Sufficient Synchronization Conditions for Resistively and Memristively Coupled Oscillators of FitzHugh-Nagumo-type

Robin Lautenbacher[1†], Bakr Al Beattie[2†], Karlheinz Ochs[2] and Ralf Köhl[1,3]

[1]Christian-Albrechts-University Mathematics Seminar Kiel, Germany Email: {lautenbacher, koehl}@math.uni-kiel.de.

[2]Ruhr-University Bochum Chair of Digital Communication Systems Bochum, Germany Email: {bakr.albeattie, karlheinz.ochs}@rub.de.

[3]Christian-Albrechts-University Kiel Nano, Surface and Interface Science Kiel, Germany.

[†]These authors contributed equally to this work.

Abstract

We study the synchronization behavior of a class of identical FitzHugh-Nagumo-type oscillators under adaptive coupling. We describe the oscillators by a circuit model and we provide a sufficient synchronization condition that relies on the shape of the nonlinear conductance's (i, u)-curve and the connectivity of the adaptive coupling network. The coupling network is allowed to be time-variant, state-dependent and locally adaptive, where we treat memristive coupling elements as a special case. We provide a physical interpretation of synchronization in terms of power dissipation and investigate the sharpness of our condition.

Keywords: Synchronization, oscillators, memristors, nonlinear circuits, network dynamics.

1 Introduction

Networks of coupled oscillators, linear and nonlinear, receive large amounts of interest from researchers across various disciplines, due to their relevance to for instance neural networks [1] but also technical applications such as power systems [2]. In the context of neuromorphic computing, synchronization plays a key role in designing novel and power-efficient technologies. Understanding the cause behind the emergence of synchronous states in oscillator networks has, for example, aided in designing these systems in a way that lets them naturally solve optimization problems [3–6]. Synchronization of oscillatory neural networks can also be used to solve tasks like image recognition [7] and gait pattern classification [8]. Hardware implementations of neural networks are of great interest [9], as they among other things are more energy efficient than their digital counterpart. The combination of neural networks with memristors [10, 11] exhibits remarkable properties such as fault tolerance [9] and event-triggered synchronization [12]. The design of such networks is an active area of research [13–15]. It is hence desirable to understand synchronization on a level relatively close to the hardware, for instance in terms of electrical parameters [16]. For that reason, well-interpretable synchronization conditions play a major role both in system design as well as robustness analysis. Numerous synchronization conditions, both necessary and sufficient, exist from a mathematical point of view. Necessary conditions have been approached for systems of linearly coupled oscillators via the Master stability function method [17], while other approaches leading to sufficient conditions are for instance based on contraction theory [18–22], QUAD-conditions [23–25] or semi-passivity [26–28]. The approaches based on semi-passivity usually construct suitable Lyapunov functions, where a standard choice consists of quadratic Lyapunov functions but also non-smooth candidates have been considered [29–31]. Most of these conditions do not allow an immediate physical interpretation and are sometimes hard to apply to actual physical systems.

The FitzHugh-Nagumo oscillator (FHNO) is a relevant oscillator for neuromorphic applications because it is technically realizable [32] and shows biologically plausible behavior [33]. For instance, phenomena like spike-timing dependent axon growth can be realized in neuromorphic circuits with FHNOs [34]. Its key ingredient is the nonlinear conductance, whose (i, u)-curve differs from that of the standard FHN-model in practical applications. Our main goal in this paper is to derive a sufficient synchronization criterion for identical diffusively coupled FHNOs that is phrased in terms of conditions on the coupling network and the (i, u)-curve. We take the more general perspective and study an electrical circuit with a general nonlinear conductance so that our condition also applies to other models such as the van-der-Pol oscillator [35, 36]. We consider diffusive coupling with time-varying, state-dependent and locally adaptive coupling strength, where our main application consists of FHNOs coupled by ideal memristors.

Similar to [31] we use an approach based on semi-passivity but we choose a rather standard Lyapunov function which is a quadratic function of the state variables. Apart from the generality of how the coupling strength is allowed to evolve over time, the novelty in our approach lies in the way we bound its time derivative, which is reminiscent of the strategy in [25] but without the need for the individual oscillators' vector fields to be Lipschitz or to satisfy a QUAD-condition. We exploit that the nonlinear conductance is semi-passive in the following sense: The dissipated power at the conductance is always positive for large enough magnitudes of applied voltage and negative differential conductance occurs only for voltages with magnitude smaller than some possibly large threshold. We observed that synchronization can be derived without much problems if the (i, u)-curve is strictly monotonically increasing and the coupling network is connected. Furthermore, we observed that (i, u)-curves of semi-passive conductances can be made strictly monotonic by adding a linear term. Our sufficient criterion essentially states that for all times the algebraic connectivity of the coupling network needs to be larger than the slope of this linear term for the oscillators to synchronize. The Lyapunov-function used includes the power dissipated by the coupling network as a term, so that this power tends to zero if our sufficient condition is met. Specialized to the standard unitless FHN-model and static diffusive coupling, it turns out that our criterion is sharper than that of [31] based on semi-passivity and coincides with [20, cor. 4.1] and [29], whose derivation is based on contraction theory and Lyapunov's method, respectively. The main result of [20] aims at a different setting than ours in that it applies to clustered synchronization of a variety of heterogeneous oscillators subject to time-varying but not adaptive or state-dependent diffusive coupling. The result of [29] applies to static linear coupling but allows for a large class of couplings, not just diffusive ones. We discuss in section 6.2 the reason why our bound coincides with that of [20] and [29].

An advantage of our criterion is that it spells out in terms of the ideal circuit's parameters, which makes it easily applicable to electronical models of the FHNO such as the ones presented in [32]. Its application only requires knowledge of the nonlinear conductance's (i, u)-curve and the coupling network.

In summary, we shed some light on a physically-interpretable sufficient synchronization condition of identical diffusively coupled FHNOs that is more specialized w.r.t. the oscillators but as sharp as the condition in [20] and has three distinct advantages:

1. It gives practitioners an easy way of ensuring the occurrence of synchronization in dependence on the system's nonlinearity.
2. It has a physical interpretation that is also embedded into the associated mathematical analysis, namely the minimization of dissipated power.
3. It also applies to locally adaptive, state-dependent couplings including memristors or nonlinear couplings

The paper is structured as follows. In section 2 we introduce the framework of dissipative and semi-passive systems. In section 3 we recapitulate the FitzHugh-Nagumo oscillator and describe a generalization of it, discuss the electrical coupling, and derive a compact unitless description. In Sec. 4, we derive an explicit sufficient condition for synchronization. As an application, we spell out this condition for FHNOs coupled by ideal memristors and static linear conductances in section 5. We demonstrate its correctness on an example in Sec. 6, where we also discuss how our result for static diffusive coupling is related to the existing literature. Finally, Sec. 7 summarizes the main contributions of this work and gives an outlook on further research in this area.

Notation: Throughout the text, vector and matrix objects are typeset with bold symbols. For instance, we denote by $\mathbf{0}$ the zero-vector of a given vector space. Given the amount of necessary notation, we included a glossary at the end.

2 Interconnected Semi-Passive Systems

Consider the input-affine dynamical system

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{B}(\boldsymbol{z})\boldsymbol{x}\,, \quad \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{z})\,, \quad \boldsymbol{z}(t_0) = \boldsymbol{z}_0\,, \qquad (1)$$

with (Lipschitz) continuous functions $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{B} : \mathbb{R}^n \to \mathbb{R}^{n\times k}$, $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^k$. We refer to the objects as the state vector $\boldsymbol{z} \in \mathbb{R}^n$, input vector $\boldsymbol{x} \in \mathbb{R}^k$, and output vector $\boldsymbol{y} \in \mathbb{R}^k$, where we have dropped the time argument for the sake of brevity. The input vector $\boldsymbol{x}$ is assumed to be a continuous and bounded function of time.

Definition 1. (C^r-dissipativity, cp. [27, def. 5]) The system (1) is called **C^r-dissipative** (in the sense of Willems [37]), with so-called **supply rate** $w : \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}$, if it is r times continuously differentiable, denoted as $w \in C^r\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}\right)$, and there exists a so-called **storage function** $S \in C^r\left(\mathbb{R}^n, \mathbb{R}_{\geq 0}\right)$ such that,

$$S\left(\boldsymbol{z}(t)\right) - S\left(\boldsymbol{z}_0\right) \leq \int_{t_0}^{t} w\left(\boldsymbol{z}(\tau), \boldsymbol{x}(\tau)\right) \mathrm{d}\tau\,,$$

for all[1] $\boldsymbol{x} \in C^0\left(\mathbb{R}, \mathbb{R}^k\right) \cap L^\infty\left(\mathbb{R}, \mathbb{R}^k\right)$, $\boldsymbol{z}_0 \in \mathbb{R}^n$ and $t_0 \leq t < t_\mathrm{e}$, where t_e is the upper time limit for a solution $\boldsymbol{z}$ of (1) to exist given the input $\boldsymbol{x}$ and initial conditions $\boldsymbol{z}_0$.

Typically, the storage function is given by the system's total energy whose change over time is bounded by the power supplied to the system. Passive and semi-passive systems are C^r-dissipative systems with respect to a specific supply rate.

Definition 2. (Passivity, cp. [27, def. 8]) The system (1) is called **passive**, if it is C^r-dissipative with supply rate $w(\boldsymbol{z}, \boldsymbol{x}) = \boldsymbol{x}^\mathrm{T}\boldsymbol{y} = \boldsymbol{x}^\mathrm{T}\boldsymbol{h}(\boldsymbol{z})$ and its storage function S satisfies $S(\boldsymbol{0}) = 0$.

Definition 3. (Semi-passivity, cp. [27, def. 9]) The system (1) is called **semi-passive** if it is C^r-dissipative with supply rate

$$w(\boldsymbol{z}, \boldsymbol{x}) = \boldsymbol{x}^\mathrm{T}\boldsymbol{y} - H(\boldsymbol{z})\,, \qquad (2)$$

for some function $H : \mathbb{R}^n \mapsto \mathbb{R}$ that satisfies:
There exists $\rho_0 > 0$ and a function $\rho : \mathbb{R} \setminus (-\rho_0, \rho_0) \mapsto \mathbb{R}_{\geq 0}$ such that for all $\boldsymbol{z} \in \mathbb{R}^n$ with $\|\boldsymbol{z}\| \geq \rho_0$ one has $H(\boldsymbol{z}) \geq \rho\left(\|\boldsymbol{z}\|\right)$. If the function ρ can be chosen to be positive then the system is called **strictly semi-passive**.

Note that the crucial part in the definition of (strict) semi-passivity is that H is nonnegative (positive) outside the ρ_0-ball around 0 but that it is allowed to be negative inside. In the sense of the above definition, a physical system is passive if its change in energy w.r.t. time is less than or equal to the power injected, as the scalar product of input $\boldsymbol{x}$ and output $\boldsymbol{y}$ usually have the unit of power. In principle, this applies to all physical systems, but in input-output-modeling it is often convenient to *not* count the power supply of active components as input. A semi-passive system is then roughly a system whose active components have a finite power supply, so that they can inject power into the system for a limited range of operating points (above: inside the ρ_0-ball for some $\rho_0 > 0$) but dissipative behavior dominates elsewhere.

Semi-passivity applies to open systems that can interact with the environment. We obtain an isolated system by considering N systems of type (1) interconnected by a generalized diffusive[2] coupling:

$$\dot{\boldsymbol{z}}_\mu = \boldsymbol{f}(\boldsymbol{z}_\mu) + \boldsymbol{B}(\boldsymbol{z}_\mu)\boldsymbol{x}_\mu\,, \qquad (3a)$$

$$\boldsymbol{y}_\mu = \boldsymbol{h}(\boldsymbol{z}_\mu)\,, \qquad (3b)$$

$$\boldsymbol{x}_\mu = -\sum_{\nu=1}^{N} a_{\mu\nu}(t, \boldsymbol{z}, c_{\mu\nu})\left[\boldsymbol{y}_\mu - \boldsymbol{y}_\nu\right]\,, \qquad (3c)$$

$$\dot{c}_{\mu\nu} = k_{\mu\nu}\left(\boldsymbol{y}_\mu, \boldsymbol{y}_\nu, t, c_{\mu\nu}\right)\,, \qquad (3d)$$

with state-variables $\boldsymbol{z}_\mu$, inputs $\boldsymbol{x}_\mu$ and outputs $\boldsymbol{y}_\mu$ for $\mu = 1, \ldots, N$, where $\boldsymbol{z} = \left[\boldsymbol{z}_1^T, \ldots, \boldsymbol{z}_N^\mathrm{T}\right]^\mathrm{T}$ denotes the stacked state vectors. The coupling weights $a_{\mu\nu} = a_{\nu\mu} \geq 0$ are nonnegative and symmetric for all arguments. The dependence of the coupling $a_{\mu\nu}$ on the state vector's of all subsystems is a very general case and for most applications dependence on $\boldsymbol{z}_\mu, \boldsymbol{z}_\nu$ suffices as most coupling mechanism are local in nature. Since our synchronization condition's proof also works for the more general case, we decided to work in this setting. We assume that the outputs and inputs are of the same dimension k so that the difference $\boldsymbol{y}_\mu - \boldsymbol{y}_\nu$ makes sense, but at this point we do not assume this for the dimension of the state spaces, which we denote by $n_1, \ldots, n_N$. The coupling weights are allowed to be functions of time t as well as the systems' state variables and can furthermore be locally adaptive in the sense that $a_{\mu\nu}$ depends on a state variable $c_{\mu\nu}$ that evolves over time. We

[1] The set of 0 times continuously differentiable functions C^0 denotes just the set of continuous functions, whereas L^∞ denotes the set of bounded functions.

[2] We call this coupling *generalized diffusive* because the input is proportional to the difference of the outputs, although the coupling strength is adaptive and allowed to depend on the subsystems' state.

call this a locally adaptive coupling because the evolution law of $a_{\mu\nu}$ depends only on time, $c_{\mu\nu}$ and the outputs $\boldsymbol{y}_\mu$ and $\boldsymbol{y}_\nu$.

