{out2} = \frac{N_0}{2B} \ , \tag{139}$$

because $G = 1$ in Eqs. (133) and (137). On the other hand, considering that the light in the coherent state includes the vacuum fluctuations with the half-photon energy, the SN ratio of the input signal is written as[25]

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{in} = \frac{N_0}{B} \ . \tag{140}$$

Comparing Eqs. (139) and (140), we find that the SN ratio is degraded by 3 dB owing to 3-dB beam splitting necessary for the simultaneous IQ measurement. However, even if we introduce the optical pre-amplifier into the coherent receiver, the degradation of the SN ratio is maintained at n_{sp}, as seen from Eqs. (138) and (140). In the ideal case that $n_{sp} = 1$, the pre-amplifier noise does not degrade the SN ratio. This is because 3-dB beam splitting after pre-amplification does not reduce the SN ratio, whereas 3-dB beam splitting without pre-amplification reduces the SN ratio by 3 dB [8].

It should be also noted that Eq. (137) does not include Δf. The homodyne detection is the linear detection process with respect to the signal, and the signal can be filtered in the electrical domain without optical filtering as far as $B < \Delta f/2$. On the contrary, DD is the square-law detection process, and the receiver noise includes Δf, as shown by Eq. (124). Optical filtering is indispensable for DD receivers even if the directly detected signal is tightly filtered in the electrical domain.

9 Equivalent Input Noise and NF

In this section, we reconsider the NF of optical amplifiers, using the concept of equivalent input noise. As shown by Fig. 10 (a), the average of the signal electric field, $\overline{a}_{s,in}$, is amplified by $\sqrt{G}$, and the noise electric field, $a_{n,out}$, is generated. Hence, the amplifier output is given by

$$a_{s,out} = \sqrt{G}\overline{a}_{s,in} + a_{n,out} \ . \tag{141}$$

If the equivalent input noise[26] is defined by

$$a_{n,eq} = \frac{a_{n,out}}{\sqrt{G}} \ , \tag{142}$$

we can regard that the signal and equivalent input noise are incident on a virtual noise-free amplifier, as shown by Fig. 10 (b) and then, the output, given by Eq. (141), is generated from the noise-free amplifier. Therefore, the NF of the amplifier, which represents the degradation of the SN ratio by

[25] Note that the spectral density of the vacuum fluctuations is $1/2$, and the bandwidth is $2B$.
[26] This is also called the input-referred noise.

amplification, is given by the energy ratio between the equivalent input noise and actual input noise, written as

$$\mathrm{NF} = \frac{\overline{|a_{n,eq}|^2}}{\overline{|a_{s,in}|^2}} \ . \tag{143}$$

When $G \gg 1$, the equivalent input noise has the energy of n_{sp} photons per mode, while the actual input noise, which is the vacuum fluctuations, has the half-photon energy. Hence, from Eq. (143), we find that the NF is generally given by $2n_{sp}$.

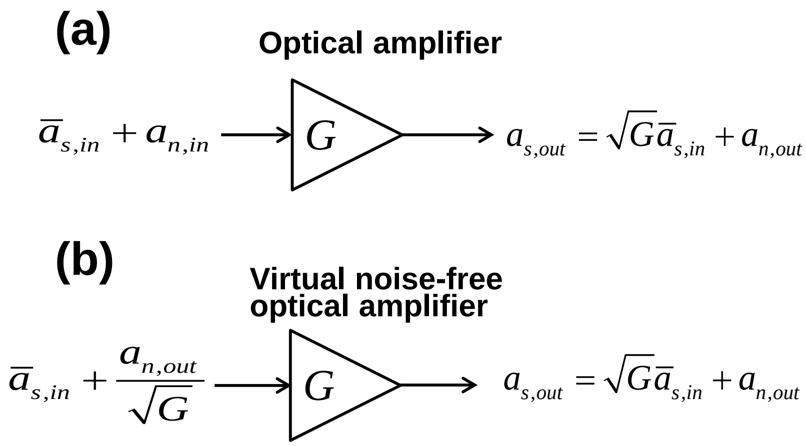

Figure 10: Definition of NF using the equivalent input noise. (a): An actual amplifier. The signal is amplified, and the noise is generated simultaneously. (b): A virtual noise-free amplifier. The signal and equivalent input noise are incident on the virtual amplifier, which is noise-free. Consequently, the NF is defined as the ratio of power between the equivalent input noise and actual input noise.

