Why is MIMO Capacity in a Fading Environment Higher than in an AWGN Environment

Yasir Ahmed and Jeffrey H. Reed

Virginia Tech, Blacksburg, VA 24061

www.raymaps.com

***Abstract* - A wireless channel suffers from two fundamental impairments; fading and noise. While fading is multiplicative, noise is additive. It is well-known that higher the noise, lower is the signal to noise ratio and lower the capacity. However, fading can be helpful in increasing the capacity when using multiple transmit and receive antennas. In this paper, we give an intuitive explanation for this. Anybody with a background in linear algebra and matrices can understand this.**

It is well known that Shannon Capacity in a fading environment is greater than in an AWGN environment when using multiple transmit and receive antennas [1]. How is this possible? How can randomness added by a fading channel increase its capacity? There are lots of papers on this subject which try to explain the problem in terms of rank, determinant and eigen decomposition of the channel matrix [1]-[2]. We try to address this problem using simple linear algebra. Let's assume the following signal model [3] for a Multi Input Multi Output (MIMO) antenna system.

$$x = Hs + w$$

Here s is the N_T by 1 signal vector, H is the N_R by N_T channel matrix and w is the N_R by 1 noise vector. The received signal vector is represented by x which has dimensions of N_R by 1, same as the noise vector. This is the model for a single snapshot in time. If we have a multiple time slots for transmission and reception than the signal matrix s would have multiple columns and so would x and w. In expanded form the MIMO signal model can be written as (assuming N_R=4 and N_T=4):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

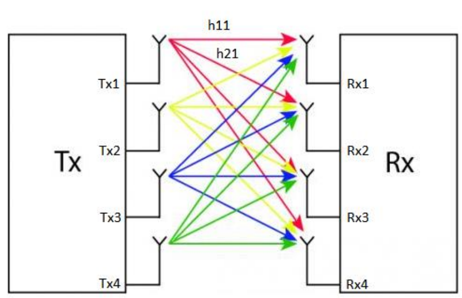

Fig 1. 4x4 MIMO Channel Model

Here $h_{12}$is the channel path between receive antenna 1 and transmit antenna 2. Similarly h_{21} is the channel path between receive antenna 2 and transmit antenna 1. In general h_{ij} is the channel coefficient between the ith receive and jth transmit antenna. Let's assume that there is no amplitude variation and there is no phase rotation introduced by the channel, then all entries of the channel matrix would be equal to unity. Let's further assume that the four transmitted symbols are $A, B, C\ and\ D$ (a modulation like PAM is assumed where the symbols lie on the real axis). Then in an AWGN channel we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$x_1 = A + B + C + D + w_1$$

$$x_2 = A + B + C + D + w_2$$

$$x_3 = A + B + C + D + w_3$$

$$x_4 = A + B + C + D + w_4$$

Let's assume that symbols are drawn from a PAM constellation with eight levels $(\pm 1, \pm 3, \pm 5, \pm 7)$ such that $A = 1, B = 3, C = 5\ and\ D = -1$. Let's further assume that there is no noise i.e. vector w is all zeros (the slightly more complex case when the noise vector is not all zeros is discussed below). The received signal would then look like:

$$x_1 = A + B + C + D = 8$$

$$x_2 = A + B + C + D = 8$$

$$x_3 = A + B + C + D = 8$$

$$x_4 = A + B + C + D = 8$$

From linear algebra we know that to find four unknowns (real numbers in this case) we need four independent equations. It is not mathematically possible to find the values of the symbols $A, B, C\ and\ D$ from the above set of equations. To simplify the above equations we have removed AWGN but even in presence of AWGN we will have the same predicament. To demonstrate this we have assumed that the noise vector has the following structure for a particular time stamp; $w = [0.1, 0.2, 0.4, -0.1]^T$.

$$x_1 = A + B + C + D + w_1 = 8.1$$

$$x_2 = A + B + C + D + w_2 = 8.2$$

$$x_3 = A + B + C + D + w_3 = 8.4$$

$$x_4 = A + B + C + D + w_4 = 7.9$$

The problem has become even more complex now with the addition of AWGN noise as we have an underdetermined system (more unknowns than number of equations). This shows that in the absence of fading there is no way to separate the signals. Spatial Multiplexing (SM) is not possible in this scenario; although MIMO theory tells us that for the four transmit and four receive case a multiplexing gain of four can be achieved. Now let's assume that we have a 4x4 MIMO fading channel, which can be substituted in the above linear model. A particular snapshot for this channel is shown below (with each entry generated as the amplitude of a circularly symmetric, complex, Gaussian Random Variable [4]).

$$H = \begin{bmatrix} 0.1 & 0.7 & 0.6 & 0.3 \\ 0.2 & 0.4 & 1.1 & 2.3 \\ 0.9 & 0.8 & 1.1 & 0.5 \\ 0.3 & 0.5 & 1.5 & 0.6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.7 & 0.6 & 0.3 \\ 0.2 & 0.4 & 1.1 & 2.3 \\ 0.9 & 0.8 & 1.1 & 0.5 \\ 0.3 & 0.5 & 1.5 & 0.6 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$x_1 = 0.1A + 0.7B + 0.6C + 0.3D = 4.9$$

$$x_2 = 0.2A + 0.4B + 1.1C + 2.3D = 4.6$$

$$x_3 = 0.9A + 0.8B + 1.1C + 0.5D = 8.3$$

$$x_4 = 0.3A + 0.5B + 1.5C + 0.6D = 8.7$$

Now the four variables namely $\mathrm{A, B, C}$ and D can be determined from the four independent linear equations. So we see that fading has this beneficial effect that what were four dependent equations are converted into four independent equations and this can be verified by checking the rank of the channel matrix. We have assumed that channel coefficients are real but in general the channel coefficients are complex quantities which when multiplied result in change of amplitude and phase of the received signal. We have further assumed that the Signal to Noise Ratio is infinite that is there is zero noise.

In case there is Additive White Gaussian Noise in addition to fading our estimation of the signal vector might not be exact and this may result in symbol errors or bit errors. The most favored method of signal estimation is Maximum Likelihood but it becomes quite complex as the number of antennas and constellation size increases. Another method which is usually adopted to estimate the signal vector is through inversion of the channel matrix. If the exact inverse does not exist we can use Moore Penrose Pseudoinverse which can be implemented in MATLAB using pinv.

Some food for thought: is there a way to arrange the equations such that four variables $\mathrm{A, B, C}$ and D can be found in absence of fading? Maybe we can compromise on the rate of transmission?

Note: A relatively slowly changing channel can be estimated by using training sequences but there is no way noise can be estimated as it changes from sample to sample. Noise is white i.e. it has a flat spectrum extending from $-\infty$ to $+\infty$.

References

[1] Foschini G.J., Gans M.J., “On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas,” Wireless Personal Communication, 6:311–335 (1998)
[2] Van Trees H.L., “Optimum Array Processing: Part IV of Detection, Estimation and Modulation Theory,” Wiley (2002)
[3] Kay S.M., “Fundamentals of Statistical Signal Processing—Estimation Theory,” Prentice Hall (1997)
[4] Alamouti S.M., “A Simple Transmit Diversity Technique for Wireless Communications,” IEEE Journal on Selected Areas in Communications, 16(8): 1451–1458 (1998)