We describe the coupled subsystems in terms of a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$ nodes $\mathcal{V}$, one for each subsystem, and $N_\mathcal{E} = |\mathcal{E}|$ edges of respective weights $a_{\mu\nu}$ depending on the variables t, $\boldsymbol{z}$ and $c_{\mu\nu}$. For each unoriented edge $\{\mu, \nu\}$ select an orientation, denoted by (μ, ν) if the edge is oriented from μ to ν or (ν, μ) otherwise. Let $\boldsymbol{N} \in \mathbb{Z}^{N \times N_\mathcal{E}}$ be the incidence matrix of the resulting directed graph $\mathcal{G}$ with the elements

$$n_{\mu e} = \begin{cases} +1 & \text{if } e = (\mu, \nu) \text{ for some } \nu \in \mathcal{V} \\ -1 & \text{if } e = (\nu, \mu) \text{ for some } \nu \in \mathcal{V} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We collect the edge weights in the diagonal matrix $\boldsymbol{D} \in \mathbb{R}^{N_\mathcal{E} \times N_\mathcal{E}}$, i.e., for the edge $e = \{\mu, \nu\}$ one has $D_{ee} = a_{\mu\nu}$. We define the Laplacian matrix as

$$\boldsymbol{\Gamma} = \boldsymbol{N}\boldsymbol{D}\boldsymbol{N}^\mathrm{T} . \quad (5)$$

It is a standard fact of algebraic graph theory [38] that $\boldsymbol{\Gamma}$ can alternatively be defined as

$$\Gamma_{\mu\mu} = \sum_{\nu \neq \mu} a_{\mu\nu} \quad \text{and} \quad \Gamma_{\mu\nu} = -a_{\mu\nu} , \quad \text{for} \quad \mu \neq \nu , \quad (6)$$

which coincides with (5). Throughout this paper we sort the eigenvalues of $\boldsymbol{\Gamma}$ in ascending order, i.e.,

$$0 = \lambda_1\{\boldsymbol{\Gamma}\} \leq \lambda_2\{\boldsymbol{\Gamma}\} \leq \cdots \leq \lambda_n\{\boldsymbol{\Gamma}\}, \quad (7)$$

where the first inequality is strict if and only if the coupling graph is connected. Note that the eigenvalues of $\boldsymbol{\Gamma}$ are always real because $\boldsymbol{\Gamma}^\mathrm{T} = \boldsymbol{\Gamma}$ and nonnegative because $\boldsymbol{\Gamma}$ is diagonally dominant by (6). For later use, we define the two (stacked) input and output vectors

$$\boldsymbol{x} = \left[\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_N^\mathrm{T}\right]^\mathrm{T} \quad \text{and} \quad \boldsymbol{y} \quad = \left[\boldsymbol{y}_1^T, \ldots, \boldsymbol{y}_N^\mathrm{T}\right]^\mathrm{T} ,$$

respectively, which are related according to (3c) by

$$\boldsymbol{x} = -\boldsymbol{L}\boldsymbol{y} \quad \text{with} \quad \boldsymbol{L} = \boldsymbol{\Gamma} \otimes \mathbf{1}_k , \quad (8)$$

where $\mathbf{1}_k$ denotes the unit matrix of dimension k. Note that $\boldsymbol{L}$ is positive semi-definite because $\boldsymbol{\Gamma}$ is. The μ-th input $\boldsymbol{x}_\mu$ is then obtained as

$$\boldsymbol{x}_\mu = -\boldsymbol{e}_\mu^\mathrm{T}\boldsymbol{L}\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{e}_\mu = e_\mu \otimes \mathbb{1}_k ,$$

with e_μ being the μ-th unit vector in $\mathbb{R}^N$ and $\mathbb{1}_k \in \mathbb{R}^k$ the vector of ones.

By the first part of [27, lem. 1], one has that the interconnection of semi-passive systems by a diffusive coupling mechanism without additional inputs results in a system whose trajectories are bounded. Unfortunately, [27, lem. 1] only applies to systems of type (3) for static coupling. Proposition 4 below is a generalization of [27, lem. 1] that also applies to more general diffusive couplings as in (3). We provide a proof in the appendix and we will use that the solutions are bounded in the proof of theorem 7, even though the bound itself is not explicitly required.

Proposition 4. *Assume:*

1. *The systems* (3a) *are semi-passive for $\mu = 1, \ldots, N$ with radially unbounded*[3] *storage functions and such that the function ρ_μ bounding H_μ from below (see the dissipation inequality in def. 3) is a strictly monotonically increasing and unbounded function.*
2. *The coupling weights $a_{\mu\nu}(t, \boldsymbol{z}, c_{\mu\nu})$ are nonnegative for all $(t, \boldsymbol{z}, \boldsymbol{c})$.*
3. *The vector fields $k_{\mu\nu}$ are such that solutions to* (3d) *exist for all times regardless of $\boldsymbol{y}$.*

Then solutions to (3) *exist for all times $t \geq t_0$ and are such that $\boldsymbol{z}(t)$ is bounded.*

Assume that the N subsystems in (3) are identical and therefore of the same dimension $n_\mu = n$ for all $\mu = 1, \ldots, N$. If the initial values of the subsystems are also identical, then the solution to the coupled system (3) is that the state vectors $\boldsymbol{z}_\mu$ of the subsystems are equal to each other at all times, i.e., $\boldsymbol{z}_\mu(t) = \boldsymbol{z}_\nu(t)$ and such that $\boldsymbol{z}_\mu(t)$ is a solution to the subsystem (1) with zero-input and an appropriate initial value. We call such solutions **synchronous**.

Definition 5. (Synchronization manifold) The (partial) **synchronization manifold** to the system (3) with identical subsystems of dimension n is defined as

$$\mathcal{S} := \left\{\boldsymbol{z} \in \mathbb{R}^{nN} \mid \boldsymbol{z}_\mu = \boldsymbol{z}_\nu, \quad \forall\, \mu, \nu = 1, \ldots, N\right\} .$$

Synchronous solutions are automatically contained in the synchronization manifold. Since we require the state vectors of all oscillators to be identical, this definition of synchrony does not include clustered synchronization or phase-locked solutions with phase differences other than 0. We call this a *partial* synchronization manifold, because

[3] We call a function $S : \mathbb{R}^n \mapsto \mathbb{R}$ radially unbounded if $S(\boldsymbol{z}) \to \infty$ whenever $\|\boldsymbol{z}\| \to \infty$.

only the state variables $\boldsymbol{z}$ synchronize whereas the couplings' state variables $\boldsymbol{c}$ may not.

3 Diffusively Coupled Oscillators of FitzHugh-Nagumo type

3.1 FitzHugh-Nagumo Oscillator

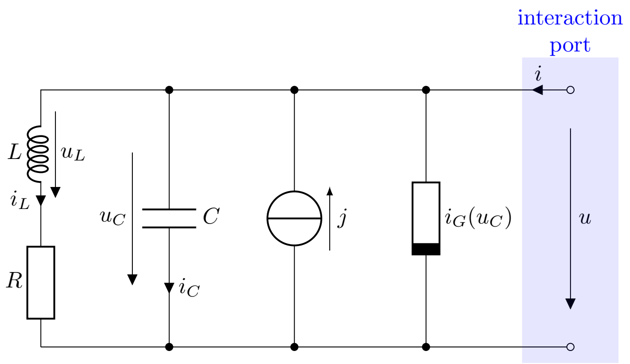

Fig. 1: Equivalent circuit of a FitzHugh-Nagumo oscillator.

The FitzHugh-Nagumo oscillator (FHNO) is a technically realizable [32] and biologically plausible neuronal oscillator [33]. In this section, we briefly recapitulate the electrical model depicted in Fig. 1 which will be used to describe the FHNO throughout this paper. We will show that this model satisfies the definition of a strictly semi-passive system as we need this property to apply prop. 4 later. The differential equations associated with the circuit can be deduced from Kirchhoff's laws and the constitutive relations of the circuit elements:

$$C\dot{u}_C = i_C = j - i_G(u_C) - i_L + i\,, \quad u_C(t_0) = u_{C,0}\,, \tag{9a}$$
$$L\dot{i}_L = u_L = u_C - Ri_L\,, \qquad i_L(t_0) = i_{L,0}\,. \tag{9b}$$

Here, u_C and i_L are state-space quantities corresponding to a capacitor voltage and an inductor current, respectively, while $u_{C,0}$ and $i_{L,0}$ denote their initial values at the starting time t_0, respectively. The current i denotes an external excitation current, while the current j represents a (bias) supply current. The electrical parameters C, L, R denote a capacitance, an inductance, and a resistance, respectively. Lastly, $i_G : \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear conductance function, which has been realized by a tunnel diode in the past [33, 35]. In the following, we work with the cubic nonlinearity

$$i_G(u_C) = G_0\left[\frac{u_C^3}{3U_0^2} - u_C\right]\,, \tag{10}$$

where the conductance G_0 and voltage U_0 are normalization constants. Even though we will stick to the above nonlinear conductance function in our examples, our results hold for more general functions. We will refer to this more general case as an oscillator of FitzHugh-Nagumo-type.

The FHNO can be written as a system of type (1), i.e., an input affine system with one-dimensional input given by the current i and one-dimensional output given by the voltage u_C. We introduce the following quantities in order to recast (9) in terms of unitless variables:

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad Z_0 = \sqrt{\frac{L}{C}}, \quad I_0 = G_0U_0, \tag{11a}$$
$$z_1 = \frac{u_C}{Z_0I_0}, \quad z_2 = \frac{i_L}{I_0}, \quad \tau = \omega_0 t, \tag{11b}$$
$$\iota = \frac{j}{I_0}, \quad \beta = \frac{R}{Z_0}, \quad f_G(z_1) = \frac{1}{I_0}i_G\left(Z_0I_0z_1\right), \tag{11c}$$

where G_0, U_0 are the parameters of (10) which transforms to

$$f_G(z_1) = \alpha\left[\frac{\alpha^2}{3}z_1^3 - z_1\right]\,, \quad \alpha = G_0Z_0\,. \tag{11d}$$

In terms of the unitless parameters of (11) and in dependence on τ, (9) can be written as

$$\boldsymbol{z}' = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{B}(\boldsymbol{z})\boldsymbol{x}\,, \qquad \boldsymbol{z}(\tau_0) = \boldsymbol{z}_0, \text{ with} \tag{12a}$$
$$\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad \boldsymbol{x} = [i/I_0], \qquad \boldsymbol{y} = [z_1]\,, \text{ and} \tag{12b}$$
$$\boldsymbol{f}(\boldsymbol{z}) = \begin{bmatrix} -f_G(z_1) - z_2 + \iota \\ z_1 - \beta z_2 \end{bmatrix}, \qquad \boldsymbol{B}(\boldsymbol{z}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{12c}$$

where we denote $\boldsymbol{z}' = \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{z}$.

Lemma 6. *The system (12) is strictly semi-passive and admits a radially unbounded storage function for all functions $f_G : \mathbb{R} \mapsto \mathbb{R}$ satisfying*

$$\lim_{u\to\pm\infty} f_G\left(z_1\right) = \pm\infty\,,$$

and all $\iota \in \mathbb{R}$. Here, the function ρ from definition 3 can be chosen as

$$\rho(\|\boldsymbol{z}\|) = c_0\|\boldsymbol{z}\| \quad \text{for} \quad c_0 > 0\,.$$

Proof Take the system's energy as storage function:

$$S(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}\,. \tag{13}$$

Now, compute its derivative w.r.t. τ:

$$S'(\boldsymbol{z}) = \left[\nabla_{\boldsymbol{z}}S(\boldsymbol{z})\right]^{\mathrm{T}}\boldsymbol{z}'$$

$$= -\beta z_2^2 - z_1 \left[f_G(z_1) - \iota \right] + z_1 x \,,$$

where $\nabla_{\boldsymbol{z}}$ denotes the gradient of a function w.r.t. the vector $\boldsymbol{z}$. Since $\boldsymbol{y} = [z_1]$, we see that we have obtained an equation similar to the supply rate of a semi-passive system (2). Thus, define

$$H(\boldsymbol{z}) = \beta z_2^2 + z_1 \left[f_G(z_1) - \iota \right]$$

and observe

$$\lim_{\|\boldsymbol{z}\|\to\infty} H(\boldsymbol{z}) = \infty \,, \text{ since} \tag{14a}$$

$$\lim_{z_1\to\pm\infty} z_1 \left[f_G(z_1) - \iota \right] = \infty \,. \tag{14b}$$

We conclude that H is radially unbounded and positive outside some ball around the origin and hence, the FHNO (12) is strictly semi-passive as there must exist a function ρ as in def. 3. In more detail, (14b) implies that there exist $r_0, c_0 > 0$ such that

$$z_1 \left[f_G(z_1) - \iota \right] \geq c_0 |z_1| \,, \quad \forall |z_1| \geq r_0 \,.$$

A similar estimate holds for the residual term, βz_2^2. Hence, there exist $r_1, c_1 > 0$ such that

$$H(\boldsymbol{z}) \geq c_1 \left[|z_1| + |z_2| \right] \,, \quad \text{for all} \quad |z_1| + |z_2| \geq r_1 \,.$$

Since all norms on $\mathbb{R}^n$ are equivalent, there exist $c_2, r_2 > 0$ such that

$$H(\boldsymbol{z}) \geq c_2 \|\boldsymbol{z}\| \,, \quad \forall \, \|\boldsymbol{z}\| \geq r_2 \,.$$

□

The physical meaning behind lemma 6 is that the system (9) is semi-passive as long as the nonlinear conductance's behavior is eventually passive in the following sense: For voltages with magnitude larger than some finite threshold the conductance always dissipates power; power injection can only occur for voltage magnitudes below that threshold. The characterization of the FHNO as a semi-passive system provided by lemma 6 allows us to apply prop. 4 to networks of such oscillators coupled by a generalized diffusive coupling mechanism as in (3).

3.2 Coupled FHN-Oscillators

Now that we have characterized the FHNOs as strictly semi-passive systems, we discuss the generalized diffusive coupling network used to connect the oscillators, where we use the graph-theoretical language developed in section 2. Every pair of adjacent oscillators is coupled by an adaptive coupling as depicted in Fig. 2. Hence, to every undirected edge $\{\mu, \nu\} \in \mathcal{E}$ there is a (positive) conductance $W_{\mu\nu}$ which we allow to depend on time t, the voltage $v_{\mu\nu}$ across the edge $\{\mu, \nu\}$ and an edge variable $c_{\mu\nu}$. We also associate a locally Lipschitz-continuous function $k_{\mu\nu} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to each edge that describes the time-evolution of $c_{\mu\nu}$:

$$\dot{c}_{\mu\nu} = k_{\mu\nu}\left(t, v_{\mu\nu}, c_{\mu\nu}\right) \,. \tag{15}$$

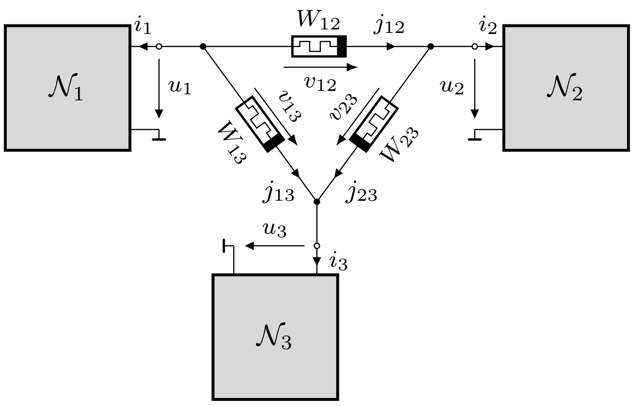

Fig. 2: Example of three coupled FHNOs $\mathcal{N}_1, \ldots, \mathcal{N}_3$ with the output voltages u_μ and input currents i_μ. A memristor symbol is used to indicate the adaptive coupling in (17). At every coupling we have the interaction current $j_{\mu\nu} = W_{\mu\nu}(t, v_{\mu\nu}, c_{\mu\nu}) v_{\mu\nu}$.

We call such a time evolution for $W_{\mu\nu}$ locally adaptive because it is driven by local information, namely the edge voltage $v_{\mu\nu}$ and the edge variable $c_{\mu\nu}$.

We denote by $\boldsymbol{u} = [u_1, u_2, \ldots, u_N]^{\mathrm{T}}$ and $\boldsymbol{i} = [i_1, i_2, \ldots, i_N]^{\mathrm{T}}$ the port quantities of the interaction ports and introduce the vectors $\boldsymbol{v} \in \mathbb{R}^{N_\mathcal{E}}$, $\boldsymbol{j} \in \mathbb{R}^{N_\mathcal{E}}$ and $\boldsymbol{c} \in \mathbb{R}^{N_\mathcal{E}}$ containing the voltage differences $v_{\mu\nu}$, the interaction currents $j_{\mu\nu}$, and edge variables $c_{\mu\nu}$ respectively. Using the graph's incidence matrix $\boldsymbol{N} \in \mathbb{Z}^{N \times N_\mathcal{E}}$ the Kirchhoff equations governing the coupling network spell out as follows:

$$\boldsymbol{i} = -\boldsymbol{N}\boldsymbol{j} \,, \quad \boldsymbol{v} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u} \,. \tag{16a}$$

Collecting the edge weights in a diagonal matrix denoted by $\boldsymbol{W}_{\mathrm{d}} \in \mathbb{R}^{N_\mathcal{E} \times N_\mathcal{E}}$ we can simultaneously state Ohm's law for every coupling conductance as:

$$\boldsymbol{j} = \boldsymbol{W}_{\mathrm{d}}\boldsymbol{v} \,, \text{ with } \boldsymbol{W}_{\mathrm{d}} = \boldsymbol{W}_{\mathrm{d}}\left(t, \boldsymbol{v}, \boldsymbol{c}\right) . \tag{16b}$$

Combination of (16b) with (16a) yields the specialized version of (8):

$$\boldsymbol{i} = -\boldsymbol{W}\boldsymbol{u} \,, \quad \text{with} \quad \boldsymbol{W} = \boldsymbol{W}\left(t, \boldsymbol{v}, \boldsymbol{c}\right) = \boldsymbol{N}\boldsymbol{W}_{\mathrm{d}}\boldsymbol{N}^{\mathrm{T}} \,. \tag{17}$$

The current directions are chosen such that a negative sign emerges in the coupling formula (17). Note that the conductance needs not be linear despite the appearance of (17). The dependence of $\boldsymbol{W}$ on the voltage differences $\boldsymbol{v}$ allows for nonlinear conductances where the shape of (17) only guarantees that a zero voltage difference results in a zero current.