10 Optical Transmission Using Laser Amplifier Chain

In long-distance optical fiber transmission systems, the laser amplifier chain, shown by Fig. 11, plays an important role. In such systems, each laser amplifier compensates for the span loss and extends the transmission distance; however, we cannot avoid the accumulation of the amplifier noise. In the following, we discuss how the amplifier noise is accumulated in the laser amplifier chain.

We focus on the noise in the real part of the electric field. The noise in the amplifier output, which consists of the vacuum fluctuations and ASE, attenuates by Γ in each span. Meanwhile, the vacuum fluctuations, having the variance of $(1-\Gamma)/4$, are merged from the loss. Then, the variance of the total noise is expressed by

$$\begin{aligned} \left\langle \Delta\hat{a}_1(t)^2 \right\rangle_{span} &= \left[\frac{1}{4} + \frac{1}{2}(G-1)\, n_{sp}\right]\Gamma + \frac{1}{4}(1-\Gamma) \\ &= \frac{1}{4} + \frac{1}{2}(1-\Gamma)\, n_{sp} \ , \end{aligned} \tag{144}$$

where we use $G\Gamma = 1$. Summing up the noise generated by each span, we have the variance of the noise at the output, given by

$$\left\langle \Delta\hat{a}_1(t)^2 \right\rangle_{out} = \frac{1}{4} + \frac{1}{2}(1-\Gamma)\, n_{sp} n \ . \tag{145}$$

Provided that the span length is made shorter and the number of spans resultantly increases, the peak optical power is maintained at a lower level, preventing from the Kerr nonlinearity of fibers [3]. Meanwhile, the the total ASE noise increases in proportion to n. It means that n_{sp} of the single amplifier should be replaced with $n_{sp}n$ in the laser amplifier chain.

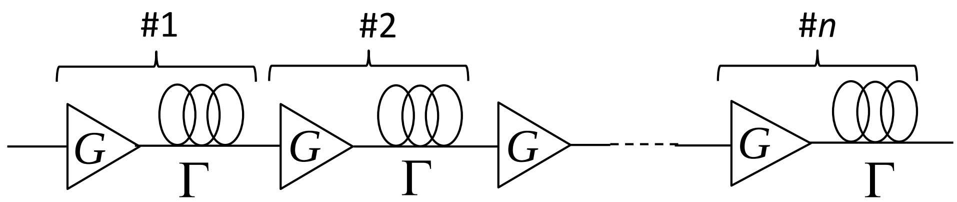

Figure 11: Laser amplifier chain, where the span loss, Γ, is compensated for, using the amplifier gain G in every span.

11 Laser Noise

In this section, we analyze the intensity-noise characteristics of semiconductor lasers including the amplitude squeezing [9][10] by using the classical rate equations described in Sec. 5. This approach enables us to obtain a clear understanding of the amplitude-squeezing mechanism. Moreover, the FM noise and spectral width of semiconductor lasers are also discussed.

11.1 Rate Equations for photons and electrons

The number of electrons, n_e, in the active region and the number of photons, s, inside the cavity obey the following classical rate equations for the single-mode oscillation of semiconductor lasers [11]:

$$\frac{dn_e}{dt} = p - \frac{n_e}{\tau_s} - (E_{cv} - E_{vc})\, s + f_n \ , \tag{146}$$

$$\frac{ds}{dt} = (E_{cv} - E_{vc})\, s + \frac{\beta n_e}{\tau_s} - \frac{s}{\tau_{ph}} + f_s \ . \tag{147}$$

In Eq.(146), p represents the pump rate. When the laser is driven by a current, J, it means the injected number of electrons per second, and is written as

$$p = \frac{J}{q} \ . \tag{148}$$

The parameter, τ_s, is the carrier lifetime, originating from spontaneous carrier recombination processes including spontaneous emission, $E_{cv}s$ is the induced emission rate, and $E_{vc}s$ is the absorption rate. In Eq. (147), the photon lifetime, τ_{ph}, is written as

$$\frac{1}{\tau_{ph}} = \frac{1}{\tau_{p1}} + \frac{1}{\tau_{p2}} \ , \tag{149}$$

where τ_{p1} represents the lifetime determined from the output coupling, and τ_{p2} is the lifetime determined from the internal loss. The parameter, $\beta(\ll 1)$, shows that only a small portion of spontaneous emission (the second term of the right-hand side of Eq. (146)) contributes to the lasing mode. In the following, we ignore this term including β.

In the stationary state, we find from Eqs. (146) and (147) that

$$\overline{p} = \frac{\overline{n}_e}{\tau_s} + \left(\overline{E}_{cv} - \overline{E}_{vc}\right)\overline{s} \ , \tag{150}$$

$$\overline{E}_{cv} - \overline{E}_{vc} = \frac{1}{\tau_{ph}} \ . \tag{151}$$

On the other hand, the noise terms, f_n and f_s, are divided separately, corresponding to each process in the right-hand side of the rate equations, as

$$f_n = f_{n1} + f_{n2} + f_{n3} \ , \tag{152}$$

$$f_s = f_{s1} + f_{s2} + f_{s3} \ . \tag{153}$$

The noise characteristics of lasers are analyzed by Fourier transform of the rate equations. In the analyses, Fourier transforms of $f_k(t)$ is written as $F_k(f)$, and its power spectral density is represented by $S_k(f)$. The relation between the Fourier transform, $F_k(f)$, and the power spectral density, $S_k(f)$ is generally given by

$$S_k(f) = \lim_{T\to\infty} \frac{\langle F_k^*(f)F_k(f)\rangle}{T} \ , \tag{154}$$

where T denotes the measurement time interval.

Using the discussions in 5.2, we can derive theoretical expressions of the spectral densities as follows. The noise term, f_{n1}, stems from the pumping process, and its two-sided spectral density is given by

$$S_{n1}(f) = \begin{cases} \overline{p} & \text{when the pump noise is associated as in optically pumped lasers,} \\ 0 & \text{when the pump noise is suppressed as in constant-current driven lasers.} \end{cases} \tag{155}$$

The spontaneous carrier recombination process (the second term of the right-hand side of Eq. (147)) generates the noise term, f_{n2}, whose power spectral density is

$$S_{n2}(f) = \frac{\overline{n}_e}{\tau_s} \ . \tag{156}$$

Moreover, f_{n3} and f_{s1}, are caused by the induced emission and absorption, and satisfy

$$f_{n3} = -f_{s1} \ . \tag{157}$$

This negative correlation between f_{n3} and f_{s1} originates from the fact that the emission (absorption) of a photon always follows the recombination (generation) of an electron-hole pair. Their power spectral densities are given by

$$S_{n3}(f) = S_{s1}(f) = \left(\overline{E}_{cv} + \overline{E}_{vc}\right)\overline{s} \ . \tag{158}$$

The noise terms, f_{s2} and f_{s3}, are associated with the output coupling and the internal loss, respectively. The power spectral densities of these noises are written as

$$S_{s2}(f) = \frac{\overline{s}}{\tau_{p1}} \ , \tag{159}$$

$$S_{s3}(f) = \frac{\overline{s}}{\tau_{p2}} \ . \tag{160}$$

Small deviations, Δn_e and Δs, from the stationary values, $\overline{n}_e$ and $\overline{s}$, obey

$$\frac{d\Delta n_e}{dt} = -\left(\frac{1}{\tau_s} + A\overline{s}\right)\Delta n_e - \frac{\Delta s}{\tau_{ph}} + f_n \ , \tag{161}$$

$$\frac{d\Delta s}{dt} = A\overline{s}\Delta n_e + f_s \ , \tag{162}$$

where A is the differential gain defined as

$$A = \left. \frac{\partial \left(E_{cv} - E_{vc} \right)}{\partial n_e} \right|_{n_e = \overline{n}_e} . \tag{163}$$

We apply the adiabatic approximation to Eqs. (161) and (162) to demonstrate the noise generation mechanism clearly. Using the following equation obtained from Eq. (161):

$$\Delta n_e = \frac{-\dfrac{\Delta s}{\tau_{ph}} + f_n}{\dfrac{1}{\tau_s} + A\overline{s}} , \tag{164}$$

we find that Δs obeys the differential equation given by

$$\frac{d\Delta s}{dt} = -\frac{A\overline{s}}{\tau_{ph}\left(A\overline{s} + \dfrac{1}{\tau_s} \right)} \Delta s + \frac{A\overline{s}}{A\overline{s} + \dfrac{1}{\tau_s}} f_n + f_s . \tag{165}$$

This approximation holds in the frequency range well below the relaxation resonance.