In order to state the coupled version of (12) we define the vectors of normalized capacitor voltages and normalized inductor currents and normalized supply currents

$$\boldsymbol{\zeta}_1 = [z_{\mu,1}]_{\mu=1}^N, \quad \boldsymbol{\zeta}_2 = [z_{\mu,2}]_{\mu=1}^N, \quad \boldsymbol{\iota} \in \mathbb{R}^N, \tag{18}$$

as well as a parameter matrix $\boldsymbol{\beta} \in \mathbb{R}^{N\times N}$ and the unitless analog to $\boldsymbol{W}$:

$$\boldsymbol{\beta} = \beta \mathbf{1}, \quad \boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c}) = Z_0 \boldsymbol{W}(t, \boldsymbol{v}, \boldsymbol{c}) , \tag{19a}$$

with $t = \omega_0^{-1}\tau$, $\boldsymbol{v} = Z_0 I_0 \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\zeta}_1$, where β is from (11) and $\mathbf{1}$ denotes the unit matrix. Furthermore, we define vector-valued functions

$$\boldsymbol{f}_G : \mathbb{R}^N \to \mathbb{R}^N, \quad \text{and} \quad \tilde{\boldsymbol{k}} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N_\mathcal{E}} \to \mathbb{R}^{N_\mathcal{E}}, \tag{19b}$$

where the μ-th element of $\boldsymbol{f}_G$ is f_G evaluated at $z_{\mu,1}$, while the $\mu\nu$-th element of $\tilde{\boldsymbol{k}}$ is $\omega_0^{-1} k_{\mu\nu}$ evaluated at $t = \omega_0^{-1}\tau$, $v_{\mu\nu} = Z_0 I_0 [z_{\mu,1} - z_{\nu,1}]$, and $c_{\mu\nu}$ for $\{\mu,\nu\}$ ranging over the edges $\mathcal{E}$ of the coupling graph $\mathcal{G}$. Using these definitions, we obtain the ODE describing the coupled FHN-type oscillators w.r.t. a generalized diffusive coupling parametrized by normal time τ:

$$\boldsymbol{\zeta}_1' = \boldsymbol{\iota} - \boldsymbol{f}_G(\boldsymbol{\zeta}_1) - \boldsymbol{\zeta}_2 - \boldsymbol{\Gamma}\boldsymbol{\zeta}_1 \tag{20a}$$
$$\boldsymbol{\zeta}_2' = \boldsymbol{\zeta}_1 - \boldsymbol{\beta}\boldsymbol{\zeta}_2 \tag{20b}$$
$$\boldsymbol{c}' = \tilde{\boldsymbol{k}}(\tau, \boldsymbol{\zeta}_1, \boldsymbol{c}) . \tag{20c}$$

This is an instance of an ODE of type (3) in unitless form.

4 Main Result

We first state a sufficient condition for the synchronization of N FHNOs described in (12) with a coupling as in (20). We will prove this result throughout the rest of this section. In accordance with def. 5, we say that the states $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N$ of N oscillators synchronize if $\lim_{t\to\infty} \|\boldsymbol{z}_\mu - \boldsymbol{z}_\nu\| = 0$ for all $\mu, \nu = 1, \ldots, N$, which excludes clustered synchronization.

Theorem 7. *Consider N identical, diffusively coupled FitzHugh-Nagumo-type oscillators (20). Let $\mathcal{G}$ denote the weighted undirected graph with time-variant and locally adaptive weights associated to the coupling network and let $\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})$ denote the associated Laplacian matrix of $\mathcal{G}$. The states $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N$ of the oscillators synchronize if the following conditions hold:*

1. *the coupling graph $\mathcal{G}$ is connected,*
2. *the normalized nonlinear conductance function $f_G : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{z_1\to\pm\infty} f_G(z_1) = \pm\infty$ and $-\frac{\mathrm{d}f_G}{\mathrm{d}z_1}$ admits a global maximum $K := \max_{z_1\in\mathbb{R}} \left\{-\frac{\mathrm{d}f_G}{\mathrm{d}z_1}\right\}$;*
3. *the algebraic connectivity $\lambda_2\{\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})\}$, i.e., the smallest nonzero eigenvalue of $\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})$, satisfies*

$$\lambda_2\{\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})\} > \max\{0, K\} \quad \forall\ (\tau, \boldsymbol{z}, \boldsymbol{c});$$

4. *The coupling's evolution law $\tilde{\boldsymbol{k}}$ in (20c) is such that solution to (20c) exist for all $\tau \geq \tau_0$ regardless of the inputs $\boldsymbol{x}$.*

The initial conditions of the system do not matter in the above theorem. As a consequence the synchronization manifold is globally asymptotically stable and the individual oscillators all converge to the same state.

4.1 Preparations

We begin with providing a candidate for a (weak) quadratic Lyapunov function, to which we show that it is decreasing along the solutions of (20) under some assumptions on the coupling graph and the nonlinearity f_G. Its decrease along the solutions of (20) will be key to deduce synchronization, because its 0-locus coincides with the synchronization manifold. To this end we denote by $\mathbb{1}_N^\perp$ the subspace of vectors perpendicular to $\mathbb{1}_N$, and introduce the orthogonal projection matrix

$$\boldsymbol{P} = \mathbf{1}_N - \frac{1}{N}\mathbb{1}_N \mathbb{1}_N^T \tag{21}$$

to $\mathbb{1}_N^\perp$. One has $\boldsymbol{P}\mathbb{1}_N = \mathbf{0}$ and $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{1}_N^\perp$. Recall that the stacked state vectors are denoted by $\boldsymbol{z} = \left[\boldsymbol{z}_1^T \ \ldots \ \boldsymbol{z}_N^T\right]^T$. The following computations will be easier if one works with the stacked normalized voltages and currents $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ defined in (18) instead. The two are related by a permutation matrix denoted by

$$\boldsymbol{\Pi} \in \mathbb{R}^{2N\times 2N} : \qquad \boldsymbol{\Pi}\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{bmatrix} . \tag{22}$$

Lemma 8. *Let the graph $\mathcal{G}$ describing the coupling network be connected and assume that the normalized supply currents $\boldsymbol{\iota}$ are identical, i.e., $\boldsymbol{\iota} = \iota\mathbb{1}$ in (20), where $\mathbb{1}$ denotes the vector of ones. Define $V : \mathbb{R}^{2N} \to \mathbb{R}$ as*

$$V(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{z}, \quad \textit{with} \quad \boldsymbol{M} = \boldsymbol{\Pi}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \boldsymbol{\Pi} . \tag{23}$$

Then $V(\boldsymbol{z}) = 0$ if and only if $\boldsymbol{\zeta}_1 = \bar{z}_1\mathbb{1}$ and $\boldsymbol{\zeta}_2 = \bar{z}_2\mathbb{1}$ for $\bar{z}_1, \bar{z}_2 \in \mathbb{R}$, i.e., $V(\boldsymbol{z}) = 0 \Leftrightarrow \boldsymbol{z} \in \mathcal{S}$, where $\mathcal{S}$ is the synchronization manifold to (20) as in definition 5.
Suppose there exists a monotonically increasing function $f_\mathrm{m} : \mathbb{R} \to \mathbb{R}$ and a constant $K_1 \geq 0$ such that,

$$f_G(z_1) = f_\mathrm{m}(z_1) - K_1 z_1 \quad \textit{for all} \quad z_1 \in \mathbb{R}\,. \tag{24}$$

In this case, if $\lambda_2\{\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})\} \geq K_1$ for all $(\tau, \boldsymbol{z}, \boldsymbol{c})$ then $V'(\boldsymbol{z}) \leq 0$ along the solutions of (20) , where $\lambda_2\{\boldsymbol{\Gamma}\}$ refers to the second-smallest eigenvalue of $\boldsymbol{\Gamma}$.

Proof First, $\boldsymbol{P}$ is a positive semi-definite matrix and its defect is equal to 1. Since $\boldsymbol{M}$ is similar to a block diagonal matrix whose two blocks are positive multiples of $\boldsymbol{P}$, the matrix $\boldsymbol{M}$ is positive semi-definite with defect 2. Thus, $V(\boldsymbol{z}) \geq 0$ with equality only when $\boldsymbol{\zeta}_1 = \bar{z}_1\mathbb{1}$ and $\boldsymbol{\zeta}_2 = \bar{z}_2\mathbb{1}$ with $\bar{z}_1, \bar{z}_2 \in \mathbb{R}$ independent of each other. Denote

$$\boldsymbol{f}_\mathrm{m} : \mathbb{R}^N \to \mathbb{R}^N, \quad \boldsymbol{\zeta}_1 \mapsto \left[f_\mathrm{m}(z_{\mu,1})\right]_{\mu=1}^N\,. \tag{25}$$

A substitution of $\boldsymbol{z}'$ by (20), considering that $\boldsymbol{\iota} = \iota\mathbb{1}$ is in the kernel of $\boldsymbol{P}$, that $\boldsymbol{P}$ is symmetric, and that $\boldsymbol{\beta} = \beta\mathbf{1}$, provides the derivative of $V(\boldsymbol{z})$ w.r.t. τ along a solution $\boldsymbol{z}$ to (20):

$$\begin{aligned} V'(\boldsymbol{z}) &= \left[\nabla_{\boldsymbol{z}} V(\boldsymbol{z})\right]^\mathrm{T} \boldsymbol{z}' = \boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{\zeta}_1' + \boldsymbol{\zeta}_2^\mathrm{T}\boldsymbol{P}\boldsymbol{\zeta}_2' \\ &= \boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\left[-\boldsymbol{f}_G(\boldsymbol{\zeta}_1) - \boldsymbol{\zeta}_2 + \boldsymbol{\iota} - \boldsymbol{\Gamma}\boldsymbol{\zeta}_1\right] \\ &\quad + \boldsymbol{\zeta}_2^\mathrm{T}\boldsymbol{P}\left[\boldsymbol{\zeta}_1 - \beta\boldsymbol{\zeta}_2\right] \\ &= -\boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{f}_G(\boldsymbol{\zeta}_1) - \boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{\Gamma}\boldsymbol{\zeta}_1 - \beta\boldsymbol{\zeta}_2^\mathrm{T}\boldsymbol{P}\boldsymbol{\zeta}_2\,. \end{aligned} \tag{26}$$

As a preparatory step, we use $\boldsymbol{P}^2 = \boldsymbol{P}$ as well as its symmetry to compute

$$\begin{aligned} &\boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{f}_\mathrm{m}(\boldsymbol{\zeta}_1) = \left[\boldsymbol{P}\boldsymbol{\zeta}_1\right]^\mathrm{T}\boldsymbol{P}\boldsymbol{f}_\mathrm{m}(\boldsymbol{\zeta}_1) \\ &= \frac{2}{N^2}\sum_{\mu<\nu}\left[z_{\mu,1} - z_{\nu,1}\right]\left[f_\mathrm{m}(z_{\mu,1}) - f_\mathrm{m}(z_{\nu,1})\right]\,. \end{aligned}$$

Since f_m is strictly monotonically increasing, each term in the sum is always nonnegative. We obtain

$$\begin{aligned} \boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{f}_G(\boldsymbol{\zeta}_1) &= \boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{f}_\mathrm{m}(\boldsymbol{\zeta}_1) - K_1\boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{\zeta}_1 \\ &\geq -K_1\boldsymbol{\zeta}_1^\mathrm{T}\boldsymbol{P}\boldsymbol{\zeta}_1\,. \end{aligned}$$

This implies that $V'(\boldsymbol{z})$ has the upper bound

$$V'(\boldsymbol{z}) \leq -\boldsymbol{z}^\mathrm{T}\boldsymbol{\Pi}^\mathrm{T}\boldsymbol{F}\boldsymbol{\Pi}\boldsymbol{z}\,, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{P}\boldsymbol{\Gamma} - K_1\boldsymbol{P} & \mathbf{0} \\ \mathbf{0} & \beta\boldsymbol{P} \end{bmatrix}\,.$$

Hence, for $V'(\boldsymbol{z}) \leq 0$ we require $\boldsymbol{F}$ to be positive semi-definite. The positive semi-definiteness of the lower right block of $\boldsymbol{F}$ is ensured, since $\beta > 0$ and $\boldsymbol{P} \geq 0$. We therefore find

$$\boldsymbol{F} \geq \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{P}\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c}) - K_1\boldsymbol{P} \geq 0\,. \tag{27}$$

Now, since $\boldsymbol{\Gamma}$ is a Laplacian matrix of a connected graph its kernel is one-dimensional and spanned by $\mathbb{1}_N$ [38]. Therefore, we conclude that $\boldsymbol{\Gamma}$ and $\boldsymbol{P}$ have a common eigenbasis (and hence commute) as $\boldsymbol{P}$ acts like the identity on $\mathbb{1}_N^\perp$ and also sends $\mathbb{1}_N$ to 0. Thus, the upper left block can be analyzed by the eigenvalues of $\boldsymbol{\Gamma}$, where we recall the chosen order (7) of the eigenvalues, and we find

$$\left[\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c}) - K_1\mathbf{1}_N\right]\boldsymbol{P} \geq \mathbf{0} \quad \Leftrightarrow \quad \lambda_2\{\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})\} \geq K_1\,.$$

□

The condition (24) on the normalized conductance function requires some clarification concerning when it holds. This is the goal of the next lemma.

Lemma 9. *Let $f_G : \mathbb{R} \to \mathbb{R}$ be a (at least once) differentiable function such that*

$$\lim_{z_1 \to \pm\infty} f_G(z_1) = \pm\infty\,.$$

Furthermore, let its derivative with respect to z_1 be a function that is bounded from below and set

$$K := \max_{z_1 \in \mathbb{R}}\left\{-\frac{\mathrm{d}}{\mathrm{d}z_1} f_G(z_1)\right\}\,.$$

Then f_G satisfies the conditions of lemma 8 with $K_1 = K + \epsilon$ for $\epsilon > 0$ arbitrarily small, and is such that (12) is semi-passive with unbounded storage function.

Proof With $K_1 = K + \epsilon$ one has that $f_\mathrm{m} : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\mathrm{m}(z_1) = f_G(z_1) + K_1 z_1 \quad \forall\, z_1 \in \mathbb{R}$$

is strictly monotonically increasing because its derivative w.r.t. z_1 is strictly positive. Lemma 6 ensures that (12) is strictly semi-passive with unbounded storage function. By construction, f_m and f_G satisfy (24) so that f_G satisfies the conditions of lemma 8. □

4.2 Proof of theorem 7

Now, we will exploit the above results in order to show that the synchronization manifold is globally asymptotically stable under the conditions of theorem 7.

The strategy is as follows: We show that the bound on V' obtained in lemma 8 can be spelled out in terms of a vector $\hat{\boldsymbol{z}}$, defined below in (28), that is perpendicular to the kernel of $\boldsymbol{M}$ from (23). We then show that $\boldsymbol{z} \in \mathcal{S}$ if and only if $\hat{\boldsymbol{z}} = 0$. We spell out the bound on V' in terms of the norm of $\hat{\boldsymbol{z}}$ which we use to bound its integral. Afterwards we use a variation of Barbalat's lemma (provided in the appendix, see lemmas 15 and 16) to conclude

$$\lim_{t \to \infty} \hat{\boldsymbol{z}}(t) = \mathbf{0}\,,$$

which implies that $\boldsymbol{z} \in \mathcal{S}$. In order to apply Barbalat's lemma we must show that the solutions are uniformly continuous and it is there where we will use semi-passivity of the coupled system.

Under the conditions of theorem 7, the conditions of lemma 8 are satisfied according to lemma 9. We had seen that then

$$V'(\boldsymbol{z}) \leq -\boldsymbol{\zeta}_1^{\mathrm{T}} \left[\boldsymbol{P\Gamma} - K_1 \boldsymbol{P}\right] \boldsymbol{\zeta}_1 - \beta \boldsymbol{\zeta}_2^{\mathrm{T}} \boldsymbol{P} \boldsymbol{\zeta}_2 \,.$$

Lemma 8 characterizes the synchronization manifold as the linear subspace consisting of those $\boldsymbol{z} \in \mathbb{R}^{2N}$ for which $\boldsymbol{\zeta}_1 = \bar{z}_1 \mathbb{1}$ and $\boldsymbol{\zeta}_2 = \bar{z}_2 \mathbb{1}$. In terms of $\boldsymbol{P}$ and the permutation matrix $\boldsymbol{\Pi}$ from (22) one has that

$$\mathcal{S} = \ker \left\{ \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \boldsymbol{\Pi} \right\} .$$

Since $\mathcal{S}$ is a linear subspace of $\mathbb{R}^{2N}$ it is always possible to split up the state vector $\boldsymbol{z}$ into a part $\boldsymbol{z}_{\mathrm{s}}$ in $\mathcal{S}$ and a part $\hat{\boldsymbol{z}}$ orthogonal to it. By the above this split is facilitated by constant projections:

$$\boldsymbol{z}_{\mathrm{s}}(\tau) = \boldsymbol{\Pi}^{-1} \begin{bmatrix} \boldsymbol{P} - \boldsymbol{1} & 0 \\ 0 & \boldsymbol{P} - \boldsymbol{1} \end{bmatrix} \boldsymbol{\Pi} \boldsymbol{z}(\tau), \tag{28a}$$

$$\hat{\boldsymbol{z}}(\tau) = \boldsymbol{\Pi}^{-1} \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \boldsymbol{\Pi} \boldsymbol{z}(\tau) \,. \tag{28b}$$

In terms of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ this looks a bit simpler:

$$\begin{bmatrix} \boldsymbol{\zeta}_{1,\mathrm{s}}(\tau) \\ \boldsymbol{\zeta}_{2,\mathrm{s}}(\tau) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} - \boldsymbol{1} & 0 \\ 0 & \boldsymbol{P} - \boldsymbol{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta}_1(\tau) \\ \boldsymbol{\zeta}_2(\tau) \end{bmatrix} = \begin{bmatrix} \bar{z}_1(\tau) \mathbb{1}_N \\ \bar{z}_2(\tau) \mathbb{1}_N \end{bmatrix}, \tag{29a}$$

$$\begin{bmatrix} \hat{\boldsymbol{\zeta}}_1(\tau) \\ \hat{\boldsymbol{\zeta}}_2(\tau) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta}_1(\tau) \\ \boldsymbol{\zeta}_2(\tau) \end{bmatrix} , \tag{29b}$$

where

$$\bar{z}_1(\tau) = \frac{1}{N} \sum_{\mu=1}^{N} z_{\mu,1}(\tau) \quad \text{and} \quad \bar{z}_2(\tau) = \frac{1}{N} \sum_{\mu=1}^{N} z_{\mu,2}(\tau) \tag{29c}$$

are the averages of the state variables taken over the ensemble of N oscillators. One has that $\boldsymbol{z} \in \mathcal{S}$ if and only if $\hat{\boldsymbol{z}} = 0$ which is equivalent to $\hat{\boldsymbol{\zeta}}_1 = \hat{\boldsymbol{\zeta}}_2 = 0$.

As $\boldsymbol{\Gamma}$ is the (symmetric) Laplacian matrix to a connected graph its only eigenvector with eigenvalue 0 is $\mathbb{1}_N$. Even though $\boldsymbol{\Gamma}$ may vary over (normal) time, its kernel does not change but is always spanned by $\mathbb{1}_N$. Hence, $\boldsymbol{\zeta}_{1,\mathrm{s}}(\tau)$ and $\boldsymbol{\zeta}_{2,\mathrm{s}}(\tau)$ defined above are always contained in the kernel of $\boldsymbol{\Gamma}$. Furthermore, $\boldsymbol{z}_s$ and $\hat{\boldsymbol{z}}$ are given by linear combinations of the entries of $\boldsymbol{z}$ with constant coefficients as can be seen from (28). We can now express the bound on V' via $\hat{\boldsymbol{z}}$ and exploit that $\hat{\boldsymbol{z}}$ can be decomposed as a function of the eigenvectors of $\boldsymbol{\Gamma}$ excluding the eigenvector corresponding to the eigenvalue zero. Here, we have that

$$\begin{aligned} V'(\boldsymbol{z}) &\leq -\boldsymbol{\zeta}_1^{\mathrm{T}} \left[\boldsymbol{P\Gamma} - K_1 \boldsymbol{P}\right] \boldsymbol{\zeta}_1 - \beta \boldsymbol{\zeta}_2^{\mathrm{T}} \boldsymbol{P} \boldsymbol{\zeta}_2 \\ &\leq -\left[\lambda_2 \left\{\boldsymbol{\Gamma}\right\} - K_1\right] \left\|\hat{\boldsymbol{\zeta}}_1\right\|^2 - \beta \left\|\hat{\boldsymbol{\zeta}}_2\right\|^2 . \end{aligned}$$

Hence, by our theorem's assumption

$$\lambda_2\{\boldsymbol{\Gamma}(\tau, \boldsymbol{z}, \boldsymbol{c})\} > K_1 \,, \quad K_1 = \max\{0, K\} \,, \quad \forall\, (\tau, \boldsymbol{z}, \boldsymbol{c}) \,,$$

there exists a constant $c_0 > 0$ such that,

$$V'(\boldsymbol{z}) \leq -c_0 \|\hat{\boldsymbol{z}}\|^2 \,,$$

and from this, we obtain via integration that

$$V(\boldsymbol{z}(\tau)) - V(\boldsymbol{z}_0) \leq -c_0 \int_0^\tau \|\hat{\boldsymbol{z}}(s)\|^2 \mathrm{d}s \,. \tag{30}$$

As $V \geq 0$, the integral is bounded from above by $c_0^{-1} V(\boldsymbol{z}_0)$ and from below by 0 for all $\tau \in \mathbb{R}$.

According to prop. 4 the solutions of the diffusively coupled system (20) are bounded and as its right hand side is continuous we conclude that $\boldsymbol{z}'$ is bounded. This implies that $\boldsymbol{z}$ is uniformly continuous (u.c.) and as a consequence, $\|\hat{\boldsymbol{z}}\| : \mathbb{R} \to \mathbb{R}$, $\tau \mapsto \|\hat{\boldsymbol{z}}(\tau)\|$ is u.c. because $\hat{\boldsymbol{z}} = \boldsymbol{P}\boldsymbol{z}$ and linear maps are u.c. It now follows from lemma 16 that

$$\lim_{\tau \to \infty} \|\hat{\boldsymbol{z}}(\tau)\| \to 0 \quad \Rightarrow \quad \lim_{\tau \to \infty} \boldsymbol{z}(\tau) \in \mathcal{S} \,.$$

We conclude that $\mathcal{S}$ is globally asymptotically stable and therefore proved theorem 7.

5 Application to resistive and memristive coupling

We would like to consider some applications of theorem 7. We consider ideal voltage-controlled memristors as coupling elements, containing a simplification to purely resistive coupling as a special case. As we consider explicit circuit elements we will spell out the conditions of theorem 7 in terms of the circuit elements' parameters. The memristors we consider are governed by the following ODE in input-state-output form, where the variables below are parametrized w.r.t. time t and carry units with the exception of the memristor's state c:

$$\dot{c} = Q_0^{-1} i, \quad i = W(c) u, \quad c \in [0, 1], \tag{31a}$$

$$W(c) = W_0 + c \left[W_1 - W_0\right] . \tag{31b}$$

Here, $W(c)$ is the memductance with lower bound W_0 and upper bound W_1 as the state variable c is restricted to the interval $[0,1]$. In the above description, the input u is the voltage across the device, while the current i is both the output as well as the quantity driving the evolution of the memristor's state. We consider the case where each edge $\{\mu,\nu\} \in \mathcal{E}$ of the coupling graph $\mathcal{G}$ is realized by a memristor of the above type and such that the constants Q_0, W_0 and W_1 may depend on the edge $\{\mu,\nu\}$, indicated by replacing W with $W_{\mu\nu}$ and Q_0 with $Q_{\mu\nu,0}$, etc., which we also collect in a diagonal matrix $\boldsymbol{Q}_\mathrm{d} \in \mathbb{R}^{N_\mathcal{E}\times N_\mathcal{E}}$. Solutions to (31) exist for all times regardless of the input u as the memristor's state variable is restricted[4] to the interval $[0,1]$. We collect the memductances $W_{\mu\nu}(c_{\mu\nu})$ in a diagonal matrix $\boldsymbol{W}_\mathrm{d}(\boldsymbol{c}) \in \mathbb{R}^{N_\mathcal{E}\times N_\mathcal{E}}$ and define the coupling matrix as in (17) to be

$$\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{c}) = \boldsymbol{N}\boldsymbol{W}_\mathrm{d}(\boldsymbol{c})\boldsymbol{N}^T\,. \tag{32}$$

We would like to provide the full set of ODEs for identical FHNOs coupled by ideal memristors of type (31) in circuit quantities instead of the normalized ones used in (20). We denote by $\boldsymbol{u}_C \in \mathbb{R}^N$ the vector of capacitor voltages, $\boldsymbol{i}_L \in \mathbb{R}^N$ the vector of inductor currents, $\boldsymbol{j} = \mathbb{1}_N$ the vector of supply currents, and $\boldsymbol{R}_0 = R\mathbf{1}_N$, $\boldsymbol{C} = C\mathbf{1}_N$, $\boldsymbol{L} = L\mathbf{1}_N$ diagonal matrices carrying the resistances, capacitances and inductances respectively that occur in the FHNO circuit model. The ODE corresponding to (20) is then given by (cp. (12) and (11) concerning the translation between the two models)

$$\boldsymbol{C}\dot{\boldsymbol{u}}_C = \boldsymbol{j} - \boldsymbol{i}_G(\boldsymbol{u}_C) - \boldsymbol{i}_L - \boldsymbol{W}(\boldsymbol{c})\boldsymbol{u}_C\,, \tag{33a}$$

$$\boldsymbol{L}\dot{\boldsymbol{i}}_L = \boldsymbol{u}_C - \boldsymbol{R}_0\boldsymbol{i}_L\,, \tag{33b}$$

$$\dot{\boldsymbol{c}} = \boldsymbol{Q}_\mathrm{d}^{-1}\boldsymbol{W}_\mathrm{d}(\boldsymbol{c})\boldsymbol{N}^\mathrm{T}\boldsymbol{u}_C\,, \tag{33c}$$

with $\boldsymbol{W}(\boldsymbol{c})$ as in (31) and (32) and where $\boldsymbol{i}_G$ denotes the vectorized form of i_G analogous to $\boldsymbol{f}_G$ and f_G.

In order to apply theorem 7 one needs to find a lower bound to $\lambda_2\{\boldsymbol{W}(\boldsymbol{c}(t))\}$ for all $t \geq t_0$. One certainly has

$$\lambda_2\{\boldsymbol{W}(\boldsymbol{c}(t))\} \geq \lambda_{2,\min} := \min_{\boldsymbol{c}\in[0,1]^{N_\mathcal{E}}}\{\lambda_2\{\boldsymbol{W}(\boldsymbol{c})\}\}. \tag{34}$$

Since the eigenvalues of Laplacian matrices are monotonic functions of the edge-weights [38], this minimum can be computed by setting each edge weight equal to the minimal memductance $W_{\mu\nu,0}$ which is assumed at $c_{\mu\nu} = 0$. Thus,

$$\lambda_{2,\min} = \lambda_2\{\boldsymbol{W}(\boldsymbol{0})\}\,. \tag{35}$$

[4] It is possible to realize an equivalent system with $c \in \mathbb{R}$, where one augments the r.h.s. of (31) by θ-functions; then solutions to initial values $c_0 \in [0,1]$ exist for all times. Hence, one could model a system with restricted state variable with unrestricted state variables as well.

We arrive at the following corollary to theorem 7:

Corollary 10. *Consider the N identical FHNOs coupled diffusively by ideal memristors in (33). Let $\mathcal{G}$ denote the weighted undirected graph with locally adaptive weights associated to the coupling network and let $\boldsymbol{W}(\boldsymbol{c})$ denote the associated Laplacian matrix of $\mathcal{G}$ described in (32). The states of the oscillators synchronize if the following conditions hold:*

1. *the coupling graph $\mathcal{G}$ is connected,*
2. *the nonlinear conductance function $i_G : \mathbb{R} \to \mathbb{R}$ from (9) satisfies $\lim_{u\to\pm\infty} i_G(u) = \pm\infty$ and $-\frac{\mathrm{d}i_G}{\mathrm{d}u}$ admits a global maximum $\Delta G := \max_{u\in\mathbb{R}}\left\{-\frac{\mathrm{d}i_G}{\mathrm{d}u}(u)\right\}$,*
3. *the lower bound $\lambda_{2,\min} = \lambda_2\{\boldsymbol{W}(\boldsymbol{0})\}$ for the algebraic connectivity $\lambda_2\{\boldsymbol{W}(\boldsymbol{c}(t))\}$, i.e. the smallest nonzero eigenvalue of $\boldsymbol{W}(\boldsymbol{c}(t))$, satisfies*

$$\lambda_{2,\min} > \max\{0, \Delta G\}\,.$$

For the special case of static coupling described by a constant conductance matrix $\boldsymbol{W}$, one replaces $\lambda_{2,\min}$ with the algebraic connectivity $\lambda_2\{\boldsymbol{W}\}$.

The quantity ΔG in the above corollary can be interpreted as the maximal negative differential conductance in the FHNO's circuit realization. It is striking that the only parameters in the above sufficient synchronization condition are ΔG and the minimal possible connectivity $\lambda_{2,\min}$ while the other parameters of the FHNO play no role at all. We note that for most applications, indiscriminate synchronization of neural oscillators with memristive coupling is undesirable. In this case the memristors need to be chosen such that the minimal possible connectivity $\lambda_{2,\min}$ does not exceed ΔG.

For synchronized states no current flows through the coupling network and therefore synchronization coincides with minimization of power dissipated by the coupling network. The projection $\boldsymbol{P}$ used in the Lyapunov function (23) is in fact the Laplacian matrix to the unweighted complete graph. For static coupling, one can define an alternative Lyapunov function to (33) where the electrical counterpart to $\boldsymbol{\Gamma}$ is replaced by the coupling matrix $\boldsymbol{W}$. Explicitly, one can show that the storage function

$$S\left(\boldsymbol{u}_C, \boldsymbol{i}_L\right) = \boldsymbol{u}_C^\mathrm{T}\boldsymbol{W}\boldsymbol{u}_C + Z_0^2\,\boldsymbol{i}_L^\mathrm{T}\boldsymbol{W}\boldsymbol{i}_L \tag{36}$$

is a weak Lyapunov function to (33) leading to the same result as cor. 10 (for static coupling). The first term is in fact the power dissipated by the coupling network. This is another way to observe the implication "synchronization implies minimization of dissipated power in the coupling network" by having the Lyapunov function dominate the dissipated power.