11.2 Origin of shot noise

When the current J is much larger than the threshold current J_{th}, the spontaneous carrier recombination rate is negligibly smaller than the induced emission rate: $A\overline{s} \gg 1/\tau_s$. In such a high-power limit, Eq. (165) is approximated as

$$\frac{d\Delta s}{dt} = -\frac{\Delta s}{\tau_{ph}} + f_{ms} , \tag{166}$$

where the new noise term, f_{ms}, is written as

$$f_{ms} = f_n + f_s = f_{n1} + f_{s2} + f_{s3} . \tag{167}$$

Note that f_{n3} and f_{s1} are canceled out each other in Eq. (167). Hence, the photon number fluctuation inside the cavity, $\Delta s(t)$, is generated only from the pump fluctuation, f_{n1}, output coupling, f_{s2}, and internal loss, f_{s3}.

When the pump fluctuation exists, the spectral density of $f_{ms}(t)$ is given by

$$S_{ms}(f) = \overline{p} + \frac{\overline{s}}{\tau_{ph}} . \tag{168}$$

Using Eqs. (150) and (151), Eq. (168) is transformed into

$$S_{ms}(f) = \frac{2\overline{s}}{\tau_{ph}} . \tag{169}$$

The Fourier transform of $\Delta s(t)$ is obtained from Eq. (166) as

$$\Delta S(f) = \frac{F_{ms}(f)}{i2\pi f + \dfrac{1}{\tau_{ph}}} . \tag{170}$$

Then, Eqs. (169) and (170) yield the spectral density of $\Delta s(t)$ written as

$$S_{ph}(f) = \frac{2\overline{s}}{\tau_{ph}} \cdot \frac{1}{(2\pi f)^2 + \left(\dfrac{1}{\tau_{ph}} \right)^2} . \tag{171}$$

Integrating $S_{ph}(f)$ on $[-\infty, +\infty]$, we have the variance of the fluctuations of the number of photons written as

$$\overline{\Delta s^2} = \int_{-\infty}^{+\infty} S_{ph}(f)\ df = \overline{s}\ . \tag{172}$$

Equation (172) indicates the shot-noise character that the variance and average are equal.

We next consider the outgoing rate of photons from the cavity, which we can measure outside the cavity. The average outgoing rate of photons, $\overline{r}$, is given by

$$\overline{r} = \frac{\overline{s}}{\tau_{p1}}\ , \tag{173}$$

and its fluctuation can be expressed as

$$\Delta r = \frac{\Delta s}{\tau_{p1}} - f_{s2}\ . \tag{174}$$

Equation (174) is easily understood from the fact that when the outgoing rate of photons from the cavity becomes higher (lower) than the average due to the output coupling noise f_{s2}, the photon emission rate inside the cavity decreases (increases) from the average value. From Eq. (174), the Fourier transform of $\Delta r(t)$ is given by

$$\Delta R(f) = \frac{\Delta S(f)}{\tau_{p1}} - F_{s2}(f)\ . \tag{175}$$

Equations (170) and (175) yield

$$\Delta R(f) = \frac{1}{\tau_{p1}} \left(\frac{F_{n1} + F_{s2} + F_{s3}}{i2\pi f + \dfrac{1}{\tau_{ph}}} \right) - F_{s2}\ . \tag{176}$$

Using Eqs. (155), (159), and (160), (176) results in the following expression of the spectral density of $\Delta r(t)$:

$$S_R(f) = \frac{\overline{s}}{\tau_{p1}}\ , \tag{177}$$

which demonstrates that the outgoing rate of photons has the shot-noise character.

Now, we go on to the case that semiconductor lasers are driven by a constant-current source, where $f_{n1} = 0$. In such a case, the photon-number fluctuation inside the cavity is derived from Eq. (170) as

$$S_{ph}(f) = \frac{\overline{s}}{\tau_{ph}} \cdot \frac{1}{(2\pi f)^2 + \left(\dfrac{1}{\tau_{ph}}\right)^2}\ . \tag{178}$$

Integration of $S_{ph}(f)$ over $[-\infty, +\infty]$ gives us the variance of the fluctuations of the number of photons written as

$$\overline{\Delta s^2} = \int_{-\infty}^{+\infty} S_{ph}(f)\ df = \frac{\overline{s}}{2}\ . \tag{179}$$

Consequently, the amount of noise is halved from the standard shot-noise level given by Eq. (172).