If one chooses identical memristors, then $\lambda_{2,\min} = W_0\, \lambda_2(\boldsymbol{\Gamma}_0)$, where $\boldsymbol{\Gamma}_0$ describes the unweighted Laplacian matrix to the network. The algebraic connectivity $\lambda_2(\boldsymbol{\Gamma}_0)$ is bounded from above both by the vertex- as well as the edge-connectivity [39], defined as the minimal number of vertices (together with the edges connected to them) or edges respectively that one has to remove to render the graph disconnected. This is again bounded from above by the minimal degree of the vertices, i.e., one finds the vertex with the least neighbors and uses this as a bound. Of course, this can be much too conservative as it is possible to construct graphs with edge-connectivity equal to 1 but such that every neighbor has at least N_0 edges (just take two complete graphs on N_0 vertices each and join them by a single edge); but nonetheless this gives an estimate if one is unwilling to compute $\lambda_2(\boldsymbol{\Gamma}_0)$ directly.

6 Simulation Results and Discussion

6.1 Simulation Results

In the first part of this section, we describe a practical guide towards the application of the derived synchronization condition. We use the ODE in terms of circuit parameters (33) and the criterion from cor. 10, once for memristive couplings and once for purely resistive couplings. In order to compare the sharpness of our synchronization condition to the one of [20] and its predecessor [31] in section 6.2, we simulate the same example as the latter, which is depicted in Fig. 3. Here, every vertex represents a FHNO as depicted in Fig. 1, whereas every edge represents the memristive and resistive interconnections from section 5, which are chosen to be identical. The circuit parameters used within our simulations are given in Table 1. The nonlinear conductance function i_G is the one given in (10) with U_0 and G_0 as in table 1.

Circuit parameters		
$R = 4.7\,\mathrm{k\Omega}$	$j = 0\,\mathrm{A}$	$G_0 = 100\,\mu\mathrm{S}$
$C = 100\,\mathrm{nF}$	$L = 23.5\,\mathrm{H}$	$U_0 = 0.24\,\mathrm{V}$
$W_0 = 100.01\,\mu\mathrm{S}$	$W_1 = 500\,\mu\mathrm{S}$	$Q_0 = 1\,\mu\mathrm{A/s}$

Table 1: FitzHugh-Nagumo oscillator parameters.

In order to apply our synchronization condition, we must first calculate the negative derivative of i_G, which is bounded from above and has the maximum $\Delta G = G_0 = 100\,\mu\mathrm{S}$, see Fig. 4. In a practical scenario, one must measure the (i,u)-curve of the nonlinear conductance, calculate the negative derivative numerically, and find its global maximum. According to cor. 10, the next step is to verify, whether for all times and memductance states, the connectivity of the graph is greater than ΔG.

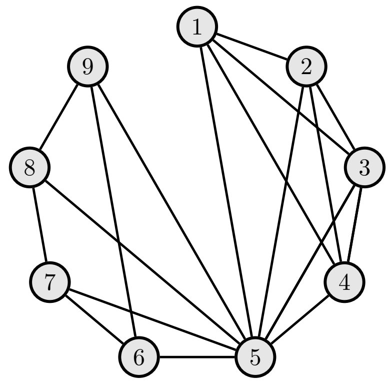

Fig. 3: Graph abstraction of the emulated example. Each vertex represents a FitzHugh-Nagumo oscillator. The edges represent the memristive interconnections depicted in Fig. 2 with identical coupling elements.

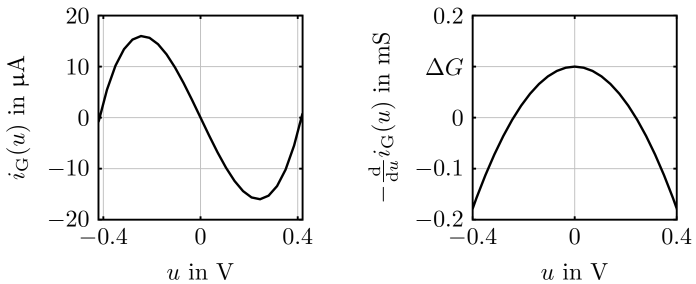

Fig. 4: The nonlinear conductance function (10) and its negative derivative w.r.t. u.

The connectivity of the unweighted graph is given by

$$\lambda_2\{\boldsymbol{\Gamma}_0\} = 1\,, \quad \text{with} \quad \boldsymbol{\Gamma}_0 = \mathbf{diag}(\boldsymbol{A}_0 \mathbb{1}) - \boldsymbol{A}_0\,,$$

where $\boldsymbol{\Gamma}_0$ denotes the unweighted graph's Laplacian and $\boldsymbol{A}_0$ is the unweighted adjacency matrix. For uniform static coupling of strength G_c, the weighted graph's Laplacian is given by $\boldsymbol{W} = G_c \boldsymbol{\Gamma}_0$ and so $\lambda_2\{\boldsymbol{W}\} = G_c$ in this case. For identical memductances as coupling elements with high-ohmic state W_0 one has by (35) that

$$\lambda_{2,\min} = \boldsymbol{W}(\mathbf{0}) = W_0 \lambda_2\{\boldsymbol{\Gamma}_0\} = W_0\,. \tag{37}$$

Hence, $\boldsymbol{W}$ satisfies the inequality of corollary 10 if

$$G_c > \Delta G = 100\,\mu\mathrm{S}, \quad W_0 > \Delta G = 100\,\mu\mathrm{S}\,. \tag{38}$$

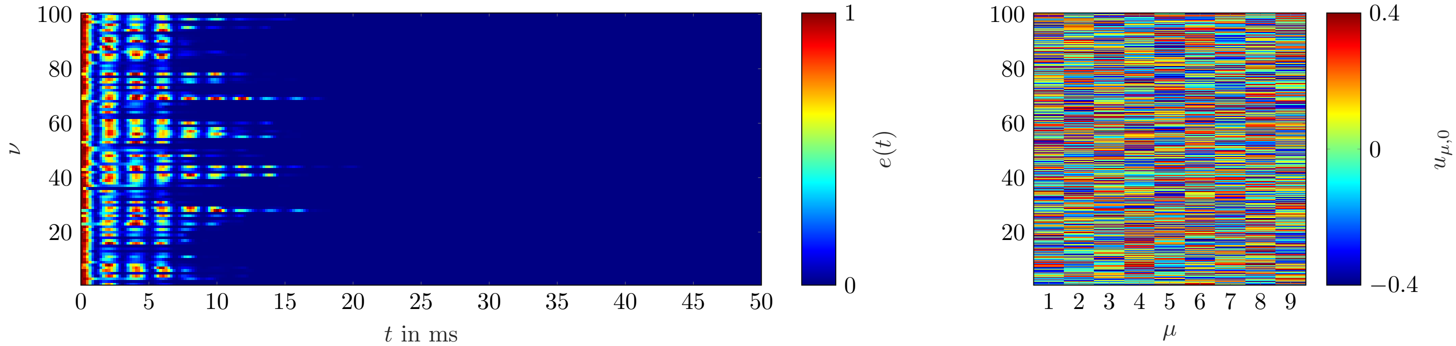

Fig. 5: Left: The plot shows the synchronization error (39) over time for 100 identical networks with different initializations. Right: 100 randomly chosen initializations of the system with coupling topology as in Fig. 3, where μ labels the oscillators.

We stress that it is not necessary to pick the coupling weights uniformly. Any coupling graph $\boldsymbol{W}$ with $\lambda_2\{\boldsymbol{W}(\mathbf{0})\} > 100\,\mu\mathrm{S}$ works here.

We display in Fig. 5 the results of a simulation for 100 copies of the system with memristive coupling for the case of $W_0 = 100.01\,\mu\mathrm{S}$ just above the boundary of our condition. The memristors are initialized identically in the high-ohmic state to give the system the "slowest" start possible. The oscillators themselves are initialized randomly such that $i_{L,0} = 0\,\mathrm{A}$ and $u_{C,0}$ is distributed uniformly in $[-400, 400]$ mV, where 400 mV is the maximal amplitude of the uncoupled FHNO's stable limit cycle. The quantity plotted is

$$e(t) = \frac{1}{e_{\max}} \sum_{\mu=2}^{N} \left[u_{C,\mu}(t) - u_{C,1}(t)\right]^2, \tag{39}$$

with $e_{\max}$ chosen s.t. $\max_t e(t) = 1$. This serves as a measure for the synchronization error although it neglects the second state variable i_L but has the advantage that it is also a measure for the power distributed by the coupling network. We observe that eventually all FHNOs synchronize but that the time required can vary. With the chosen parameters the period of a single oscillator is about 2 ms so that synchronization occurs within at most 10 oscillation cycles for the chosen range of initial values. We also observe that the power dissipated by the coupling network is not monotonically decreasing but exhibits a damped oscillation in magnitude with a similar period as the FHNO.

We have also tested the distance between our sufficient synchronization condition from the (unknown) necessary one by running a series of simulations for different (uniform) static coupling conductances on the graph depicted in Fig. 3 with the parameters in Table 1. The results are displayed in Fig. 6, where the boundary of our condition $G_\mathrm{c} = 100\,\mu\mathrm{S}$ is placed at the very top of the scale. We

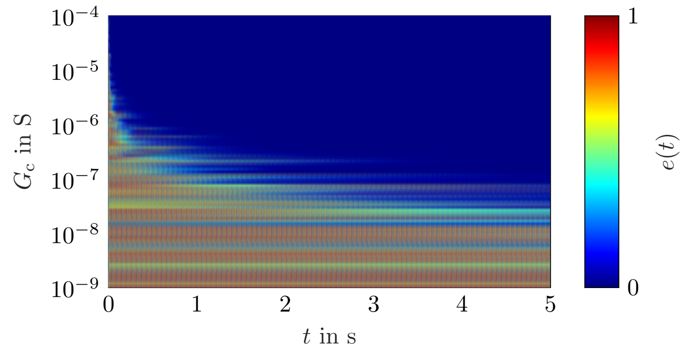

Fig. 6: The synchronization error (39) over time for static coupling and a wide range of conductances. The y-axis starts at the boundary G_c = 100 µS of the sufficient condition. Synchronization is achieved for coupling conductances orders of magnitude lower than required but with reduced and varying speed.

observe that synchronization is (eventually) achieved for conductances three orders of magnitude smaller than the criterion requires. However, we also observe that the time at which synchronization is achieved increases drastically from a few oscillation cycles to more than 1000.

While we did not observe a large variance in synchronization speed in the memristive scenario, we did so in the static case for situations where the criterion was not met. We think that this suggests that the gap between sufficient and necessary conditions for synchronization needs additional exploration. We also think that our sufficient condition should be further refined and augmented by criteria that guarantee a certain synchronization speed, potentially in dependence of the initial values, as it may be unacceptable to have a required synchronization time of a factor more than a 1000 time larger then the system's time scale defined by the FHNOs period.

6.2 Comparison with the literature

To the best of our knowledge there exists no sufficient synchronization condition for memristively coupled FHNOs beyond the situation of two memristively coupled oscillators [40] although the phenomenon has been studied numerically on numerous occasions for two or more oscillators not necessarily of FHN-type [41–44]. While there exist sufficient conditions towards systems coupled by a time-variant coupling matrix [20, 21], the time-variance there is not allowed to be depend on the oscillators' state or additional state variables which excludes both nonlinear coupling elements as well as locally adaptive ones. A few types of diffusive, locally adaptive coupling have been studied in [23, 45], where the coupling strength is only allowed to be increasing so that our general form of admissible coupling and its evolution law also adds to the existing literature. For purely static, linear coupling the picture is more complicated, which is why we discuss this topic to more extent below and summarize our discussion in table 2. Given the vast amount of literature on the topic we can of course only include a selection of publications. We begin with comparing the sharpness of our synchronization condition for static diffusive coupling to the one presented in [31] as this is also based on semi-passivity, albeit working with a non-smooth Lyapunov function. Afterwards we analyze why our result coincides with the bound obtained from contraction theory more recently [20, 21]. We also relate it to QUAD-conditions [25] and conclude with a direct approach based on Lyapunov's method [29]. Although [29] was published almost 30 years ago, it still sets the bar of the sharpest sufficient condition; it is reached but not outperformed by most of the other works, including the specialization of our own result to the static linear case.

The following unitless version of the FitzHugh-Nagumo model is used in [19]:

$$\dot{x} = -\left[\frac{x^3}{3} - x\right] + I + u\,, \tag{40a}$$

$$\dot{y} = \varepsilon[x + a - by]\,, \tag{40b}$$

where $b, \varepsilon, a > 0$ and I are constants and u denotes the input. We remark that while our model (12) allows more general nonlinearities in x than (11d) as well as a dependence of $\dot{x}$ on y, it only treats the case $a = 0$. The relevant ingredient for our criterion is the nonlinearity

$$f_G(z_1) = \frac{1}{3}z_1^3 - z_1 \tag{41}$$

in (12). Since the maximum of $-\frac{\mathrm{d}}{\mathrm{d}z_1}f_G$ evaluates to 1 (the slope of f_G is extremal at the origin), we have the condition $\lambda_2\{\boldsymbol{\Gamma}\} > 1$. The condition presented in [31] spells out as

$$\lambda_2\{\boldsymbol{\Gamma}\} \geq 1 + \varepsilon + \frac{\beta_1^2}{3}, \tag{42}$$

where β_1 is the bound on x resulting from the fact that the trajectories are ultimately bounded. According to [31] common values (of the parameters relevant for the criterion) for biologically plausible firing behavior of the FHN-oscillators are $\varepsilon \approx \frac{1}{12}$ and $\beta_1 \approx 2$ which would require the connectivity to fulfill $\lambda_2\{\boldsymbol{\Gamma}\} > 2.41$. But even without explicit values we observe that our condition is sharper than (42) which is always strictly greater than 1.

The bound $\lambda_2\{\boldsymbol{\Gamma}\} > 1$ for the above FHN-model (40) is the sharpest available sufficient condition for synchronization and has also been obtained in [20, cor. 4.1] with the methods of contraction theory, in [21, thm. 30] as part of the generalization of contraction theory to so-called semi-contracting systems, and in [29] by a direct application of Lyapunov's method.

The general sufficient condition of [20] is spelled out in terms of the log-matrix norm $\mu_{2,P}$ and it is also obtained for more general log-matrix (semi-)norms in [21]. According to [21] (this is the phrasing of [22, thm. 5.19]) the diffusively coupled system

$$\dot{\boldsymbol{x}}_\mu = \boldsymbol{f}\left(t, \boldsymbol{x}_\mu\right) - \sum_\nu a_{\mu\nu}\left[\boldsymbol{x}_\mu - \boldsymbol{x}_\nu\right],\ \mu \in \{1, \ldots, N\}\,. \tag{43}$$

synchronizes if there exist $p \in [1, \infty]$, positive definite $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ and $\varepsilon > 0$ such that for every $(t, \boldsymbol{x}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ one has

$$\mu_{p,\boldsymbol{Q}}\left(D\boldsymbol{f}(t, \boldsymbol{x})\right) \leq \lambda_2\left\{\boldsymbol{L}\right\} - \varepsilon, \tag{44}$$

where $\boldsymbol{L}$ is the Laplacian matrix to the weighted undirected graph with adjacency matrix $\boldsymbol{A}$ and $\mu_{p,\boldsymbol{Q}}$ denotes the weighted log-matrix-norm. The only one needed explicitly in the following is $\mu_{2,\boldsymbol{Q}}(\boldsymbol{M}) = \lambda_{\max}\left(\boldsymbol{Q}\frac{\boldsymbol{M}+\boldsymbol{M}^{\mathrm{T}}}{2}\boldsymbol{Q}^{-1}\right)$. Now by the Demidovic-lemma ([46], cp. [22, lem. 3.1]) the Jacobian $D\boldsymbol{f}$ satisfies $\mu_{2,\boldsymbol{Q}^{\frac{1}{2}}}\left(D\boldsymbol{f}(x)\right) \leq C$ for some positive-definite matrix $\boldsymbol{Q}$ if and only if $\boldsymbol{f}$ satisfies the one-sided Lipschitz condition

$$\left[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\right]^{\mathrm{T}}\boldsymbol{Q}\left[\boldsymbol{x} - \boldsymbol{y}\right] \leq C\left\|\boldsymbol{x} - \boldsymbol{y}\right\|^2_{2,\boldsymbol{Q}^{\frac{1}{2}}} \tag{45}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where $\|\boldsymbol{x}\|_{2,\boldsymbol{Q}} := \|\boldsymbol{Q}\boldsymbol{x}\|_2$. One can check that the nonlinear function f_G satisfies the one-sided Lipschitz condition

$$-\left[f_G(x) - f_G(y)\right]\left[x - y\right] \leq K_1\left[x - y\right]^2, \tag{46}$$

where K_1 is as in (24), i.e., such that $f_{\mathrm{m}}(z_1) = f_G(z_1) + K_1 z_1$ is strictly monotonic increasing. Due to the choice of coordinate transformation used to describe the unitless model, this translates to the following condition on $\boldsymbol{f}$:

$$[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})]^{\mathrm{T}} [\boldsymbol{x} - \boldsymbol{y}] \leq K_1 \|\boldsymbol{x} - \boldsymbol{y}\|_2^2 \,. \tag{47}$$

Therefore, our condition $\lambda_2\{\boldsymbol{\Gamma}\} > \max\{0, K\}$ with $K = \max_{z_1 \in \mathbb{R}} \left\{ -\frac{\mathrm{d}f_G}{\mathrm{d}z_1}(z_1) \right\}$ implies that (44) is satisfied due to the equivalence of one-sided Lipschitz-conditions and bounds on log-matrix-norms established by the Demidovic-lemma.

QUAD-conditions, which have been used for instance in [23–25, 45], are a variation of one-sided Lipschitz-conditions:

Definition 11. Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, let $\boldsymbol{\Delta}$ be a diagonal matrix and $\omega > 0$ a real number. One says that $\boldsymbol{f}$ is $QUAD(\boldsymbol{\Delta}, \omega)$ if it satisfies the inequality

$$\begin{aligned} [\boldsymbol{x} - \boldsymbol{y}]^{\mathrm{T}} [\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})] \leq [\boldsymbol{x} - \boldsymbol{y}]^{\mathrm{T}} \boldsymbol{\Delta} [\boldsymbol{x} - \boldsymbol{y}] \\ - \omega [\boldsymbol{x} - \boldsymbol{y}]^{\mathrm{T}} [\boldsymbol{x} - \boldsymbol{y}] \,. \end{aligned}$$

There is an intimate connection between QUAD-conditions and contraction theory summarized in [25]. Denote by $\boldsymbol{\Lambda}_0 = -\boldsymbol{\Gamma}$ the negative Laplacian to (43), let $\boldsymbol{f}$ be $QUAD(\boldsymbol{\Delta}_0, \omega)$ for some $\omega > 0$ and introduce the following matrices:

$$\boldsymbol{\Lambda} \coloneqq \boldsymbol{\Lambda}_0 \otimes \mathbf{1}_n, \;\; \boldsymbol{\Pi} \coloneqq \boldsymbol{P}_N \otimes \mathbf{1}_n, \;\; \boldsymbol{\Delta} \coloneqq \mathbf{1}_N \otimes \boldsymbol{\Delta}_0, \tag{48}$$

where $\boldsymbol{P}_N$ is the orthogonal projection to the complement of $\mathbb{1}_N$. According to [24, thm. 2]) the network of oscillators (43) synchronizes if the matrix $[\mathbf{\Pi\Delta} + \mathbf{\Pi\Lambda}]$ is negative semi-definite. The negative semi-definiteness of the matrix $[\mathbf{\Pi\Delta} + \mathbf{\Pi\Lambda}]$ can be characterized by an inequality between the maximal eigenvalue of $\boldsymbol{\Delta}_0$ and the algebraic connectivity of $\boldsymbol{\Gamma}$ via a standard argument. To our surprise we could not find this in the literature so we record a proof for the reader's convenience.

Proposition 12. *The matrix* $[\mathbf{\Pi\Delta} + \mathbf{\Pi\Lambda}]$ *is negative semi-definite if and only if the largest eigenvalue* $\lambda_{\max}\{\boldsymbol{\Delta}_0\}$ *of* $\boldsymbol{\Delta}_0$ *and the algebraic connectivity* $\lambda_2\{\boldsymbol{\Gamma}\}$ *satisfy* $\lambda_{\max}\{\boldsymbol{\Delta}_0\} \leq \lambda_2\{\boldsymbol{\Gamma}\}$.

Proof As $\boldsymbol{\Delta}_0$ is diagonal and $\boldsymbol{\Lambda}_0 = -\boldsymbol{\Gamma}$ is symmetric both possess a basis of eigenvectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$, $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ respectively, where the $\boldsymbol{e}_i$ are the unit vectors of $\mathbb{R}^n$. Since $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_0 \otimes \mathbf{1}_n$ and $\boldsymbol{\Delta} = \mathbf{1}_N \otimes \boldsymbol{\Delta}_0$ commute a joint basis of eigenvectors of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Delta}$ is given by $\{\boldsymbol{v}_\mu \otimes \boldsymbol{e}_j, \mu = 1, \ldots, N, j = 1, \ldots, n\}$. One has

$$[\boldsymbol{\Delta} + \boldsymbol{\Lambda}] [\boldsymbol{v}_\mu \otimes \boldsymbol{e}_j] = [\Delta_{0,jj} - \lambda_\mu\{\boldsymbol{\Gamma}\}] [\boldsymbol{v}_\mu \otimes e_j] \tag{49}$$

for all $\mu = 1, \ldots, N$ and $j = 1, \ldots, n$. Furthermore, as $\boldsymbol{\Gamma}$ is a Laplacian matrix, the eigenvectors to the eigenvalues $\lambda_2\{\boldsymbol{\Gamma}\}, \ldots, \lambda_N\{\boldsymbol{\Gamma}\}$ are orthogonal to the kernel of $\boldsymbol{P}_N$ and therefore $\boldsymbol{P}_N \boldsymbol{v}_\mu = \boldsymbol{v}_\mu$ for all $\mu \geq 2$, which leads to

$$\begin{aligned} [\boldsymbol{\Pi\Delta} + \boldsymbol{\Pi\Lambda}] [\boldsymbol{v}_\mu \otimes \boldsymbol{e}_j] = \\ \begin{cases} [\Delta_{0,jj} - \lambda_\mu\{\boldsymbol{\Gamma}\}] \boldsymbol{v}_\mu \otimes \boldsymbol{e}_j & \forall \mu \geq 2 \\ 0 & \text{for } i = 1. \end{cases} \end{aligned}$$

Thus, $[\boldsymbol{\Pi\Delta} + \boldsymbol{\Pi\Lambda}]$ is negative semi-definite if and only if $\Delta_{0,jj} - \lambda_\mu\{\boldsymbol{\Gamma}\} \leq 0$ for all $\mu \geq 2$ and all $j = 1, \ldots, n$. Since $\lambda_2\{\boldsymbol{\Gamma}\}$ is the smallest eigenvalue apart from $\lambda_1\{\boldsymbol{\Gamma}\}$ this is equivalent to

$$\lambda_{\max}\{\boldsymbol{\Delta}_0\} \leq \lambda_2\{\boldsymbol{\Gamma}\} \,. \tag{50}$$

□

In order to compare our synchronization condition to the one of [24, thm. 2] based on QUAD-conditions we need to derive QUAD-estimates for the FHNO in dependence of the parameters. We derive the following estimate in the appendix:

Lemma 13. *Consider the FitzHugh-Nagumo oscillator in the form*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \boldsymbol{f}\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -ax^3 + bx - c_0 y + j \\ c_1 x - c_2 y \end{bmatrix} \tag{51}$$

with $a, b, c_0, c_1, c_2 \geq 0$ *and constant current injection* j. *Then* $\boldsymbol{f}$ *is QUAD for* $\boldsymbol{\Delta}, \omega$ *such that* $\Delta_{11} - \omega \geq b + \frac{|c_1 - c_0|}{2}$ *and* $\Delta_{22} - \omega \geq -c_2 + \frac{|c_1 - c_0|}{2}$.

In the case of static diffusive coupling one can now compare the criteria obtained from QUAD-conditions and ours. As ω can be chosen arbitrarily small one has that f is $QUAD(\Delta, \omega)$ for all Δ such that $\Delta_{11} > b + \frac{|c_1 - c_0|}{2}$ and $\Delta_{22} > -c_2 + \frac{|c_1 - c_0|}{2}$ and as all parameters are nonnegative $\Delta_{11} \geq \Delta_{22}$. By the use of Proposition 12 this results in the sufficient condition

$$b + \frac{|c_1 - c_0|}{2} > \lambda_2(\boldsymbol{\Gamma}) \,. \tag{52}$$

Our condition $K = \max_{z_1 \in \mathbb{R}} \left(-\frac{\mathrm{d}f_G(z_1)}{\mathrm{d}z_1} \right) < \lambda_2(\boldsymbol{\Gamma})$ spells out in terms of the above parameters as $b < \lambda_2\{\boldsymbol{\Gamma}\}$ since the slope of the conductance function is extremal at the origin. Hence, our condition appears to be sharper in this case, because it does not depend on the cross terms proportional to c_0, c_1 that occur in the above inequalities. However, we had also seen that a coordinate transformation can ensure $c_1 = c_0$ so that the two in fact coincide if the ODE is set up appropriately.

Lastly, we would like to present one of the sufficient synchronization conditions from Wu and Chua [29] dating

back to 1995. In this work the authors provide a rather general and powerful framework to study synchronization of identical oscillators subject to static linear coupling with particular emphasis on allowing different classes of coupling matrices. Their results are therefore divided according to those classes of coupling matrices and we only cite the case that applies to the system (43):
Assume there exist a positive definite matrix $\boldsymbol{V}$, a diagonal matrix $\boldsymbol{T} = \operatorname{diag}(t_1, \dots, t_n)$ and a continuous, nondecreasing function $c : \mathbb{R} \to \mathbb{R}$ satisfying $c(0) = 0$ and $c(s) > 0$ for all $s \neq 0$. If [29, eq. 17] holds, i.e., if

$$[\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{V} [\boldsymbol{f}(\boldsymbol{x},t) - \boldsymbol{f}(\boldsymbol{y},t) - \boldsymbol{T}[\boldsymbol{x}-\boldsymbol{y}]] \tag{53a}$$

$$\leq -c(\|\boldsymbol{x}-\boldsymbol{y}\|) \tag{53b}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and t then (43) synchronizes if $\lambda_2\{\boldsymbol{L}\} > t_i$ for all i (This is [29, thm. 7, cor. 3]). One observes that (53) is a generalized form of QUAD-condition as it is more flexible due to the matrix $\boldsymbol{V}$. Based on the previous example one sees that this provides a more powerful criterion than [24, thm. 2] based on QUAD-conditions as one has a more tractable method to sharpen the bound by variation of $\boldsymbol{V}$. One can check that (12) satisfies (53) for

$$\boldsymbol{V} = \mathbf{1} \text{ and all } \boldsymbol{T} = \operatorname{diag}(t_1, 0) \tag{54}$$

with $t_1 \geq K$. Thus, for $t_1 = K$ one recovers the condition $\lambda_2\{\boldsymbol{L}\} > K$ from thm. 7 for the static case which coincided with [20, cor. 4.1]. This is again no coincidence, as (53) and the one-sided Lipschitz-condition (45) are similar and at least for diagonal matrices lead to equivalent results. To see this, spell out (53) with, for simplicity, diagonal $\boldsymbol{V} = \boldsymbol{Q}$:

$$\begin{aligned}
&[\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{Q} [\boldsymbol{f}(\boldsymbol{x},t) - \boldsymbol{f}(\boldsymbol{y},t)] \\
&\leq [\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{Q}\boldsymbol{T} [\boldsymbol{x}-\boldsymbol{y}] - c(\|\boldsymbol{x}-\boldsymbol{y}\|) \\
&\leq [\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{Q}\boldsymbol{T} [\boldsymbol{x}-\boldsymbol{y}] \\
&\leq \max_i\{t_i\} \|\boldsymbol{x}-\boldsymbol{y}\|^2_{2,\boldsymbol{Q}^{\frac{1}{2}}} ,
\end{aligned}$$

which is a one-sided Lipschitz-condition (45) with $C = t_{\max} = \max_i\{t_i\}$.
Conversely assume that (45) holds for diagonal, positive definite $\boldsymbol{Q}$ and $C > 0$, then

$$[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})]^{\mathrm{T}} \boldsymbol{Q} [\boldsymbol{x}-\boldsymbol{y}] \leq C \|\boldsymbol{x}-\boldsymbol{y}\|^2_{2,\boldsymbol{Q}^{\frac{1}{2}}} \tag{55}$$

$$= C [\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{Q} [\boldsymbol{x}-\boldsymbol{y}] \leq [\boldsymbol{x}-\boldsymbol{y}]^{\mathrm{T}} \boldsymbol{T}_0 \boldsymbol{Q} [\boldsymbol{x}-\boldsymbol{y}] \tag{56}$$

for $\boldsymbol{T}_0 = \operatorname{diag}(t_1, \dots, t_n)$ such that $t_i \geq C$. Hence, (53) holds for all $\boldsymbol{T} = \boldsymbol{T}_0 + \varepsilon \mathbf{1}$ with $\varepsilon > 0$ as $-\varepsilon\|\boldsymbol{x}-\boldsymbol{y}\|^2$ provides a function $-c(\|\boldsymbol{x}-\boldsymbol{y}\|)$ with the required properties. The sharpest choice consists of $t_i = C$ so that $t_{\max} = C$ which is the relevant part of $\boldsymbol{T}$ towards the synchronization condition. In conclusion, although (53) and (45) are not fully equivalent due the higher flexibility in the matrix $\boldsymbol{T}$ they lead to the same synchronization condition, because only the maximal eigenvalue of $\boldsymbol{T}$ matters.

It is fascinating that the sharpness of sufficient synchronization conditions for oscillators coupled by static linear coupling has not increased over the last 30 years. We wonder, if this is only due to the similarities of the used methods, or if there is something deeper to be learned here.

7 Conclusion and Outlook

This work is dedicated to the derivation of a sufficient synchronization condition for a network of (identical) diffusively coupled FitzHugh-Nagumo oscillators with time-variant, state-dependent and locally adaptive coupling, which among others includes coupling by memristors and/or nonlinear conductances. We started by briefly reviewing the necessary theory of dissipative and semi-passive systems and provided a description of uncoupled and diffusively coupled FitzHugh-Nagumo-type oscillators by ordinary differential equations fitting in this framework. Then, we provided a Lyapunov function candidate V and derived conditions on the oscillators' nonlinear conductance relative to the connectivity of the network for $\dot{V}$ to be decreasing along the solutions of the system. Our condition on the nonlinear conductance function i_G is that it admits a modification by a linear term $i_{\mathrm{m}}(u) = i_G(u) + \Delta G\, u$ such that i_{m} is strictly monotonically increasing. We then showed that the oscillators synchronize globally, i.e., from any initial state, if the connectivity $\lambda_2\{\boldsymbol{W}\}$ of the network exceeds the slope ΔG of this modification for all times. To examine the validity and sharpness of this condition, we conducted a few numerical experiments. Finally, we showed that our synchronization condition specialized to linear static coupling is as sharp as those from the literature [20, 21, 29] but also easy to use for practitioners who need these conditions to be given in terms of parameters of the ideal circuit.