On the other hand, the outgoing rate of photons is given from Eqs. (170) and (175) by

$$\Delta R(f) = \frac{1}{\tau_{p1}} \left(\frac{F_{s2} + F_{s3}}{i2\pi f + \dfrac{1}{\tau_{ph}}} \right) - F_{s2} \ . \tag{180}$$

Equations (159), (160), and (180) yield the spectral density of $\Delta r(t)$ given by

$$S_R(f) = S_{R1}(f) + S_{R2}(f) \ , \tag{181}$$

where

$$S_{R1}(f) = \frac{\overline{s}}{\tau_{p1}} \cdot \frac{(2\pi f \tau_{ph})^2}{(2\pi f \tau_{ph})^2 + 1} \ , \tag{182}$$

$$S_{R2}(f) = \frac{\overline{s}}{\tau_{p1}} \cdot \frac{\tau_{p1}}{\tau_{p1} + \tau_{p2}} \cdot \frac{1}{(2\pi f \tau_{ph})^2 + 1} \ . \tag{183}$$

The black and blue curves in Fig. 12 show S_{R1} and S_{R2}, respectively. In the ideal case where no internal loss exists, that is, $\tau_{p2} \to \infty$, S_{R2} vanishes, and only S_{R1} remains. We find that the spectral density of S_{R1} becomes less than the standard shot noise level, $\overline{s}/\tau_{p1}$, in the frequency range below the cutoff frequency of $1/2\pi\tau_{ph}$, and the amplitude squeezing is achieved. On the contrary, when the internal loss is increased larger than the output coupling loss, that is, $\tau_{p2} \ll \tau_{p1}$, S_R approaches the standard shot noise level. It should be noted that when the pump noise exists, the noise spectral density is the shot-noise level, $\overline{s}/\tau_{p1}$, independent of f.

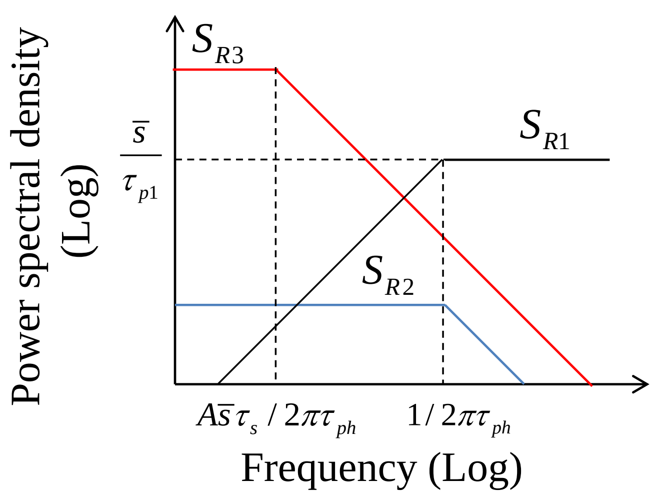

Figure 12: Intensity noise spectra when semiconductor lasers are driven by a constant-current source. S_{R1} is the intrinsic only caused by output coupling. S_{R2} is the noise induced by internal loss. S_{R3} is the noise caused by spontaneous carrier recombination. The cut-off characteristics of theses spectra are approximated by line graphs.

11.3 Intensity noise due to spontaneous carrier recombination

In 11.2, we have derived the noise spectrum in the high-power limit where $A\overline{s} \gg 1/\tau_s$. In the opposite limit where $A\overline{s} \ll 1/\tau_s$, the excess noise is generated as shown in what follows. In this case, Eq. (165)

can be approximated as

$$\frac{d\Delta s}{dt} = -\frac{A\bar{s}\tau_s}{\tau_{ph}}\Delta s + f_s \ . \tag{184}$$

The spectral density of f_s is given from Eqs. (151), (158), (159), and (160) as

$$S_s(f) = \frac{2n_{sp}\bar{s}}{\tau_{ph}} \ , \tag{185}$$

where n_{sp} is the spontaneous emission factor defined by

$$n_{sp} = \frac{\overline{E}_{cv}}{\overline{E}_{cv} - \overline{E}_{vc}} \ . \tag{186}$$