In future research, we would like to sharpen our synchronization condition and to loosen the condition on $\lambda_2\{\boldsymbol{W}\}$ for state-dependent and adaptive coupling. In the current form the condition does not take into account that only a minor part of phase space is reached by a trajectory but the criterion considers the minimum of $\lambda_2\{\boldsymbol{W}\}$ over all times and phase space. We would therefore like to spell out this conditions with respect to the initial values in the sense that we want to study how the minimum over a given trajectory evaluates and depends on the initial value. We also plan to investigate if our approach can

	Coupling type	Time variance	Clusters	General oscillators	Main bound
Semi-passivity as in thm. 7	diffusive, nonlinear passive	adaptive	No	No	$1 < \lambda_2 \{\mathbf{L}\}$
Direct Lyapunov approach	general linear	static	No	Yes	$1 < \lambda_2 \{\mathbf{L}\}$
Contraction Theory	diffusive, linear	time-variant	Yes	Yes	$1 < \lambda_2 \{\mathbf{L}\}$
QUAD-conditions	diffusive, linear	(adaptive)	No	Yes	$1 < \lambda_2 \{\mathbf{L}\}^*$

Table 2: A comparison of the different approaches to synchronization mentioned, which for the static case all compare the algebraic connectivity $\lambda_2\{\boldsymbol{L}\}$ to some other parameter that is set to 1 in the column "main bound". The parentheses and the asterisk in the last line indicate that the results of [24, 25] include only a small class of adaptive couplings and that although the main bound is determined by $\lambda_2 \{\mathbf{L}\}$, the original exposition does not state this explicitly.

be adapted to the case of heterogeneous oscillators and clustered synchronization.

References

[1] Arbib MA, Amari S, Arbib PH. The Handbook of Brain Theory and Neural Networks. A Bradford book. MIT Press; 2003. Available from: https://books.google.de/books?id=Av6qWhtw0-EC.

[2] Dörfler F, Bullo F. Synchronization in complex networks of phase oscillators: A survey. Automatica. 2014 06;50. https://doi.org/10.1016/j.automatica.2014.04.012.

[3] Ahmed I, Chiu PW, Moy W, Kim CH. A Probabilistic Compute Fabric Based on Coupled Ring Oscillators for Solving Combinatorial Optimization Problems. IEEE Journal of Solid-State Circuits. 2021;56(9):2870–2880. https://doi.org/10.1109/JSSC.2021.3062821.

[4] Csaba G, Porod W. Coupled oscillators for computing: A review and perspective. Applied Physics Reviews. 2020;7(1):011302. https://doi.org/10.1063/1.5120412. https://doi.org/10.1063/1.5120412.

[5] Chou J, Bramhavar S, Ghosh S, Herzog W. Analog Coupled Oscillator Based Weighted Ising Machine. Scientific Reports. 2019;9(1):14786. https://doi.org/10.1038/s41598-019-49699-5.

[6] Ochs K, Al Beattie B, Jenderny S. An Ising Machine Solving Max-Cut Problems based on the Circuit Synthesis of the Phase Dynamics of a Modified Kuramoto Model. In: 2021 IEEE International Midwest Symposium on Circuits and Systems (MWSCAS); 2021. p. 982–985.

[7] Abernot M, Gil T, Jiménez M, Núñez J, Avellido MJ, Linares-Barranco B, et al. Digital Implementation of Oscillatory Neural Network for Image Recognition Applications. Frontiers in Neuroscience. 2021;15. https://doi.org/10.3389/fnins.2021.713054.

[8] Michaelis D, Ochs K, Jenderny S. A Memristive Circuit for Gait Pattern Classification Based on Self-Organized Axon Growth. In: 2021 IEEE International Midwest Symposium on Circuits and Systems (MWSCAS); 2021. p. 162–165.

[9] Johnson AP, Liu J, Millard AG, Karim S, Tyrrell AM, Harkin J, et al. Homeostatic Fault Tolerance in Spiking Neural Networks: A Dynamic Hardware Perspective. IEEE Transactions on Circuits and Systems I: Regular Papers. 2018;65(2):687–699. https://doi.org/10.1109/TCSI.2017.2726763.

[10] Chua L. Memristor-The missing circuit element. IEEE Trans Circuits Theor. 1971;18(5):507–519. https://doi.org/10.1109/TCT.1971.1083337.

[11] Chua LO, Kang SM. Memristive devices and systems. Proc IEEE. 1976;64(2):209–223. https://doi.org/10.1109/PROC.1976.10092.

[12] Li H, Fang JA, Li X, Rutkowski L, Huang T. Event-Triggered Synchronization of Multiple Discrete-Time Markovian Jump Memristor- Based Neural Networks With Mixed Mode-Dependent Delays. IEEE Transactions on Circuits and Systems I: Regular Papers. 2022;69(5):2095–2107. https://doi.org/10.1109/TCSI.2022.3149535.

[13] Duan S, Hu X, Dong Z, Wang L, Mazumder P. Memristor-based cellular nonlinear/neural network: design, analysis, and applications. IEEE transactions on neural networks and learning systems. 2015;26(6):1202–1213. https://doi.org/10.1109/TNNLS.2014.2334701.

[14] Zhang Y, Wang X, Friedman EG. Memristor-Based Circuit Design for Multilayer Neural Networks. IEEE

Transactions on Circuits and Systems I: Regular Papers. 2018;65(2):677–686. https://doi.org/10.1109/TCSI.2017.2729787.

[15] Li M, Hong Q, Wang X. Memristor-based circuit implementation of Competitive Neural Network based on online unsupervised Hebbian learning rule for pattern recognition. Neural Computing and Applications. 2022;34(1):319–331. https://doi.org/10.1007/s00521-021-06361-4.

[16] Ochs K, Michaelis D, Solan E, Feketa P, Schaum A, Meurer T. Synthesis, Design, and Synchronization Analysis of Coupled Linear Electrical Networks. IEEE Transactions on Circuits and Systems I: Regular Papers. 2020;67(12):4521–4532. https://doi.org/10.1109/TCSI.2020.3002672.

[17] Pecora LM, Carroll TL. Master Stability Functions for Synchronized Coupled Systems. Phys Rev Lett. 1998 Mar;80:2109–2112. https://doi.org/10.1103/PhysRevLett.80.2109.

[18] Lohmiller W, Slotine JJE. On Contraction Analysis for Non-linear Systems. Autom. 1998;34:683–696.

[19] Aminzare Z, Sontag E. Synchronization of Diffusively-Connected Nonlinear Systems: Results Based on Contractions with Respect to General Norms. IEEE Transactions on Network Science and Engineering. 2014;1:91–106.

[20] Aminzare Z, Dey B, Davison EN, Leonard NE. Cluster Synchronization of Diffusively Coupled Nonlinear Systems: A Contraction-Based Approach. Journal of Nonlinear Science. 2020;30. Published April 2018. https://doi.org/10.1007/s00332-018-9457-y.

[21] Jafarpour S, Cisneros-Velarde P, Bullo F. Weak and Semi-Contraction for Network Systems and Diffusively Coupled Oscillators. IEEE Transactions on Automatic Control. 2022;67(3):1285–1300. https://doi.org/10.1109/TAC.2021.3073096.

[22] Bullo F. Contraction Theory for Dynamical Systems. 1.1 ed. Kindle Direct Publishing; 2023. Available from: https://fbullo.github.io/ctds.

[23] De Lellis P, di Bernardo M, Garofalo F. Synchronization of complex networks through local adaptive coupling. Chaos: An Interdisciplinary Journal of Nonlinear Science. 2008 09;18(3). 037110. https://doi.org/10.1063/1.2944236. https://pubs.aip.org/aip/cha/article-pdf/doi/10.1063/1.2944236/13859891/037110_1_online.pdf.

[24] DeLellis P, diBernardo M, Garofalo F. Novel decentralized adaptive strategies for the synchronization of complex networks. Automatica. 2009;45(5):1312–1318. https://doi.org/https://doi.org/10.1016/j.automatica.2009.01.001.

[25] Delellis P, Di Bernardo M, Russo G. On QUAD, Lipschitz, and Contracting Vector Fields for Consensus and Synchronization of Networks. Circuits and Systems I: Regular Papers, IEEE Transactions on. 2011 04;58:576 – 583. https://doi.org/10.1109/TCSI.2010.2072270.

[26] Pogromski AY. Passivity based design of synchronizing systems. International Journal of Bifurcation and Chaos. 1998;8:295–319.

[27] Pogromsky A, Glad T, Nijmeijer H. On diffusion driven oscillations in coupled dynamical systems. International Journal of Bifurcation and Chaos. 1999 04;9:629–644. https://doi.org/10.1142/S0218127499000444.

[28] Steur E, Tyukin IY, Nijmeijer H. Semi-passivity and synchronization of diffusively coupled neuronal oscillators. Physica D: Nonlinear Phenomena. 2009;238:2119–2128.

[29] Wu CW, Chua LO. Synchronization in an array of linearly coupled dynamical systems. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications. 1995;42(8):430–447. https://doi.org/10.1109/81.404047.

[30] Dörfler F, Bullo F. On the Critical Coupling for Kuramoto Oscillators. SIAM Journal on Applied Dynamical Systems. 2011;10(3):1070–1099. https://doi.org/10.1137/10081530X. https://doi.org/10.1137/10081530X.

[31] Davison EN, Dey B, Leonard NE. Synchronization bound for networks of nonlinear oscillators. In: 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton); 2016. p. 1110–1115.

[32] Petrovas A, Lisauskas S, Slepikas A. Electronic Model of FitzHugh-Nagumo Neuron. Elektronika ir Elektrotechnika. 2012 Jun;122(6):117–120. https://doi.org/10.5755/j01.eee.122.6.1835.

[33] Nagumo J, Arimoto S, Yoshizawa S. An Active Pulse Transmission Line Simulating Nerve Axon. Proceedings of the IRE. 1962;50(10):2061–2070. https://doi.org/10.1109/JRPROC.1962.288235.

[34] Ochs K, Michaelis D, Jenderny S. Synthesis of an Equivalent Circuit for Spike-Timing-Dependent Axon Growth: What Fires Together Now Really Wires Together. IEEE Trans Circuits Syst I Regul Pap. 2021;68(9):3656–3667. https://doi.org/10.1109/TCSI.2021.3093172.

[35] FitzHugh R. Impulses and Physiological States in Theoretical Models of Nerve Membrane. Biophysical Journal. 1961;1(6):445–466. https://doi.org/https://doi.org/10.1016/S0006-3495(61)86902-6.

[36] Ringkvist M, Zhou Y. On the dynamical behaviour of FitzHugh–Nagumo systems: Revisited. Nonlinear Analysis-theory Methods & Applications - NONLINEAR ANAL-THEOR METH APP. 2009 10;71:2667–2687. https://doi.org/10.1016/j.na.2009.01.149.

[37] Willems JC. Dissipative dynamical systems part I: General theory. Archive for Rational Mechanics and Analysis. 1972;45(5):321–351. https://doi.org/10.1007/BF00276493.

[38] Biggs N. Algebraic Graph Theory. 2nd ed. Cambridge Mathematical Library. Cambridge University Press; 1974.

[39] Fiedler M. Algebraic connectivity of graphs. Czechoslovak Mathematical Journal. 1973;23:298–305.

[40] Korneev I, Semenov V, Vadivasova T. Synchronization of Periodic Self-Oscillators Interacting via Memristor-Based Coupling. International Journal of Bifurcation and Chaos. 2020 06;30:2050096. https://doi.org/10.1142/S0218127420500960.

[41] Ignatov M, Hansen M, Ziegler M, Kohlstedt H. Synchronization of two memristively coupled van der Pol oscillators. Applied Physics Letters. 2016 02;108(8):084105. https://doi.org/10.1063/1.4942832. https://pubs.aip.org/aip/apl/article-pdf/doi/10.1063/1.4942832/13766521/084105_1_online.pdf.

[42] Wang C, Lv M, Alsaedi A, Ma J. Synchronization stability and pattern selection in a memristive neuronal network. Chaos: An Interdisciplinary Journal of Nonlinear Science. 2017 11;27(11):113108. https://doi.org/10.1063/1.5004234. https://pubs.aip.org/aip/cha/article-pdf/doi/10.1063/1.5004234/14000021/113108_1_online.pdf.

[43] Ma J, Xu W, Zhou P, Zhang G. Synchronization between memristive and initial-dependent oscillators driven by noise. Physica A: Statistical Mechanics and its Applications. 2019;536:122598. https://doi.org/https://doi.org/10.1016/j.physa.2019.122598.

[44] Korneev IA, Ramazanov IR, Semenov VV, Slepnev AV, Vadivasova TE. Synchronization of Traveling Waves in Memristively Coupled Ensembles of FitzHugh–Nagumo Neurons With Periodic Boundary Conditions. Frontiers in Physics. 2022;10. https://doi.org/10.3389/fphy.2022.886476.

[45] De Lellis P, di Bernardo M, Garofalo F. In: Chiuso A, Fortuna L, Frasca M, Rizzo A, Schenato L, Zampieri S, editors. Decentralized Adaptive Control for Synchronization and Consensus of Complex Networks. Berlin, Heidelberg: Springer Berlin Heidelberg; 2009. p. 27–42. Available from: https://doi.org/10.1007/978-3-642-03199-1_2.

[46] Demidovič BP. Dissipativity of a nonlinear system of differential equations. Uspekhi Matematicheskikh Nauk. 1961;16:216.

[47] Barbalat I. Systems d'equations differentielles d'oscillations nonlinéaires. Revue Roumaine de Mathématique Pures et Appliquées. 1959;4:267–270.

[48] Tao G. A simple alternative to the Barbalat lemma. IEEE Transactions on Automatic Control. 1997;42(5):698–. https://doi.org/10.1109/9.580878.

[49] Sun Z.: A gathering of Barbalat's lemmas and their (unsung) cousins. Available from: https://doi.org/10.48550/arXiv.2301.00466.

Statements and Declarations

Funding

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 434434223 – SFB 1461 and the Christian-Albrechts-University of Kiel in preparation of the Cluster of Excellence-Initiative "Networked Matter".

Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

The first two authors contributed equally to this work. All authors read and approved the final manuscript.

Data Availability

The datasets generated in section 6 are available from the corresponding author on reasonable request.

Nomenclature

$\mathbf{1}_n \in \mathbb{R}^{n\times n}$ is the unit matrix.
$\mathbb{1}_n \in \mathbb{R}^n$ is the vector whose every entry is equal to 1.
$\boldsymbol{z}_\mu$: state vector of subsystem μ; $z_{\mu,1},\ldots,z_{\mu,n_\mu}$: its components; n_μ: dimension of subsystem μ.
N: the number of subsystems.
$\boldsymbol{x}$ and $\boldsymbol{y}$: input and output vectors of dim. k except for section 6.2, where they just denote generic vectors.
ω: supply rate, a term which occurs in definitions 1, 2 and 3.
S: storage function, see def. 1.
$a_{\mu\nu}$: coupling strength of the gen. diffusive coupling in (3). Depends on time t, an edge variable $c_{\mu\nu}$ and $\boldsymbol{z}_\mu$, $\boldsymbol{z}_\nu$.
$k_{\mu\nu}$: describes the time evolution of $c_{\mu\nu}$, see (3).
$\mathcal{G}$: The coupling graph with N vertices $\mathcal{V}$ and $N_\mathcal{E}$ edges $\mathcal{E}$.
$\boldsymbol{N} \in \mathbb{R}^{N\times N_\mathcal{E}}$: incidence matrix of $\mathcal{G}$ to some arbitrary orientation.
$\boldsymbol{D} \in \mathbb{R}^{N_\mathcal{E}\times N_\mathcal{E}}$: diag. matrix carrying the edge weights of $\mathcal{G}$.
$\boldsymbol{\Gamma}$: The Laplacian matrix to $\mathcal{G}$, defined in (5).
$\boldsymbol{z}$: The vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N$ stacked by subsystem. $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined the same way, see sec. 2.
$\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$: stacked vectors of the first, resp. second, component of the $\boldsymbol{z}_\mu$, see (18) and (22).
$\mathcal{S}$ denotes the synchronization manifold, see def. 5.
$\boldsymbol{W} \in \mathbb{R}^{N\times N}$: conductance matrix from the electrical model in 3.2; related to $\boldsymbol{W}_\mathrm{d} \in \mathbb{R}^{N_\mathcal{E}\times N_\mathcal{E}}$, the matrix carrying the edge conductances via (17) and to $\boldsymbol{\Gamma}$ via (19a).
u_C and i_L: variables of the FHNO-circuit model.
j, C, L, R: scalar constants of the model.
$i_G : \mathbb{R} \to \mathbb{R}$: nonlinear conductance function.
z_1, z_2: variables in the unitless model.
$\tau = \omega_0 t$: Normal time used in the unitless model.
ι, α, β, γ: scalar constants of the unitless model.
$f_G : \mathbb{R} \to \mathbb{R}$: unitless conductance function.
$\boldsymbol{f}_G$: vectorized version of f_G, see (19).
$\lambda_\mu\{\boldsymbol{A}\}$: μ-th eigenvalue of $\boldsymbol{A}$ in ascending order.
$\boldsymbol{\beta} = \beta\mathbf{1}$: parameter matrix in unitless model (19).
K: constant associated to f_G and relevant in thm. 7.
$\mathbb{1}_N^\perp \subset \mathbb{R}^N$: space of vectors orthogonal to $\mathbb{1}_N$
$\boldsymbol{P}$ or $\boldsymbol{P}_N$: orthogonal projection to $\mathbb{1}_N^\perp$
$\boldsymbol{\Pi} \in \mathbb{R}^{2N\times 2N}$: relates the stacked vector $\boldsymbol{z}$ to $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$.
V: Lyapunov function, bilinear form w.r.t. $\boldsymbol{M}$, see (23)
K_1: constant associated to f_G in (24)
$\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{\zeta}}_1$, $\hat{\boldsymbol{\zeta}}_2$: nonsynchronous parts of the respective vectors.
$\boldsymbol{\zeta}_{1,\mathrm{s}}$, $\boldsymbol{\zeta}_{2,\mathrm{s}}$: synchronous part of $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$.

A Appendix

A.1 A preparatory lemma to prop. 4

In order to prove prop. 4 one needs the following lemma:

Lemma 14. *Consider N strictly semi-passive systems of dimensions $n_1,\ldots,n_N$ with storage functions S_μ and functions H_μ and ρ_μ as in def. 3. If the $\rho_\mu : \mathbb{R} \to \mathbb{R}_{\geq 0}$ are unbounded from above and strictly monotonically increasing there exists a nonnegative function $\rho : \mathbb{R}^{n_\mathrm{tot}} \to \mathbb{R}_{\geq 0}$ and a constant $r > 0$ such that*

$$H(\boldsymbol{z}) \geq \rho(\boldsymbol{z}) \qquad \forall \|\boldsymbol{z}\| \geq r\,, \text{ where} \tag{57a}$$

$$n_\mathrm{tot} = \sum_{\mu=1}^N n_\mu\,, \qquad H(z) = \sum_{\mu=1}^N H_\mu(\boldsymbol{z}_\mu)\,. \tag{57b}$$

Proof We cannot assume for $\|\boldsymbol{z}\|$ arbitrarily large that this also holds for all $\|\boldsymbol{z}_\mu\|$. Thus, we cannot use all individual bounds simultaneously but need to consider the worst case, i.e., only one $\|\boldsymbol{z}_\mu\|$ is large enough for H_μ to be bounded from below by ρ_μ. Set $h_\mu = \inf_{\boldsymbol{z}_\mu} H_\mu(\boldsymbol{z}_\mu)$ then

$$H(\boldsymbol{z}) = \sum_{\mu=1}^N H_\mu(\boldsymbol{z}_\mu) \geq H_\nu(\boldsymbol{z}_\nu) + \sum_{\mu\neq\nu}^N h_\mu \tag{58}$$

holds for all ν and all $\boldsymbol{z}$. Without loss of generality assume that $h_\mu \leq 0$ as this only provides an even lower bound on $H(\boldsymbol{z})$. Let $\varepsilon_\nu > 0$ and choose $\tilde{r}_\nu > 0$ such that

$$\rho_\nu(\tilde{r}_\nu) = \varepsilon_\nu + \sum_{\mu\neq\nu}^N h_\mu \quad \text{and}$$

$$\rho_\nu(x) \geq \varepsilon_\nu + \sum_{\mu\neq\nu}^N h_\mu\,, \quad \forall x \geq \tilde{r}_\nu\,.$$

This is possible because all ρ_μ are strictly monotonically increasing and unbounded. We furthermore are free to choose $\tilde{r}_\nu \geq r_\nu$, where $r_\nu > 0$ is the constant fulfilling

$$H_\nu(\boldsymbol{z}_\nu) \geq \rho_\nu(\|\boldsymbol{z}_\nu\|)\,, \quad \forall \|\boldsymbol{z}_\nu\| > r_\nu\,.$$

For $\nu = 1,\ldots,N$ set

$$\tilde{\rho}_\nu(x) = \begin{cases} \rho_\nu(x) - \sum_{\mu\neq\nu} h_\mu\,, & \text{for } x > \tilde{r}_\nu \\ \varepsilon_\nu\,, & \text{for } x \leq \tilde{r}_\nu \end{cases} \tag{59}$$

and observe that $\tilde{\rho}_\nu$ is positive and monotonically increasing. One then has for $\|\boldsymbol{z}_\nu\| > \tilde{r}_\nu$ that

$$H_\nu(\boldsymbol{z}_\nu) \geq \rho_\nu(\|\boldsymbol{z}_\nu\|) = \tilde{\rho}_\nu(\|\boldsymbol{z}_\nu\|) - \sum_{\mu\neq\nu} h_\mu \tag{60}$$

and therefore

$$H_\nu(\boldsymbol{z}) \geq H_\nu(\boldsymbol{z}_\nu) + \sum_{\mu\neq\nu} h_\mu \geq \tilde{\rho}_\nu(\|\boldsymbol{z}_\nu\|) \ , \quad \forall\, \boldsymbol{z} \in \mathbb{R}^{n_{\mathrm{tot}}} \ ,$$

such that $\|\boldsymbol{z}_\nu\| > \tilde{r}_\nu$. Now set

$$\tilde{\rho}(\boldsymbol{z}) = \min_\nu \tilde{\rho}_\nu(\|\boldsymbol{z}_\nu\|) \quad \text{and} \quad \tilde{r} = \sqrt{\sum_{\mu=1}^N \tilde{r}_\mu^2}$$

which implies $\tilde{\rho}(\boldsymbol{z}) \leq \tilde{\rho}_\nu(\|\boldsymbol{z}_\nu\|)$ by definition.

Now for $\|\boldsymbol{z}\| > \tilde{r}$ there exists ν such that $\|\boldsymbol{z}_\nu\| > \tilde{r}_\nu$ and hence

$$H(\boldsymbol{z}) \geq H_\nu(\boldsymbol{z}_\nu) + \sum_{\mu\neq\nu} h_\mu \geq \tilde{\rho}_\nu(\|\boldsymbol{z}_\nu\|) \geq \tilde{\rho}(\boldsymbol{z}) \tag{61}$$

for all $\|\boldsymbol{z}\| > \tilde{r}$. This shows the lemma with $\rho = \tilde{\rho}$ and $r = \tilde{r}$. □

A.2 Proof of Proposition 4

We don't assume the systems to be identical or have equal dimension, therefore define $n_{\mathrm{tot}} = \sum_{\mu=1}^N n_\mu$, where n_μ is the state space dimension of the μ-th subsystem. Each individual subsystem has a storage function S_μ to which there exists a function H_μ as in def. 3 such that along the trajectories $\boldsymbol{z}_\mu(t)$ one has

$$\dot{S}_\mu(\boldsymbol{z}_\mu) \leq \boldsymbol{x}_\mu^T \boldsymbol{y}_\mu - H_\mu(\boldsymbol{z}_\mu) \ .$$

By assumption $\boldsymbol{L} \geq \boldsymbol{0}$, so that for

$$S(\boldsymbol{z}) = \sum_{\mu=1}^N S_\mu(\boldsymbol{z}_\mu) \quad \text{and} \quad H(\boldsymbol{z}) = \sum_{\mu=1}^N H_\mu(\boldsymbol{z}_\mu)$$

one has

$$\begin{aligned} \dot{S} &\leq \sum_{\mu=1}^N \boldsymbol{x}_\mu^{\mathrm{T}} \boldsymbol{y}_\mu - H_\mu(\boldsymbol{z}_\mu) \\ &= -\boldsymbol{y}^{\mathrm{T}} \boldsymbol{L}(t, \boldsymbol{z}, \boldsymbol{c}) \boldsymbol{y} - H(\boldsymbol{z}) \\ &\leq -H(\boldsymbol{z}) \ , \end{aligned}$$

along the solutions of (3), which holds for all $(t, \boldsymbol{z}, \boldsymbol{c})$. This implies the integral inequality:

$$S(\boldsymbol{z}(t)) - S(\boldsymbol{z}(t_0)) \leq -\int_{t_0}^t H(\boldsymbol{z}(\tau))\, \mathrm{d}\tau \ . \tag{62}$$

We want to use the above inequality to show that $\boldsymbol{z}(t)$ is bounded for all t and in order to do so we need to turn bounds for H_μ by ρ_μ into a bound for H by a single nonnegative function ρ outside some ball $B_r(0) \subset \mathbb{R}^{n_{\mathrm{tot}}}$. Now as all storage functions S_μ are radially unbounded, the same is true for S and therefore to $C > 0$ there exists $R > 0$ such that

$$\{\boldsymbol{z} \in \mathbb{R}^{n_{\mathrm{tot}}} \mid S(\boldsymbol{z}) \leq C\} \subset B_R(0), \tag{63}$$

where $B_R(0)$ denotes the ball of radius R in $\mathbb{R}^{n_{\mathrm{tot}}}$ centered at 0. We may assume that $R \geq r > 0$ with r as in lemma 14. We wish to ignore what happens inside $B_R(0)$ by setting

$$\tilde{S}(\boldsymbol{z}) := \begin{cases} 0 & \text{if } \|\boldsymbol{z}\| \leq R \\ S(\boldsymbol{z}) & \text{otherwise.} \end{cases} \ , \tag{64a}$$

$$\tilde{\rho}(\boldsymbol{z}) := \begin{cases} 0 & \text{if } \|\boldsymbol{z}\| \leq R \\ \rho(\boldsymbol{z}) & \text{otherwise.} \end{cases} \ . \tag{64b}$$

Then $\tilde{S}$ is also radially unbounded since it is equal to S outside a compact region. Furthermore, from inequality (62) one deduces

$$\tilde{S}(\boldsymbol{z}(t)) - \tilde{S}(\boldsymbol{z}(t_0)) \leq -\int_{t_0}^t \tilde{\rho}(\boldsymbol{z}(\tau))\, \mathrm{d}\tau \tag{65}$$

along the solutions of (3). For trajectories outside the ball $B_R(0)$ this follows directly from (62) and for trajectories inside the ball this is trivially satisfied. For trajectories that cross the ball's boundary exactly once at time t_1 one can split the integral as

$$\int_{t_0}^t \tilde{\rho}(\boldsymbol{z}(\tau))\, \mathrm{d}\tau = \int_{t_0}^{t_1} \tilde{\rho}(\boldsymbol{z}(s))\, \mathrm{d}\tau + \int_{t_1}^t \tilde{\rho}(\boldsymbol{z}(\tau))\, \mathrm{d}\tau \ ,$$

to obtain

$$\tilde{S}(\boldsymbol{z}(t)) - \tilde{S}(\boldsymbol{z}(t_0)) \leq \int_0^{t_1} \tilde{\rho}(\boldsymbol{z}(\tau))\, \mathrm{d}\tau + \int_{t_1}^t \tilde{\rho}(\boldsymbol{z}(\tau))\, \mathrm{d}\tau.$$

This shows that (65) holds for any trajectory of (3) as one can subdivide the integrals in the above manner for more than one crossing time as well. Now as $\tilde{\rho}$ is nonnegative this shows

$$\tilde{S}(\boldsymbol{z}(t)) \leq \tilde{S}(\boldsymbol{z}(t_0)) \ , \quad \forall\, t \geq t_0 \ ,$$

and together with $\tilde{S}$ being radially unbounded this shows that $\boldsymbol{z}(t)$ is bounded for all t which concludes the proof.

A.3 Barbalat's Lemma and Derivatives

Lemma 15. *(Barbalat's lemma [47]) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous function such that*

$$\lim_{t\to\infty} \int_0^t f(\tau)\,\mathrm{d}\tau$$

exists and is finite, then

$$\lim_{t\to\infty} f(t) = 0\,.$$

In our setting it is more convenient to use a cousin of Barbalat's lemma due to [48] (see also [49] for a nice collection of different versions of Barbalat's lemma and where to find them).

Lemma 16. *(Cp. [48]) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and square-integrable, i.e., such that*

$$\int_{-\infty}^{+\infty} f^2(t)\,\mathrm{d}t < \infty\,.$$

Then one has

$$\lim_{t\to\infty} f(t) = 0.$$

If f is differentiable, one may replace "f uniformly continuous" by "$\dot{f}$ being bounded".

A.4 Proof of the QUAD-estimate in lemma 13

One computes

$$\begin{aligned}
&\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}^{\mathrm{T}} \left[\boldsymbol{f}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) - \boldsymbol{f}\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)\right] = \\
&[x_1 - x_2]\left[-a\left[x_1^3 - x_2^3\right] + b\,[x_1 - x_2] - c_0\,[y_1 - y_2]\right] \\
&\quad + [y_1 - y_2]\,[c_1\,[x_1 - x_2] - c_2\,[y_1 - y_2]] \\
= &\quad -a\,[x_1 - x_2]\left[x_1^3 - x_2^3\right] + b\,[x_1 - x_2]^2 \\
&\quad + [c_1 - c_0]\,[x_1 - x_2]\,[y_1 - y_2] - c_2\,[y_1 - y_2]^2 \\
\overset{(i)}{\leq} &\quad -a\,[x_1 - x_2]\left[x_1^3 - x_2^3\right] + \left[b + \tfrac{|c_1 - c_0|}{2}\right][x_1 - x_2]^2 \\
&\quad + \left[-c_2 + \tfrac{|c_1 - c_0|}{2}\right][y_1 - y_2]^2 \\
\overset{(ii)}{\leq} &\left[b + \tfrac{|c_1 - c_0|}{2}\right][x_1 - x_2]^2 + \left[-c_2 + \tfrac{|c_1 - c_0|}{2}\right][y_1 - y_2]^2 \\
= &\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b + \frac{|c_1 - c_0|}{2} & 0 \\ 0 & -c_2 + \frac{|c_1 - c_0|}{2} \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix},
\end{aligned}$$

where at (i) it is used that $2ab \leq a^2 + b^2$ and at (ii) that $[x_1 - x_2]\left[x_1^3 - x_2^3\right] \geq 0$ for all $x_1, x_2 \in \mathbb{R}$. This shows that $\boldsymbol{f}$ is $QUAD(\boldsymbol{\Delta}, \omega)$ for all $\boldsymbol{\Delta}, \omega$ such that

$$\Delta_{11} - \omega \geq b + \frac{|c_1 - c_0|}{2} \quad \text{and} \quad \Delta_{22} - \omega \geq -c_2 + \frac{|c_1 - c_0|}{2}.$$