The output rate of photons is given by Eq. (174), but the second term can be omitted by the approximation used in this section. Equations (174), (184), and (185) yield the spectrum of the fluctuation of the outgoing rate of photons written as

$$S_{R3}(f) = \frac{2n_{sp}\tau_{ph}}{A^2\tau_s^2\tau_{p1}\bar{s}} \cdot \frac{1}{1 + \left(2\pi f \dfrac{\tau_{ph}}{A\bar{s}\tau_s}\right)^2} \ , \tag{187}$$

which is the conventional expression for the intensity noise of lasers [12]. The red curve in Fig. 12 shows $S_{R3}(f)$.

In this way, we can classify the semiconductor-laser noise into the three categories, S_{R1}, S_{R2}, and S_{R3}. The noise at any bias current can approximately be expressed as the superposition of the three noises. The noise S_{R3} can be suppressed by operating low-threshold lasers at high bias current levels ($J \gg J_{th}$). Even in such a case, there remain S_{R1} and S_{R2}. To reduce S_{R2}, it is necessary to make the internal loss much smaller than the output coupling loss ($\tau_{p2} \gg \tau_{p1}$). The two conditions are nothing but those for obtaining the 100% quantum efficiency. The noise S_{R1} is the ultimate noise of semiconductor lasers, whose spectral density is less than the standard shot-noise level below the cutoff frequency of the passive cavity. However, it is not easy to observe such a low-noise state below the shot-noise level in actual semiconductor lasers. Only a few experimental results have been reported so far [13].

11.4 FM noise and spectral width

We need rely on the Langevin equation approach to discuss the FM noise characteristics. The Langevin equation given by Eq. (55) is adapted to lasers, as shown by

$$\frac{d\hat{a}\,(t)}{dt} = -i\Delta\omega(n_e)\hat{a}\,(t) + \frac{1}{2}\left[E_{cv}(n_e) - E_{vc}(n_e)\right]\hat{a}\,(t) - \frac{\hat{a}(t)}{2\tau_{ph}} + \hat{f}_\ell\,(t) \ , \tag{188}$$

where $\Delta\omega(n_e)$ shows the angular frequency deviation from the cavity mode. The Langevin noise term, $f_\ell(t)$, is expressed as

$$\hat{f}_\ell(t) = \hat{f}_{cv}(t) + \hat{f}_{vc}(t) + \hat{f}_{ph}(t) \ , \tag{189}$$

where $\hat{f}_{cv}(t)$, $\hat{f}_{vc}(t)$, are $\hat{f}_{ph}(t)$ stem from the induced emission, absorption, and photon lifetime of the passive cavity, respectively. The correlation relations of these noise terms are given by

$$\left\langle \hat{f}^{\dagger}_{cv}(t)\hat{f}_{cv}(t')\right\rangle = \overline{E}_{cv}\delta(t-t') \ , \tag{190}$$

$$\left\langle \hat{f}_{cv}(t)\hat{f}^{\dagger}_{cv}(t')\right\rangle = 0 \ , \tag{191}$$

$$\left\langle \hat{f}^{\dagger}_{vc}(t)\hat{f}_{vc}(t')\right\rangle = 0 \ , \tag{192}$$

$$\left\langle \hat{f}_{vc}(t)\hat{f}^{\dagger}_{vc}(t')\right\rangle = \overline{E}_{vc}\delta(t-t') \ , \tag{193}$$

$$\left\langle \hat{f}^{\dagger}_{\tau_{ph}}(t)\hat{f}_{ph}(t')\right\rangle = 0 \ , \tag{194}$$

$$\left\langle \hat{f}_{ph}(t)\hat{f}^{\dagger}_{ph}(t')\right\rangle = \frac{1}{\tau_{ph}}\delta(t-t') \ . \tag{195}$$

Using Eqs. (190)-(195), we obtain the correlation relations of the real part operator, $\hat{f}_{\ell,1}(t)$, and imaginary part operator, $\hat{f}_{\ell,2}(t)$, as

$$\left\langle \hat{f}_{\ell,1}(t)\hat{f}_{\ell,1}(t')\right\rangle = \left\langle \hat{f}_{\ell,2}(t)\hat{f}_{\ell,2}(t')\right\rangle = \frac{1}{4}\left(\overline{E}_{cv}+\overline{E}_{vc}+\frac{1}{\tau_{ph}}\right)\delta(t-t') = \frac{n_{sp}}{2\tau_{ph}}\delta(t-t') \ , \tag{196}$$

$$\frac{1}{2}\left\langle \hat{f}_{\ell,1}(t)\hat{f}_{\ell,2}(t') + \hat{f}_{\ell,2}(t)\hat{f}_{\ell,1}(t')\right\rangle = 0 \ . \tag{197}$$

These equations demonstrate that the real and imaginary parts of the Langevin force have the same energy, and heve no correlation between them[27].

Neglecting the intensity noise, we assume that $\hat{a}(t)$ is expressed as

$$\hat{a}(t) = \sqrt{\overline{s}}\exp\left[-i\Delta\hat{\phi}(t)\right] \ , \tag{198}$$

where $\Delta\hat{\phi}(t)$ is the phase fluctuations[28]. Equation (188) shows that $\Delta\hat{\phi}(t)$ satisfies

$$\frac{d\Delta\hat{\phi}(t)}{dt} = -\frac{\hat{f}_{\ell,2}(t)+\alpha\hat{f}_{\ell,1}(t)}{\sqrt{\overline{s}}} \ . \tag{199}$$

The parameter α is called the linewidth enhancement factor, defined by [14]

$$\alpha = \frac{2\dfrac{\partial\Delta\omega}{\partial n_e}}{A} \ . \tag{200}$$

Since the fluctuation of the instantaneous frequency is expressed as

$$\hat{f}_{instant}(t) = \frac{1}{2\pi}\frac{d\Delta\hat{\phi}}{dt} \ , \tag{201}$$

its spectral density is given from Eqs. (196), (197), and (199) by[29]

$$S_{FM}(f) = \frac{n_{sp}\left(1+\alpha^2\right)}{8\pi^2\overline{s}\tau_{ph}} \ , \tag{202}$$

[27] It can be shown that the energy of noise inside the passive resonator bandwidth is given by $hf_c n_{sp}$, which is the origin of the laser noise.

[28] See Eq. (36) on the definition of the phase operator.

[29] The operators can be regarded as c-numbers. The Langevin force can be regarded as a classical Gaussian noise.

which is called the FM noise spectrum. When the FM noise spectrum is white, the field spectrum has the Lorentzian shape, whose full width at the half maximum (FWHM), δf, is $2\pi S_{FM}$ [15]. Therefore, from Eq. (202), we have

$$\delta f = \frac{n_{sp}(1+\alpha^2)}{4\pi\tau_{ph}\bar{s}} \ . \tag{203}$$

The output power, P_o, including the loss inside the cavity, is expressed as

$$P_o = \frac{hf_c\bar{s}}{\tau_{ph}} \ . \tag{204}$$

Moreover, the full spectral width of the passive resonator, $\Delta\nu$, is given by

$$\Delta\nu = \frac{1}{2\pi\tau_{ph}} \ . \tag{205}$$

Then, Eq. (203) is transformed into

$$\delta f = \frac{\pi hf_c n_{sp}(1+\alpha^2)\Delta\nu^2}{P_o} \ . \tag{206}$$

The linewidth when $n_{sp} = 1$ and $\alpha = 0$ in Eq. (206) was derived by Schawlow and Townes [16]. Therefore, this expression is commonly called the Schawlow-Townes linewidth.

12 Conclusions

This study reviewed the master equation, the quantum-mechanical Langevin equation, and the rate equation that can analyze the noise characteristics of laser amplifiers. Bridging physics and engineering, we especially took care to apply the theory to the system applications. Consequently, the derived noise formulae were finally expressed in terms of system parameters, such as a photocurent, optical bandwidth, and receiver bandwidth. Using the derived noise formulae, we discussed the sensitivity of the pre-amplified DD receiver and coherent receiver. Moreover, the NF of laser amplifiers was examined by using the concept of the equivalent input noise. The accumulation of noise along the laser-amplifier chain was also analyzed. Finally, based on the rate-equation approach, we clearly demonstrate the intensity noise and FM noise characteristics of semiconductor lasers.

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